Journal of Applied Mathematics and Physics, 2024, 12, 2691-2701
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

A Modified Transitional Korteweg-De Vries Equation: Posed in the Quarter Plane

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How to cite this paper: Bu, C. (2024) A Modified Transitional Korteweg-De Vries Equation: Posed in the Quarter Plane. *Journal of Applied Mathematics and Physics*, **12**, 2691-2701.
https://doi.org/10.4236/jamp.2024.127160

Received: June 26, 2024
Accepted: July 28, 2024
Published: July 31, 2024

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Abstract

This paper is concerned with a modified transitional Korteweg-de Vries equation $u_t + f(t)u^2u_x + u_{xxx} = 0$, $(x,t) \in \mathbf{R}^+ \times \mathbf{R}^+$ with initial value $u(x,0) = g(x) \in H^4(\mathbf{R}^+)$ and inhomogeneous boundary value $u(0,t) = Q(t) \in C^2([0,\infty))$. Under the conditions either 1) $f(t) \le 0$, $f'(t) \ge 0$ or 2) $f(t) \le -\alpha$ where $\alpha > 0$, we prove the existence of a unique global classical solution.

Keywords

Modified Transitional KdV Equation, Initial-Boundary Value Problem, Semi-Group, Local and Global Existence

1. Introduction

The Korteweg-de Vries equation (KdV)

$$u_t + uu_x + u_{xxx} = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R} \tag{1}$$

arises originally in connection with a certain regime of surface water waves (see [1]-[5] for instance). It has been extensively studied and derived as a model for undirectional propagation of small-amplitude long waves in a number of physical systems, such as the evolution of shallow water waves, ion acoustic waves, long waves in shear flows.

It is well-known that KdV equation is a soliton equation and hence has Hamiltonian structures and an infinitely number of independent motion constants in involution [6]-[8]. The existence of a unique global solution and well-posedness for the KdV equation with smooth initial data can be found in [9]-[14]. KdV equation's cousin, the modified KdV equation (mKdV)

$$u_t - 6\sigma u^2 u_x + u_{xxx} = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R} \tag{2}$$

has also been investigated [15]-[24] and shown that it also has infinitely many conserved quantities. The famous Miura transformation establishes the connection between KdV and mKdV equations, namely, a solution of the mKdV equation φ yields a solution of the KdV equation $\varphi^2 + \delta\varphi_x$. However, even though the mKdV equation is derived from the KdV equation, the symmetries of the two systems are not the same. While the KdV equation is Galilean invariant, the mKdV equation is not and solutions of the KdV equation and the mKdV equation are not in 1-1 correspondence. For the Cauchy problem of the mKdV equation, there is a unique global classical solution. In addition, for a more generalized KdV equation in the form of

$$u_t + a(u)u_x + u_{xxx} = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R}, \tag{3}$$
$$u(x,0) \in H^s(\mathbf{R}), \quad s \geq 3$$

a unique global solution in $H^s(\mathbf{R})$ can be found if $\sup a(r)/r^4 \leq 0$ as $r \to \infty$ [25] [26].

For the following generalized mKdV equation

$$u_t - 6\sigma u^p u_x + u_{xxx} = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R} \tag{4}$$

where p is a positive integer, it is found to be integrable only in two cases: $p = 1$ (KdV) and $p = 2$ (mKdV) (see [27] for details).

Historically, important evolution equations like KdV and nonlinear Schrödinger (NLS) equations have been the subject of prolific study, especially for the pure initial value problems *i.e.* the Cauchy problems. But in many applications in mathematical physics, the model leads to mixed initial-boundary value problems when the boundary value is nonzero. This is called the forced problems. As we know, the KdV equation can be used to describe long waves. In order to assess the performance of the KdV equation as a model for waves in a particular system, it might be inconvenient to consider the pure initial value problem as it may be difficult in determining the entire wave profile accurately at a given time. An approach to obtain undirectional waves to test the appurtenance of KdV, is to generate waves at one end of a homogeneous stretch of the medium in question and to allow them to propagate into the initial undisturbed medium beyond the wavemaker [28].

This model leads to the following inhomogeneous initial-boundary value problem in which global existence and well-posedness were established [29] [30]:

$$u_t - 6\sigma u u_x + u_{xxx} = 0, \quad 0 \leq x,t < \infty. \tag{5}$$
$$u(x,0) = g(x), \quad u(0,t) = Q(t).$$

For the 1D nonlinear Schrödinger equation (NLS) in a semi-infinite line with initial condition and inhomogeneous boundary condition

$$iu_t = u_{xx} + k|u|^2 u, \quad 0 \leq x,t < \infty, \tag{6}$$

$$u(x,0)=g(x),\quad u(0,t)=Q(t),$$

inverse transformation, global existence and well-posedness results can be found in [31]-[34].

Meanwhile, for the following NLS in n-dimensional space

$$i\partial_t u=\Delta u-g|p|^{p-1}u,\quad x\in\Omega\subset\mathbf{R}^n, \tag{7}$$
$$u(x,0)=\varphi(x),\quad u(x,t)=Q(x,t)\ \text{for}\ x\in\partial\Omega,$$

where $g>0$, $p>1$, $\varphi(x)\in H^1(\Omega)$, $Q\in C^3(\partial\Omega\times(-\infty,\infty))$ has compact support and satisfies compactibility conditions $\varphi(x)\equiv Q(x,0)$ on $\partial\Omega$ in the sense of traces. Then there exists a global solution
$u\in L^\infty_{loc}((-\infty,\infty);H^1(\Omega)\cap L^{p+1}(\Omega))$ for $t\in\mathbf{R}$. The PDE is understood in the sense of distribution while the boundary condition is understood as
$u(\cdot,t)-Q(\cdot,t)\in H^1_0(\Omega)$ for a.e. t. Furthermore, if $1<p<1+\frac{4}{n-2}$, this solution is unique [35].

For the following modified KdV posed in the quarter plane with initial condition and inhomogeneous boundary condition

$$u_t-6\sigma u^p u_x+u_{xxx}=0,\quad 0\le x,t<\infty \tag{8}$$
$$u(x,t)=g(x),\quad u(0,t)=P(t)$$

where $p\ge 2$ is an even integer, $\sigma>0$. The existence of a unique global classical solution in $C^0([0,\infty),H^3_0(\mathbf{R}^+))\cap C^1((0,\infty),L^2(\mathbf{R}^+))$ is proved in [36] provided that $g(x)\in H^4(\mathbf{R}^+,P(t)\in C^2([0,\infty)))$.

On the other hand, transitional KdV equation arises in the study of long solitary waves in lakes and estuaries. It propagates on the thermocline separating two layers of fluids of almost equal densities. The effect of the change in the depth of the bottom layer which the wave feels as it approaches the shore, results in the coefficient of the nonlinear term (see [37] for example). Global well-posedness for the Cauchy problem of the following transitional KdV equation was obtained in [38]:

$$\partial_t u+\partial_x^3 u+f(t)u\partial_x u=0,\quad u(x,0)=\varphi(x) \tag{9}$$

where $x,t\in\mathbf{R}$, $f\in C(\mathbf{R})$, $f'\in L^1_{loc}(\mathbf{R})$.

We notice that when f is a constant, this is the classic KdV equation.

We are primarily interested in the following modified transitional KdV equation posed in the quarter plane with inhomogeneous boundary

$$u_t+f(t)u^2u_x+u_{xxx}=0,\quad (x,t)\in\mathbf{R}^+\times\mathbf{R}^+, \tag{10}$$
$$u(x,0)=g(x)\in H^4(\mathbf{R}^+),\quad u(0,t)=Q(t)\in C^2([0,\infty)).$$

We assume compactiblity conditions $g(0)=Q(0)$,
$Q'(0)=-f(0)g^2(0)g_x(0)-g_{xxx}(0)$ hold.

The modified transitional KdV is a variation of the traditional transitional KdV. It changes the nonlinearity term from $f(t)uu_x$ to $f(t)u^2u_x$. In the sense of physics, the dispersion term in transitional KdV tends to spread out the wave and the nonlinearity term tends to localize the wave. Thus such a variation effectively changes the rate how the wave is localized. We are not aware of any results for inhomogeneous boundary value problems for the modified transitional KdV.

2. Local Existence and Uniqueness

The process of proving local and global existence-uniqueness theorem is accomplished this way. We first convert the original PDE to an integral equation in functional space, and use the semigroup theory to prove the existence of a unique local solution. Then we work on various estimates to establish the L^∞ bound for the solution based on initial-boundary data to conclude that the obtained unique local solution is indeed global.

We first utilize the standard technique of changing of variables $u = v + Q(t)\mathrm{e}^{-x}$. This substitution in (10) yields

$$v_t + f(t)v^2 v_x + v_{xxx} + (a_1 v + a_0)v_x = (Q'(t) - Q(t))\mathrm{e}^{-x} - b_2 v^2 - b_1 v - b_0 \tag{11}$$

where $a_i, b_i, 0 \le i \le 2$ depend on $f(t), Q(t)$ and $g(x)$. Furthermore, $v(x,0) = h(x) = g(x) - Q(0)\mathrm{e}^{-x}$, $v(0,t) = 0$ and $a_i, b_i \in H^2(\mathbf{R}^+)$, $0 \le i \le 2$. Let $H_0^2(\mathbf{R}^+)$ be a subspace of $H^2(\mathbf{R}^+)$ with standard Sobolev norm, then (11) is converted to a quasi-linear equation of evolution

$$\frac{\mathrm{d}v}{\mathrm{d}t} + A(t,v)v = B(t,v), \tag{12}$$

$$v(x,0) = h(x) = g(x) - Q(0)\mathrm{e}^{-x}, \quad v(0,t) = 0,$$

where

$$A(t,v)v = \left(f(t)v^2 + a_1 v + a_0\right)v_x + v_{xxx}, \tag{13}$$

$$B(t,v) = (Q'(t) - Q(t))\mathrm{e}^{-x} - b_2 v^2 - b_1 v - b_0.$$

Let $S = (1 + D^2)^{s/2}$, $s \ge 3$, $Y = H_0^3(\mathbf{R}^+)$, $X = L_0^2(\mathbf{R}^+)$ then Y is continuously and densely embedded in *X* with usual norms. Since $A(t,v) = A(v) = D^3 + b(t,v)D$ where $b(t,v) = f(t)v^2 + a_1 v + a_0$, the leading term D^3 in $A(v)$ is the generator of a contraction semi-group in *X*, skew-adjoint with $H_0^3(\mathbf{R}^+)$. The perturbing term $b(t,v)D$ is quasi-accretive and relatively bounded with respect to D^3. We consider the solution for (12) on any time interval $[0,T]$. Since $\partial_t b(t,v) \in C^1$ as $Q(t) \in C^2$, $A(v)$ is a first-order differential operator with a smooth coefficient $b(v)$. We have the following estimate

$$\begin{aligned} \|(A(v) - A(z))w\|_X &= \|(b(t,v) - b(t,z))w_x\|_X \\ &\le \|b(t,v) - b(t,z)\|_X \|w_x\|_\infty \\ &\le \alpha(T)\|v - z\|_X \|w\|_Y \end{aligned} \tag{14}$$

provided that $u(x,0) \in H^4(\mathbf{R}^+)$, $u(0,t) \in C^2([0,\infty))$ and $f \in C^1([0,\infty))$. The presence of D^3 in $A(t,v)$ does not introduce any trouble since it commutes with *S* and is independent of *v*. Since $Q \in C^2$, *f* is a locally bounded function, we see that $t \to B(t,v)$ is *X*-Lipschitz continuous for each $t \in [0,T]$. Similar to the results on abstract quasi-linear equation of evolution in [39] [40], we have the following existence theorem.

Theorem 2.1 (*Local Existence and Uniqueness*) *For the modified transitional Korteweg-de Vries Equation* (12) *posed in the quarter plane, there exists a unique classical solution* $v \in C^0\left(\left[0,T_M\right), H_0^3(\mathbf{R})\right) \cap C^1\left(\left[0,T_M\right), L^2(\mathbf{R})\right)$ *for some* $T_M > 0$ *if* $v(x,0) \in H^4(\mathbf{R})$. *Thus there is a unique local classical solution* $u \in C^0\left(\left[0,T_M\right), H^3(\mathbf{R})\right) \cap C^1\left(\left[0,T_M\right), L^2(\mathbf{R})\right)$ *for* (10) *with inhomogeneous boundary data provided that* $u(x,0) \in H^4(\mathbf{R})$.

3. Global Existence Theorem

We start the process by working on several estimates involving boundary data in (10). To prove the global existence, we need to show that $\|u\|_{H^1(\mathbf{R}^+)}$ is bounded on $\left[0,T_M\right]$ for any $T_M > 0$. Define $Q(t) = u(0,t)$, $P(t) = u_x(0,t)$, $R(t) = u_{xx}(0,t)$. We know that $Q(t) \in C^2([0,\infty))$ but no assumption is given regarding $P(t)$ and $R(t)$. However, we may safely assume that $P(t)$ and $R(t)$ are defined at least on a finite interval, which is implied by local existence. We first differentiate $\|u\|_2^2$ and $\|u_x\|_2^2$ with respect to t variable and substitute them in (10) to get

$$\begin{aligned}\partial_t \int_0^\infty u^2 \mathrm{d}x &= \int_0^\infty 2uu_t \mathrm{d}x = \int_0^\infty 2u\left(-u_{xxx} - f(t)u^2u_x\right)\mathrm{d}x \\ &= -2uu_{xx}\Big|_0^\infty + \int_0^\infty 2u_xu_{xx}\mathrm{d}x - \int_0^\infty 2f(t)u^3u_x\mathrm{d}x \\ &= 2Q(t)R(t) + (u_x)^2\Big|_0^\infty - 2f(t)\frac{1}{4}u^4\Big|_0^\infty \\ &= 2Q(t)R(t) - P^2(t) + \frac{1}{2}f(t)Q^4.\end{aligned} \tag{15}$$

$$\begin{aligned}\partial_t \int_0^\infty u_x^2 \mathrm{d}x &= \int_0^\infty 2u_xu_{xt}\mathrm{d}x = 2u_xu_t\Big|_0^\infty - \int_0^\infty 2u_{xx}u_t\mathrm{d}x \\ &= -2P(t)Q'(t) - \int_0^\infty 2u_{xx}\left(-u_{xxx} - f(t)u^2u_x\right)\mathrm{d}x \\ &= -2P(t)Q'(t) + \int_0^\infty 2u_{xx}u_{xxx}\mathrm{d}x + \int_0^\infty 2f(t)u^2u_xu_{xx}\mathrm{d}x \\ &= -2P(t)Q'(t) + (u_{xx})^2\Big|_0^\infty + \int_0^\infty 2f(t)u^2u_xu_{xx}\mathrm{d}x \\ &= -2P(t)Q'(t) - R^2(t) + \int_0^\infty 2f(t)u^2u_xu_{xx}\mathrm{d}x.\end{aligned} \tag{16}$$

Next we differentiate $-\frac{1}{6}\int_0^\infty u^4 \mathrm{d}x$ with respect to t variable to get

$$\begin{aligned}\partial_t\left(\frac{1}{6}\int_0^\infty f(t)u^4\mathrm{d}x\right) &= -\frac{1}{6}\int_0^\infty f'(t)u^4\mathrm{d}x - \frac{1}{6}\int_0^\infty f(t)4u^3u_t\mathrm{d}x \\ &= -\frac{1}{6}f'(t)\|u\|_4^4 - \frac{2}{3}\int_0^\infty f(t)u^3\left(-u_{xxx} - f(t)u^2u_x\right)\mathrm{d}x \\ &= -\frac{1}{6}f'(t)\|u\|_4^4 + \frac{2}{3}\int_0^\infty f(t)u^3u_{xxx}\mathrm{d}x + \frac{2}{3}\int_0^\infty f^2(t)u^5u_x\mathrm{d}x \\ &= -\frac{1}{6}f'(t)\|u\|_4^4 + \frac{2}{3}f(t)u^3u_{xx}\Big|_0^\infty - 2\int_0^\infty 2f(t)u^2u_xu_{xx}\mathrm{d}x + \frac{2}{3}f^2(t)\frac{u^6}{6}\Big|_0^\infty \\ &= -\frac{1}{6}f'(t)\|u\|_4^4 - \frac{2}{3}f(t)Q^3(t)R(t) - 2\int_0^\infty f(t)u^2u_xu_{xx}\mathrm{d}x - \frac{1}{9}f^2(t)Q^6(t)\end{aligned} \tag{17}$$

Add (15)-(17) to get:

$$\begin{aligned}
&\partial_t\left(\|u\|_2^2+\|u_x\|_2^2-\frac{1}{6}\int_0^\infty f(t)u^4\mathrm{d}x\right)\\
&=2Q(t)R(t)-P^2(t)+\frac{1}{2}f(t)Q^4-2P(t)Q'(t)-R^2(t)+2f(t)\int_0^\infty u^2u_xu_{xx}\mathrm{d}x\\
&\quad-\frac{1}{6}f'(t)\|u\|_4^4-\frac{2}{3}f(t)Q^3(t)R(t)-2f(t)\int_0^\infty u^2u_xu_{xx}\mathrm{d}x-\frac{1}{9}f^2(t)Q^6(t) \qquad (18)\\
&=2Q(t)R(t)-P^2(t)+\frac{1}{2}f(t)Q^4-2P(t)Q'(t)-R^2(t)\\
&\quad-\frac{1}{6}f'(t)\|u\|_4^4-\frac{2}{3}f(t)Q^3(t)R(t)-\frac{1}{9}f^2(t)Q^6(t).
\end{aligned}$$

Now we are in position to prove the following global existence theorem.

Theorem 3.1 (*Global Existence*) *For the initial-boundary value problem of modified transitional KdV* (10), $g(x)\in H^4(\mathbf{R}), Q(t)\in C^2([0,\infty))$, *there exists a unique global classical solution* $u\in C^0([0,\infty),H^3(\mathbf{R}))\cap C^1([0,\infty),L^2(\mathbf{R}))$ *under the conditions either* 1) $f(t)\le 0$, $f'(t)\ge 0$ *or* 2) $f(t)\le -\alpha$ *where* $\alpha>0$.

To prove global existence, we need to show that $\|u\|_{H^1}$ is bounded on any $[0,T_M)$.

We notice that *f* is the coefficient of the nonlinear term which controls the rate how the wave is localized. If *f* is constant, then the modified transitional KdV becomes the classic KdV equation. If $f=f(t)$ is not constant, then the rate how the wave is localized not only depends on the location, but also depends on time. The conditions 1) $f(t)\le 0$, $f'(t)\ge 0$ or 2) $f(t)\le -\alpha$ where $\alpha>0$ are necessary to control the growth rate of *u* in H^3 space to establish the global existence. We are not aware of any previous results on the model involving inhomogeneous boundary here.

First consider case 1) $f(t)\le 0$, $f'(t)\ge 0$. From (18) we see that for any $t\in[0,T_M)$

$$\begin{aligned}
&\partial_t\left(\|u\|_2^2+\|u_x\|_2^2-\frac{1}{6}\int_0^\infty f(t)u^4\mathrm{d}x\right)\\
&=2Q(t)R(t)-P^2(t)+\frac{1}{2}f(t)Q^4-2P(t)Q'(t)-R^2(t)\\
&\quad-\frac{1}{6}f'(t)\|u\|_4^4-\frac{2}{3}f(t)Q^3(t)R(t)-\frac{1}{9}f^2(t)Q^6(t) \qquad (19)\\
&\le\frac{1}{2}f(t)Q^4-2P(t)Q'(t)-P^2(t)+\left(2Q(t)-\frac{2}{3}f(t)Q^3(t)\right)R(t)-R^2(t)\\
&\le c_0+c_1P(t)-P^2(t)+c_2R(t)-R^2(t)\le m
\end{aligned}$$

for some positive number *m* which depends on c_0,c_1 and c_2 which in turn depend on $g(x),f(t),Q(t),Q'(t)$ and T_M. By integrating (19) in *t* variable and noting that $f(t)\le 0$ we obtain

$$\|u\|_{H^1}^2\le\|u\|_2^2+\|u_x\|_2^2-\frac{1}{6}\int_0^\infty f(t)u^4\mathrm{d}x\le\int_0^t m\mathrm{d}t\le mT_M \qquad (20)$$

which implies that $\|u\|_{H^1}$ is bounded on any $[0,T_M)$.

Next we turn to case 2) $f(t) \le -\alpha$ where $\alpha > 0$ with no restriction on $f'(t)$. Again, from (18) we get

$$\begin{aligned}
&\partial_t\left(\|u\|_2^2+\|u_x\|_2^2-\frac{1}{6}\int_0^\infty f(t)u^4\mathrm{d}x\right) \\
&=2Q(t)R(t)-P^2(t)+\frac{1}{2}f(t)Q^4-2P(t)Q'(t)-R^2(t) \\
&\quad-\frac{1}{6}f'(t)\|u\|_4^4-\frac{2}{3}f(t)Q^3(t)R(t)-\frac{1}{9}f^2(t)Q^6(t) \\
&\le\frac{1}{2}f(t)Q^4-2P(t)Q'(t)-P^2(t)+\left(2Q(t)-\frac{2}{3}f(t)Q^3(t)\right)R(t) \\
&\quad-R^2(t)+\frac{1}{6}\left|f'(t)\right|\cdot\|u\|_4^4 \\
&\le c_0+c_1P(t)-P^2(t)+c_2R(t)-R^2(t)+c_3\|u\|_4^4\le m+c_3\|u\|_4^4
\end{aligned} \tag{21}$$

for some positive m which depends on c_0, c_1, c_2 and c_3 which in turn depend on $g(x), f(t), f'(t), Q(t), Q'(t)$ and T_M. By integrating (21) in t variable and noting that $f(t) \le -\alpha$ we obtain

$$\begin{aligned}
\|u\|_2^2+\|u_x\|_2^2+\frac{\alpha}{6}\|u\|_4^4 &\le \|u\|_2^2+\|u_x\|_2^2-\frac{1}{6}\int_0^\infty f(t)u^4\mathrm{d}x \\
&\le\int_0^t\left(m+c_3\|u\|_4^4\right)\mathrm{d}t \\
&\le mT_M+m_0\int_0^t\left(\|u\|_2^2+\|u_x\|_2^2+\frac{\alpha}{6}\|u\|_4^4\right)\mathrm{d}t
\end{aligned} \tag{22}$$

for some positive number m_0 which depends on c_3 and α. By Gronwall's lemma, $\|u\|_2^2+\|u_x\|_2^2+\frac{\alpha}{6}\|u\|_4^4$ is bounded on any $[0, T_M)$, so is $\|u\|_{H^1}$. From Gagliardo-Nirenburg estimate [41] $\|u\|_\infty^2 \le \lambda\|u\|_2\|u_x\|_2$ for some $\lambda > 0$, we conclude that $\|u\|_\infty$ is bounded on any $[0, T_M)$.

Now consider the Cauchy problem for the linear equation

$$\mathrm{d}u/\mathrm{d}t + A(t)u = B(t), \quad 0 \le t \le T, \quad u(0) = g(x) \tag{23}$$

in a Banach space X and if one assumes that $-A(t)$ generates an analytical semigroup then the solution of (10) can be written as

$$u(t) = U(t,0)g + \int_0^t U(t,s)B(s)\mathrm{d}s \tag{24}$$

where $U(t,s) = \mathrm{e}^{-(t-s)A}$ is defined as the family of operators such that $u(t) = U(t,s)g$ is the solution of the homogeneoug differential equation $\mathrm{d}u/\mathrm{d}t + A(t)u = 0$ with the initial value $u(s) = g$. For the nonlinear case in a Banach space X:

$$\mathrm{d}u/\mathrm{d}t + A(t,u) = B(t,u), \quad 0 \le t \le T, \quad u(0) = g \tag{25}$$

we consider the linear equation $\mathrm{d}u/\mathrm{d}t + A(t, v(t))u = B(t, v(t))$, $u(0) = g$ for certain functions $t \to v(t) \in X$. If this equation has a solution $u = u(t)$, then defines a mapping $v \to u = G(v)$ and seeks a fixed point of G which will be a solution of (25). We note that (25) is similar to (12) as we take boundary data $Q(t)$ into consideration and switch v and u variables. We now can adopt arguments in [42], thinking $X = L_0^2$, and write the following as the solution to (12)

$$v(t) = U(t,0)g + \int_0^t U(t,s) f(t,v)\mathrm{d}s \tag{26}$$

where U is continuous and bounded operator. Recall from (20) and (22) that $u = v + Q(t)\mathrm{e}^{-x}$ is bounded under H^1 and L^∞ norms, thus v is also bounded under H^1 and L^∞ norms on any given interval of time $[0,T]$. Take $Y = H_0^3$ norm on both side of (26) one obtains the following inequality

$$\|v\|_Y \le c_0 + \left\| \int_0^t U(t,s) B(t,v)\mathrm{d}s \right\|_Y \le c_0 + c_1 \int_0^t \|v\|_Y \,\mathrm{d}s \tag{27}$$

Apply the Grownwall lemma on (27) one conclude that v is bounded under Y norm on any given interval of time $[0,T]$. Therefore, u is a global classical solution to the inhomogeneous initial-boundary value problem for the modified transitional KdV Equation (10). Therefore we have proved the global existence theorem.

There are many mathematical analysis and proof techniques used to prove the existence of global solutions for PDEs, such as energy methods, variational methods, or other applicable mathematical tools. But most of them are used to study pure initial value problems. When boundary value is inhomogeneous, the energy is no longer conserved. We prove the local existence via semi-group theory (which could be considered as fixed point theory in functional spaces). For the global existence, we utilize the same technique of a priori estimates as in the case of pure initial value problems, except that the estimates are a lot more complex.

Acknowledgements

This research was supported by the William R. Kenan Jr. Professorship and Wellesley College Faculty Awards.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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