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The Global Attractor and Its Dimension Estimation of Generalized Kolmogorov-Petrovlkii-Piskunov Equation

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Abstract

In this paper, the initial boundary value problem of a class of nonlinear generalized Kolmogorov-Petrovlkii-Piskunov equations is studied. The existence and uniqueness of the solution and the bounded absorption set are proved by the prior estimation and the Galerkin finite element method, thus the existence of the global attractor is proved and the upper bound estimate of the global attractor is obtained.

Keywords

Generalized Kolmogorov-Petrovlkii-Piskunov Equation, Existence of Solution, Hausdorff Dimension, Fractal Dimension

1. Introduction

Many scholars at home and abroad have studied the dynamical system theory described by mathematical physics equations, such as Navier-Stokes equation, nonlinear Schrödinger equation, KdV equations, reaction-diffusion equation, damped semilinear equation, etc, and estimated the dimension of the attractor.

In [1], Xu *et al.* studied the type KPP equations of (3 + 1) and (1 + 1) dimensions:

$$u_t - \alpha\Delta u + \mu u + v u^2 + \delta u^3 = 0$$

Which is the exact solution of the equation.

Wu studied the initial boundary value problem of generalized KPP equations in [2]:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \frac{D}{\sigma}\int_0^t \mathrm{e}^{-\frac{t-s}{\sigma}} \frac{\partial^2 u}{\partial x^2}(x,s)\mathrm{d}s + \lambda u(1-u),\ x\in(a,b),\ t\in(0,T), \\ u(x,0) = \varphi(x),\ x\in[a,b], \\ u(a,t) = \alpha(t),\ u(b,t) = \beta(t),\ t\in[0,T], \end{cases}$$

where $D>0$ is the diffusion coefficient, $\lambda>0$ is the reaction rate constant and $\sigma>0$ is the time delay constant. When $\sigma\to 0$, it became the famous KPP equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda u(1-u)$$

The traveling wave solution is studied and the propagation speed is $c = 2\sqrt{D\lambda}$.

In [3], Chen *et al.* proposed a new auxiliary equation method to find the exact traveling wave solution of the nonlinear development equation. By selecting Bernoulli equation with variable coefficient as the auxiliary ordinary differential equation, the generalized Burgers-KPP equation was solved according to the principle of homogeneous equilibrium, and the traveling wave solution of the equation was obtained. In [4], Cao *et al.* studied the stability and uniqueness of the generalized traveling wave of discrete Fisher-KPP equations with general time and space dependence. More studies on global attractors and their dimension estimation of equations can be seen in references [5]-[10].

In this paper, we consider the initial boundary value problem of the following generalized KPP equations:

$$(u_t - \Delta u_t) + \alpha(u - \Delta u) + \beta u^2 + \gamma u^{2p+1} = 0, \tag{1}$$

$$u|_{\partial\Omega} = 0, \tag{2}$$

$$u(x,0) = u_0(x) \tag{3}$$

where, $\Omega\subset R^3, \alpha>0, \beta>0, \gamma>0$, p is a natural number.

Notation is introduced for the convenience of narration: $\|\bullet\|$ represents the norm in $H_0^1(\Omega)$ space; $|\bullet|$ and $(\bullet,\bullet)$ represents the norm and inner product in $L^2(\Omega)$ space, and $|f| = (f,f)^{\frac{1}{2}} = \left(\int_\Omega f^2 \mathrm{d}x\right)^{\frac{1}{2}}$.

2. The Existence of a Global Attractor

In order to prove the existence of problems (1)-(3) global attractors, the following conclusions are needed:

Lemma 1. Let $u_0 \in L^2(\Omega)\cap H_0^1(\Omega)$, then the solution u of problems (1)-(3) is estimated as follows:

$$|u|^2 + \|u\|^2 \le \left(|u_0|^2 + \|u_0\|^2\right)\mathrm{e}^{-2\alpha t} + C,\quad t\ge t_1, \tag{4}$$

where $t_1 = -\frac{1}{2\alpha}\ln\left(\frac{C}{|u_0|^2 + \|u_0\|^2}\right)$, C is a normal number that depends on

α,β,γ, and $C=\frac{C_1}{2\alpha}\left(1-\mathrm{e}^{-2\alpha t}\right)$.

Proof. By taking the inner product of both sides of Equation (1) with u, we get:

$$\left(\left(u_t-\Delta u_t\right),u\right)+\alpha\left(\left(u-\Delta u\right),u\right)+\beta\left(u^2,u\right)+\gamma\left(u^{2p+1},u\right)=0 \tag{5}$$

Obtained from Formula (5):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^2+\|u\|^2\right)+\alpha\left(|u|^2+\|u\|^2\right)+\beta\int_\Omega u^3\mathrm{d}x+\gamma\int_\Omega u^{2p+2}\mathrm{d}x=0. \tag{6}$$

Thus, there:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^2+\|u\|^2\right)+\alpha\left(|u|^2+\|u\|^2\right)+\gamma\int_\Omega u^{2p+2}\mathrm{d}x\le\left|\beta\int_\Omega u^3\mathrm{d}x\right|\le\int_\Omega\left|\beta u^3\right|\mathrm{d}x. \tag{7}$$

By Young's Inequality, there:

$$\begin{aligned}\left|u^3\beta\right|&\le\frac{3\varepsilon^{\frac{2p+2}{3}}}{2p+2}u^{2p+2}+\frac{1}{\frac{2p+2}{2p-1}\varepsilon^{\frac{2p+2}{2p-1}}}\beta^{\frac{2p+2}{2p-1}}\\&=\frac{\gamma}{2}u^{2p+2}+\frac{2p-1}{2p+2}\left(\frac{p+1}{3}\gamma\right)^{-\frac{2p-1}{3}}\beta^{\frac{2p+2}{2p-1}}\\&=\frac{\gamma}{2}u^{2p+2}+C_0\end{aligned}$$

$$\left(\text{Let } \frac{3\varepsilon^{\frac{2p+2}{3}}}{2p+2}=\frac{\gamma}{2},C_0=\frac{2p-1}{2p+2}\left(\frac{p+1}{3}\gamma\right)^{-\frac{2p-1}{3}}\beta^{\frac{2p+2}{2p-1}}\right). \tag{8}$$

By substituting Formula (8) into Formula (7), we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^2+\|u\|^2\right)+\alpha\left(|u|^2+\|u\|^2\right)+\frac{\gamma}{2}\int_\Omega u^{2p+2}\mathrm{d}x\le C_1 \quad \left(\text{Let } C_1=\int_\Omega C_0\mathrm{d}x\right). \tag{9}$$

Due to $\frac{\gamma}{2}\int_\Omega u^{2p+2}\mathrm{d}x>0,(\gamma>0)$, so:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^2+\|u\|^2\right)+\alpha\left(|u|^2+\|u\|^2\right)\le C_1. \tag{10}$$

From Gronwall's inequality, we get:

$$|u|^2+\|u\|^2\le\left(|u_0|^2+\|u_0\|^2\right)\mathrm{e}^{-2\alpha t}+\frac{C_1}{\alpha}\left(1-\mathrm{e}^{-2\alpha t}\right)=\left(|u_0|^2+\|u_0\|^2\right)\mathrm{e}^{-2\alpha t}+C, \tag{11}$$

(among them $C=\frac{C_1}{\alpha}\left(1-\mathrm{e}^{-2\alpha t}\right)$).

Hence, $u\in L^2(\Omega)\cap H_0^1(\Omega)$.

Lemma 2. Let $u_0\in H_0^2(\Omega)$, then the solution u of problems (1)-(3) is estimated as follows:

$$\|u\|^2+\|u\|_{H_0^2}^2\le\left(\|u_0\|^2+\|u_0\|_{H_0^2}^2\right)\mathrm{e}^{-2\alpha t}+C', \quad t\ge t_2, \tag{12}$$

where $t_2=-\frac{1}{2\alpha}\ln\left(\frac{C'}{\|u_0\|^2+\|u_0\|_{H_0^2}^2}\right)$, C' is a normal number that depends on

α,β,γ , and $C^{/}=\dfrac{C_2}{\alpha}\left(1-\mathrm{e}^{-2\alpha t}\right)$.

Proof. By taking the inner product of both sides of Equation (1) with $-\Delta u$, we get:

$$\left(\left(u_t-\Delta u_t\right),-\Delta u\right)+\alpha\left(\left(u-\Delta u\right),-\Delta u\right)+\beta\left(u^2,-\Delta u\right)+\gamma\left(u^{2p+1},-\Delta u\right)=0. \quad (13)$$

Obtained from Formula (13):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|u\|_{H_0^2}^2\right)+\alpha\left(\|u\|^2+\|u\|_{H_0^2}^2\right)+\beta\left(u^2,-\Delta u\right)+\gamma\left(u^{2p+1},-\Delta u\right)=0,$$

Thus, there are:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|u\|_{H_0^2}^2\right)+\alpha\left(\|u\|^2+\|u\|_{H_0^2}^2\right)+\gamma\left(u^{2p+1},-\Delta u\right)=\beta\left(u^2,\Delta u\right). \quad (14)$$

Due to

$$\left(u^{2p+1},-\Delta u\right)=\left(\nabla u^{2p+1},\nabla u\right)=\left(2p+1\right)\left(u^{2p}\nabla u,\nabla u\right)=\left(2p+1\right)\int_\Omega u^{2p}\left|\nabla u\right|^2\mathrm{d}x\ge 0, \quad (15)$$

$$\text{So, } \gamma\left(u^{2p+1},-\Delta u\right)\ge 0 \ \text{(because } \gamma>0\text{)}. \quad (16)$$

And because:

$$\left|\beta\left(u^2,\Delta u\right)\right|=2\beta\left|\left(u\nabla u,\nabla u\right)\right|\le 2\beta\left|u\right|_{L^\infty}\|u\|^2\le C_2. \quad (17)$$

Obtained from Formula (14):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|u\|_{H_0^2}^2\right)+\alpha\left(\|u\|^2+\|u\|_{H_0^2}^2\right)+\gamma\left(u^{2p+1},-\Delta u\right)\le\left|\beta\left(u^2,\Delta u\right)\right|. \quad (18)$$

By (16)-(18), get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2+\|u\|_{H_0^2}^2\right)+\alpha\left(\|u\|^2+\|u\|_{H_0^2}^2\right)\le C_2. \quad (19)$$

From the Gronwall inequality, obtain:

$$\|u\|^2+\|u\|_{H_0^2}^2\le\left(\|u_0\|^2+\|u_0\|_{H_0^2}^2\right)\mathrm{e}^{-2\alpha t}+\frac{C_2}{\alpha}\left(1-\mathrm{e}^{-2\alpha t}\right)\le\left(\|u_0\|^2+\|u_0\|_{H_0^2}^2\right)\mathrm{e}^{-2\alpha t}+C^{/},$$

$$\text{(Ream } C^{/}=\frac{C_2}{\alpha}\left(1-\mathrm{e}^{-2\alpha t}\right)\text{)}. \quad (20)$$

Therefore, $u\in H_0^2\left(\Omega\right)$.

Theorem 1. Set a given function u_0, and $u_0\in H_0^2\left(\Omega\right)$, then the problems (1)-(3) has a unique solution *u*, such that $u\in H_0^2\left(\Omega\right)$.

Proof. 1) Existence: According to Lemma 1 and Lemma 2, the solution $u\in H_0^2\left(\Omega\right)$ of problems (1)-(3) exists.

2) Uniqueness: Let u,v be two solutions of Equation (1), and let $w=u-v$, then:

$$\left(u_t-\Delta u_t\right)+\alpha\left(u-\Delta u\right)+\beta u^2+\gamma u^{2p+1}=0, \quad (21)$$

$$\left(v_t-\Delta v_t\right)+\alpha\left(v-\Delta v\right)+\beta v^2+\gamma v^{2p+1}=0. \quad (22)$$

Obtained by (21) and (22):

$$\left(w_t-\Delta w_t\right)+\alpha\left(w-\Delta w\right)+\beta\left(u^2-v^2\right)+\gamma\left(u^{2p+1}-v^{2p+1}\right)=0, \tag{23}$$

where $w(0)=0, w\in H_0^2(\Omega)$.

Take the inner product of both sides of Equation (23) with $w(w=u-v)$, we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|w|^2+\|w\|^2\right)+\alpha\left(|w|^2+\|w\|^2\right)+\beta\left(\left(u^2-v^2\right),w\right)+\gamma\left(\left(u^{2p+1}-v^{2p+1}\right),w\right)=0.$$

Thus, have:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|w|^2+\|w\|^2\right)+\alpha\left(|w|^2+\|w\|^2\right)+\gamma\left(\left(u^{2p+1}-v^{2p+1}\right),w\right)=-\beta\left(\left(u^2-v^2\right),w\right)\\&\le\left|-\beta\left(\left(u^2-v^2\right),w\right)\right|\le\beta\int_\Omega\left|u^2-v^2\right||w|\mathrm{d}x\end{aligned} \tag{24}$$

And because:

$$\beta\int_\Omega\left|u^2-v^2\right||w|\mathrm{d}x\le C_3|w|^2\le C_3\left(|w|^2+\|w\|^2\right), \tag{25}$$

$$\gamma\left(\left(u^{2p+1}-v^{2p+1}\right),w\right)=\gamma\int_\Omega\left(u^{2p+1}-v^{2p+1}\right)(u-v)\mathrm{d}x\ge 0. \tag{26}$$

Obtained by (24)-(26):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|w|^2+\|w\|^2\right)+\alpha\left(|w|^2+\|w\|^2\right)\le C_3\left(|w|^2+\|w\|^2\right),$$

Thus, there are $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|w|^2+\|w\|^2\right)+\left(\alpha-C_3\right)\left(|w|^2+\|w\|^2\right)\le 0.$ (27)

So, from (27), we get: $w=0$, that is $u=v$.

Define. [7] Let $S(t)$ be a continuous operator semigroup,

$$S(t):H_0^1(\Omega)\to H_0^1(\Omega), S(t+\tau)=S(t)S(\tau),\quad \forall t\ge 0,\tau\ge 0;\quad S(0)=I_0.$$

If the compact set $A\subset H_0^1(\Omega)$ is satisfied:

1) Invariance: A is an invariant set under the action of a semigroup $S(t)$, *i.e.* $S(t)A=A, \forall t>0$.

2) Attraction: A attracts all bounded sets in $H_0^1(\Omega)$, that is, any bounded set $B\subset H_0^1(\Omega)$, have:

$$dist\left(S(t)B,A\right)=\sup_{x\in B}\inf_{y\in A}\left\|S(t)x-y\right\|_{H_0^1}\to 0,(t\to+\infty).$$

In particular, when $t\to+\infty$, all orbitals $S(t)u_0$ from u_0 converge to A, that is:

$$dist\left(S(t)u_0,A\right)\to 0,(t\to+\infty).$$

Then, the compact set A is called the global attractor of semigroup $S(t)$.

Theorem 2. [8] Let E be a Banach space, $\{S(t)\}(t\ge 0)$ be a family of operators, $S(t):E\to E$, $S(t+\tau)=S(t)S(\tau)$, $S(0)=I$,where I is the identity operator. Let $S(t)$ satisfy:

1) $S(t)$ is bounded, that is, $\forall R>0,\|u\|_E\le R$, then there is a constant $C(R)$ such that $\left\|S(t)u\right\|_{H_0^1}\le C(R)\left(t\in[0,+\infty)\right)$.

2) There is a bounded absorption set $B_0 \subset E$, that is, any bounded absorption set $B \subset E$, and there is a constant t_0 time such that the bounded absorption set $S(t)B \subset B_0 (t > t_0)$.

3) For $t > 0$, $S(t)$ is a completely continuous operator.

Then, the semigroup $S(t)$ has a compact global attractor *A*.

Theorem 3. If the problems (1)-(3) have *A* solution and satisfies the conditions of Lemma 2, then the problems (1)-(3) have a global attractor *A*, that is, there is a compact set $A \subset H_0^1(\Omega)$ such that:

1) $S(t)A = A, \forall t > 0$.

2) Any bounded set $B \subset H_0^1(\Omega)$, yes:

$$dist(S(t)B, A) = \sup_{x \in B} \inf_{y \in A} \|S(t)x - y\|_{H_0^1} \to 0, (t \to +\infty).$$

In particular, when $t \to +\infty$, all orbitals $S(t)u_0$ from u_0 converge to *A*, that is:

$$dist(S(t)u_0, A) \to 0, (t \to +\infty).$$

Proof. Let $u(0) = u_0 \in H_0^2(\Omega), |u_0| \le R_0, \|u_0\| \le R_1, \|u_0\|_{H_0^2} \le R_2$, and $u = S(t)u_0$, then follows from Lemma 1: $|u|^2 + \|u\|^2 \le \left(|u_0|^2 + \|u_0\|^2\right) e^{-2\alpha t} + C$.

Thus, there is:

$$|u|^2 + \|u\|^2 \le \left(R_0^2 + R_1^2\right) e^{-2\alpha t} + C. \tag{28}$$

Ream $\left(R_0^2 + R_1^2\right) e^{-2\alpha t} \le C$, then $t \ge -\frac{1}{2\alpha} \ln\left(\frac{C}{R_0^2 + R_1^2}\right)$.

Take $t_1 = -\frac{1}{2\alpha} \ln\left(\frac{C}{R_0^2 + R_1^2}\right)$, then $t \ge t_1$, Formula (28) can be written as: $|u|^2 + \|u\|^2 \le 2C$.

Similarly, from Lemma 2:

$$\|u\|^2 + \|u\|_{H_0^2}^2 \le \left(\|u_0\|^2 + \|u_0\|_{H_0^2}^2\right) e^{-2\alpha t} + C'.$$

Thus, have:

$$\|u\|^2 + \|u\|_{H_0^2}^2 \le \left(R_1^2 + R_2^2\right) e^{-2\alpha t} + C'. \tag{29}$$

Ream $\left(R_1^2 + R_2^2\right) e^{-2\alpha t} \le C'$, then $t \ge -\frac{1}{2\alpha} \ln\left(\frac{C'}{R_1^2 + R_2^2}\right)$.

Take $t_2 \ge -\frac{1}{2\alpha} \ln\left(\frac{C'}{R_1^2 + R_2^2}\right)$, then $t \ge t_2$, Formula (29) can be written as: $\|u\|^2 + \|u\|_{H_0^2}^2 \le 2C'$.

Let $B = \left\{u \in H_0^1(\Omega) : \|u\| \le \sqrt{2\left(C + C'\right)}\right\}$, and *u* is bounded in $H_0^2(\Omega)$, and $H_0^2(\Omega)$ is tightly embedded in $H_0^1(\Omega)$, so *B* is the compact absorption set in $H_0^1(\Omega)$.

Let $u_0 \in H_0^1(\Omega), v_0 \in H_0^1(\Omega)$, and u, v be the corresponding two solutions of the equation, $S(t)u_0 = u, S(t)v_0 = v$, and let $w = u - v$, then w satisfies:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|w|^2 + \|w\|^2\right) + \alpha\left(|w|^2 + \|w\|^2\right) + \beta\int_\Omega\left(u^2 - v^2\right)(u - v)\mathrm{d}x$$
$$+\gamma\int_\Omega\left(u^{2p+1} - v^{2p+1}\right)(u - v)\mathrm{d}x = 0$$

From uniqueness, can be obtained:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|w|^2 + \|w\|^2\right) + (\alpha - C_3)\left(|w|^2 + \|w\|^2\right) \le 0.$$

Thus, have:

$$|w|^2 + \|w\|^2 \le \left(|w_0|^2 + \|w_0\|^2\right)\mathrm{e}^{2(C_3 - \alpha)t}.$$

So, the operator $S(t): H_0^1(\Omega) \to H_0^1(\Omega)$ is continuous.

Thus, from Theorem 2, we know that problems (1)-(3) exist global attractors:

$$A = w(B) = \bigcap_{S\ge 0}\overline{\bigcup_{t\ge 0} S(t)B}.$$

3. The Dimension Estimation of the Global Attractor

In order to establish the Hausdoarff dimension of the problems (1)-(3) global attractor A, the upper bound of the fractal dimension. A linear variational problem for problems (1)-(3) needs to be established:

$$(v_t - \Delta v_t) + \alpha(v - \Delta v) + 2\beta uv + (2p+1)\gamma u^{2p}v = 0, \text{ (let } u_t = v\text{)},$$

$$\textit{i.e.}\ \ (I - \Delta)v_t + \alpha(I - \Delta)v + 2\beta uv + (2p+1)\gamma u^{2p}v = 0, \tag{30}$$

$$v(0) = v_0(x), \tag{31}$$

where $v_0 \in H_0^1(\Omega), u(t) = S(t)u_0$ is solution of problems (1)-(3) with $u_0 \in A$.

Lemma 3. Let $v_0 \in H_0^1(\Omega)$ and $u_0, v_0 \in A$, $S(t)u_0 \in H_0^1(\Omega)$, then the linearization problems (30) and (31) have unique solutions:

$$v(x,t) \in L^\infty\left(0,T; H_0^1(\Omega)\right), T > 0. \tag{32}$$

In addition, remember $v(t) = G(t)v_0$, then $\forall T > 0, \bar{R} \ge 0$, there is a constant E related to R and T, such that:

$$\left\|S(t)(u_0 + v_0) - S(t)u_0 - G(t)v_0\right\|_{H_0^1} \le E\|v_0\|_{H_0^1}^2, \forall t \in [0,T], \tag{33}$$

where $\|u_0\| \le \bar{R}, \|u_0 + v_0\| \le \bar{R}, u_0 + v_0 \in A$, this shows that the operator $S(t)$ is uniformly differentiable on A, and that the differential of $S(t)$ in $u_0 \in A$ at $H_0^1(\Omega)$ is:

$$DS(t)u_0 : v_0 \in H_0^1(\Omega) \to G(t)v_0 \in H_0^1(\Omega).$$

Let $W = I - \Delta$, and W is positive definite dense, then W^{-1} exists and is bounded.

So, (30) becomes:

$$Wv_t + \alpha Wv + 2\beta uv + (2p+1)\gamma u^{2p} v = 0.$$

Multiply both sides by W^{-1}, you get:

$$v_t + L(u(t))v = 0, \quad (34)$$

where $L(u(t))v = \alpha v + 2\beta W^{-1}uv + (2p+1)\gamma W^{-1}u^{2p}v$.

Let's say $v_1(t), v_2(t), \cdots, v_m(t)$ is m solutions of (30) (31), and the corresponding initial values are: $v_1(0), v_2(0), \cdots, v_m(0)$, then:

$$\begin{aligned}&\left|v_1(t)\wedge v_2(t)\wedge\cdots\wedge v_m(t)\right|_{\wedge^m H_0^1}\\ &\le \left|v_1(0)\wedge v_2(0)\wedge\cdots\wedge v_m(0)\right|_{\wedge^m H_0^1}\exp\left[-\int_0^t Tr\left(L(u(\tau))\circ Q_m(\tau)\right)\mathrm{d}\tau\right]\end{aligned}$$

Let $Q_m(t)$ represent the orthogonal projection on the space spanning $H_0^1(\Omega)$ to $\{v_1(t), v_2(t), \cdots, v_m(t)\}$.

Next, the exponential attenuation of the m dimensional volume element $\left|v_1(t)\wedge v_2(t)\wedge\cdots\wedge v_m(t)\right|$ and the dimension estimation of the global attractor A are considered. Thus, there is:

Theorem 4. Let the global attractor $A \in H_0^1(\Omega)$ of problems (1)-(3), and m satisfy: $m-1 < \left(\frac{K_3}{K_2}\right)^3 \le m$, (among which $K_2 = \alpha C_4 \lambda_1$, $K_3 = dK_1$, $d = 2\beta|u|_{L^\infty}$).

Then:

1) When $t \to +\infty$, the m dimensional volume element $\left|v_1(t)\wedge v_2(t)\wedge\cdots\wedge v_m(t)\right|$ vwill decay exponentially.

2) $\dim(A)_H \le m, \dim(A)_F \le 2m$.

Proof. Let's say $\varphi_1, \varphi_2, \cdots, \varphi_m$ is a set of orthogonal bases for $H_0^1(\Omega)$ and satisfies:

$$(W\varphi_i, \varphi_i) = 1, (W\varphi_i, \varphi_i) = \lambda_i(\varphi_i, \varphi_i), (i = 1, 2, \cdots, m),$$

where $\lambda_i (0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m; i = 1, 2, \cdots, m)$ is the eigenroot of the operator $W(W = I - \Delta)$.

The lower bound of $Tr\left(L(u(t))\circ Q_m(t)\right)$ is estimated below:

$$\begin{aligned}&Tr\left(L(u(t))\circ Q_m(t)\right)\\ &= \sum_{i=1}^{\infty}\left(L(u(t))\circ Q_m(t)\varphi_i(t), W\varphi_i(t)\right) = \sum_{i=1}^{m}\left(L(u(t))\varphi_i(t), W\varphi_i(t)\right)\\ &= \sum_{i=1}^{m}\left(\alpha\varphi_i + 2\beta W^{-1}u\varphi_i + (2p+1)\gamma W^{-1}u^{2p}\varphi_i, W\varphi_i\right)\\ &= \alpha\sum_{i=1}^{m}(\varphi_i, W\varphi_i) + 2\beta\sum_{i=1}^{m}(u\varphi_i, \varphi_i) + (2p+1)\gamma\sum_{i=1}^{m}\left(u^{2p}\varphi_i, \varphi_i\right)\end{aligned} \quad (35)$$

Again $\left(u^{2p}\varphi_i, \varphi_i\right) \ge 0$, so (35) can be written as:

$$Tr\left(L(u(t))\circ Q_m(t)\right) \ge \alpha m + 2\beta\sum_{i=1}^{m}(u\varphi_i, \varphi_i) \ge \alpha m - 2\beta|u|_{L^\infty}\sum_{i=1}^{m}(\varphi_i, \varphi_i). \quad (36)$$

Because $(W\varphi_i, \varphi_i) = \lambda_i(\varphi_i, \varphi_i)$, then there is $(\varphi_i, \varphi_i) = \frac{1}{\lambda_i}(W\varphi_i, \varphi_i) = \frac{1}{\lambda_i}$.

Therefore, Formula (36) becomes:

$$Tr\left(L\left(u\left(t\right)\right)\circ Q_m\left(t\right)\right)\geq \alpha m-2\beta\left|u\right|_{L^\infty}\sum_{i=1}^{m}\frac{1}{\lambda_i}.$$

That is $Tr\left(L\left(u\left(t\right)\right)\circ Q_m\left(t\right)\right)\geq \alpha m-d\sum_{i=1}^{m}\frac{1}{\lambda_i}$, (among others $d=2\beta\left|u\right|_{L^\infty}$). (37)

Let $\lambda_j\sim C_4\lambda_1 j^{2/3}$, then Formula (37) becomes:

$$Tr\left(L\left(u\left(t\right)\right)\circ Q_m\left(t\right)\right)\geq \alpha m-d\sum_{j=1}^{m}\frac{1}{C_4\lambda_1 j^{2/3}}=\alpha m-\frac{d}{C_4\lambda_1}\sum_{j=1}^{m}\frac{1}{j^{2/3}}. \tag{38}$$

Thus, have:

$$Tr\left(L\left(u\left(t\right)\right)\circ Q_m\left(t\right)\right)\geq \alpha m-\frac{d}{C_4\lambda_1}m^{\frac{2}{3}}\sum_{j=1}^{m}\frac{1}{j^{4/3}}\geq \alpha m-\frac{d}{C_4\lambda_1}m^{\frac{2}{3}}\sum_{j=1}^{\infty}\frac{1}{j^{4/3}}. \tag{39}$$

And $\sum_{k=1}^{m}\frac{1}{k^{4/3}}\leq\sum_{k=1}^{\infty}\frac{1}{k^{4/3}}$, and the series $\sum_{k=1}^{\infty}\frac{1}{k^{4/3}}$ converges, let's say the series $\sum_{k=1}^{\infty}\frac{1}{k^{4/3}}$ converges to the normal number K_1, so Formula (39) becomes:

$$Tr\left(L\left(u\left(t\right)\right)\circ Q_m\left(t\right)\right)\geq \alpha m-\frac{dK_1}{C_4\lambda_1}m^{\frac{2}{3}}. \tag{40}$$

When $\alpha m-\frac{dK_1}{C_4\lambda_1}m^{\frac{2}{3}}>0$, have $m^{\frac{2}{3}}\left(\alpha m^{\frac{1}{3}}-\frac{dK_1}{C_4\lambda_1}\right)>0$, *i.e.* $m>\left(\frac{dK_1}{\alpha C_4\lambda_1}\right)^3=\left(\frac{K_3}{K_2}\right)^3$, (among them $K_2=\alpha C_4\lambda_1, K_3=dK_1$), thereby having $m-1<\left(\frac{K_3}{K_2}\right)^3\leq m$.

Let $\mu_j\left(j\in N\right)$ be the Lyapunov exponent, then there is an inequality:

$$\mu_1+\mu_2+\cdots+\mu_m\leq -\alpha m+\frac{dK_1}{C_4\lambda_1}m^{\frac{2}{3}}<0,$$

So, $\mu_1+\mu_2+\cdots+\mu_m<0$, and $\frac{\mu_1+\mu_2+\cdots+\mu_m}{\left|\mu_1+\mu_2+\cdots+\mu_m\right|}\leq 1$.

Therefore, $\dim\left(A\right)_H\leq m, \dim\left(A\right)_F\leq 2m$.

4. Closing Remarks

In this paper, the existence and uniqueness of the solutions (1)-(3) of the initial boundary value problem of generalized KPP equation and the existence of the global attractor are studied. The Hausdorff dimension and fractal dimension of the global attractor are estimated.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Xu, L.P., Zhang, C.Y. and Mi, X.H. (2008) Linear Solution of a Class of Nonlinear Heat Transfer Equations. *Journal of Sichuan Normal University*, **31**, 393-396.

[2] Wu, H.W. (2009) Numerical Solutions of Generalized KPP Equations. *Computational Mathematics*, **31**, 137-150.

[3] Chen, Z.G. and Zhang, W.Z. (2010) Variable Coefficient Auxiliary Equation Method for Solving Generalized Burgers-KPP Equation. *Journal of North China Institute of Water Resources and Electric Power*, **31**, 151-154.

[4] Cao, F. and Shen, W.X. (2017) Stability and Uniqueness of Generalized Traveling Waves of Discrete Fisher-KPP Equations in Inhomogeneous Media. *Chinese Science and Mathematics*, **47**, 1787-1808. (In Chinese) https://doi.org/10.1360/N012017-00116

[5] Ghidaglia, J.-M. (1988) Weakdy Damped Fored Korteweg-De Vries Equation Behave as a Finite Dimensional Dynamical System in the Long Time. *Journal of Differential Equations*, **74**, 369-390. https://doi.org/10.1016/0022-0396(88)90010-1

[6] Guo, B.L. (1992) Nonlinear Evolution Equations. Shanghai Science and Technology Education Press, Shanghai.

[7] Zhang, R.-F. and Li, R.-G. (2005) The Global Solution for a Class of Dissipative Hasegawa-Mima Equation. *Chinese Quarterly Journal of Mathematics*, **20**, 360-366.

[8] Temam, R. (2000) Infinite Dimensional Dynamical System in Mechanics and Physics. Springer-Verlag, New York, 15-329.

[9] Zhang, R.F. and Guo, B.L. (2006) Global Attractor of Hasegava-Mima Equation. *Applied Mathematics and Mechanics*, **27**, 567-574. https://doi.org/10.1007/s10483-006-0501-1

[10] Xu, G.-G. and Lin, G.-G. (2011) Global Attractor and Its Dimension Estimation for Generalized Boussinesq Equation. *Science and Technology Information of China*, No. 10, 54-61.