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Numerical Simulation for Fredholm Integral Equation of the Second Kind

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Abstract

This work mainly focuses on the numerical simulation of the Fredholm integral equation of the second kind. Applying the idea of Gauss-Lobatto quadrature formula, a numerical method is developed. For the integral item, we give an approximation with high precision. The existence condition of the solution for the Fredholm equation is given. Furthermore, the error analyses are presented. Finally, the numerical examples verify the theoretical analysis, and show the efficiency of the algorithm we discussed.

Keywords

Integral Equation, Fredholm, Gauss-Lobatto, Numerical Method

1. Introduction

Integral equation is a very important tool, it has been more and more widely used in Mechanics, Mathematics, Thermodynamics, Physics and other fields in recent decades. As early as 1823, Norwegian mathematician N.H. Abel has obtained the integral equation of the following form when he studied the tautochrone problem. Now, the integral equation is also known as the Abel equation,

$$\int_a^x \frac{G(x,t)u(t)\,\mathrm{d}t}{(x-t)^{1-\alpha}} = f(x). \tag{1}$$

This integral equation can be regarded as a special case of Volterral's integral equation, which can also be transformed into a fractional differential equation. The theory of integral equation is established mainly by Fredholm and Volterra in the late 19th century. Their work profoundly influenced the study of integral equations in the 20th century. Many mathematical models in science and engineering can be described by integral equation model, such as anomalous diffusion problem, population prediction model, biological population ecological model, nerve pulse propagation, medical scanning, viscoelastic material simulation, heat conduction problem with memory material, *etc.*, one can refer to [1]-[6].

In this work, we will focus on the Fredholm integral equation of the second kind,

$$y(x) = \lambda \int_a^b \phi(x,t) y(t) \mathrm{d}t + f(x), \tag{2}$$

where $\phi(x,t)$ is the kernal of the integral equation, λ is the coefficient of the integral item, $f(x)$ is a free item. $y(x)$ is the unknown function, a and b are the nodes of the integral interval.

As is well known, the analytical solution of the integral equation is hard to get. Only in some special cases, such as the integral part contains degenerate kernal, the analytical solution can be presented. The numerical method is a very useful tool for solving the integral equation which is difficult to obtain the analytic solution, even the integral equation whose analytic solution does not exist at all. From different research perspectives, many scholars have proposed various numerical methods, such as finite difference method, Galerkin method, finite element method, Nyström method, allocation method, successive approximation method, *etc.*

Guo *et al.* used a finite difference scheme for solving the nonlinear time-fractional partial integro-differential equation [7]. Assari *et al.* presented a discrete Galerkin method for solving Fredholm integral equations of the second kind with logarithmic kernels [8]. Brambilla *et al.* described the implementation of a one-dimensional integral equation in a finite-element model [9]. Guoqiang and Jiong got a Nystrom solution for two-dimensional nonlinear Fredholm integral equations of the second kind, and gave an error analysis for this method [10]. Chen and Tang analyzed the convergence of the Jacobi spectral-collocation methods for Volterra integral equations [11]. Maleknejad and Sohrabi provided a numerical solution of Fredholm integral equations of the first kind by using Legendre wavelets [12]. Yusufoglu and Erbas obtained a numerical solution for Fredholm-Volterra type integral equations by interpolation and quadrature rules [13]. And other one can refer to [14]-[21].

2. Preliminaries

In this part, the properties of the Legendre polynomials will be shown. And the zeros of the Legendre polynomials will be used to construct a numerical algoruthm in the following part.

Definition 1. [22] In the interval $[-1,1]$, with the weight $\rho(x)=1$, the polynomials with the following relations are called Legendre polynomicals.

$$P_0(x) = 1, P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n \left[\left(x^2 - 1 \right)^n \right]}{\mathrm{d}x^n}, \tag{3}$$

where $n = 1, 2, 3, \cdots$.

From the above relations, it is clear that the coefficient of Legendre Polynomials $P_n(x)$ is $\frac{(2n)!}{2^n (n!)^2}$.

Property 1.

$$\int_{-1}^{1} P_m(x) P_n(x) \mathrm{d}x = \begin{cases} \frac{2}{2n+1}, & m = n, \\ 0, & m \neq n. \end{cases} \tag{4}$$

Property 2.

$$P_n(-x) = (-1)^n P_n(x). \tag{5}$$

Property 3.

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x), \tag{6}$$

where $n = 1, 2, 3, \cdots$.

Similarly, the zeros of Chebyshev polynomials and other special polynomials also can be used to cope with this problem, one can refer to [22]. Legendre polynomials are chosen here mainly because the weight coefficient is 1, which provides a great convenience for calculation.

3. Nunmerical Algorithm and Error Analysis for Fredholm Integral Equation of the Second Kind

Let's consider the Gauss-Lobatto formula in the interval $[-1,1]$

$$\int_{-1}^{1} g(t) \mathrm{d}t = \sum_{j=1}^{n} \omega_j g(t_j) + R_n[g], \tag{7}$$

where the coefficients $\omega_j, j = 1, 2, 3, \cdots$ satisfy the following relations

$$\omega_j = \begin{cases} \frac{2}{n(n+1)}, & t_j = \pm 1, \\ \frac{2}{n(n+1) P_n^2(t_j)}, & \text{else}, \end{cases} \tag{8}$$

where $P_n(x)$ are the Legendre polynomials and $t_j, j = 1, 2, 3$ are the zeros of $P_n'(x)$. The trunction error is

$$R_n[g] = -\frac{(n+1) n^3 2^{2n+1} ((n-1)!)^4 g^{(2n)}(\xi)}{(2n+1)((2n)!)^3}, \xi \in (-1,1). \tag{9}$$

If the interval is $[a,b]$, one should make the following substitution $t = \frac{(b-a)s + b + a}{2}$. Then the interval $[a,b]$ can be mapped into $[-1,1]$. And the Fredholm integral equation can be equivalently transformed to the following form

$$y(x) = \frac{\lambda(b-a)}{2} \int_{-1}^{1} \phi\left(x, \frac{(b-a)s+b+a}{2}\right) y\left(\frac{(b-a)s+b+a}{2}\right) \mathrm{d}s + f(x), \tag{10}$$

where $x \in [a,b]$, $\frac{(b-a)s+b+a}{2} \in [a,b]$.

For the Fredholm integral equation of the second kind, the hardest part to deal with is the integral term. The quadrature formula was applied to cope with the integral term. *i.e.*

$$y(x) = \tilde{\lambda}\sum_{j=1}^{n}\omega_j \phi(x, x_j) y(x_j) + f(x) + \tilde{R}_n, \tag{11}$$

where $x_j = \frac{(b-a)s_j + b + a}{2}, j = 1,2,3,\cdots$ are the nodes, $\omega_j, j = 1,2,3,\cdots$ are the coefficients, $\tilde{R}_n$ is the truncation error for the quadrature formula,

$$\tilde{R}_n = -\frac{\lambda(n+1)n^3(b-a)^{2n+1}((n-1)!)^4 G^{(2n)}(\xi)}{(2n+1)((2n)!)^3}, \xi \in (-1,1), \tag{12}$$

where $G(s) = \frac{\lambda(b-a)}{2}\phi\left(x, \frac{(b-a)s+b+a}{2}\right) y\left(\frac{(b-a)s+b+a}{2}\right)$.

If we denote $y_i = y(x_i)$, $f_i = f(x_i)$, $\phi_{ij} = \phi(x_i, x_j)$, then the following algorithm will be proposed.

$$y_i = \tilde{\lambda}\sum_{j=1}^{n}\omega_j \phi_{ij} y_j + f_i. \tag{13}$$

The coefficient matrix of the above system can be written in detailed as

$$C = \begin{pmatrix} \tilde{\lambda}\omega_1\phi_{11} & \tilde{\lambda}\omega_2\phi_{12} & \cdots & \tilde{\lambda}\omega_n\phi_{1n} \\ \tilde{\lambda}\omega_1\phi_{21} & \tilde{\lambda}\omega_2\phi_{22} & \cdots & \tilde{\lambda}\omega_n\phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\lambda}\omega_1\phi_{n1} & \tilde{\lambda}\omega_2\phi_{n2} & \cdots & \tilde{\lambda}\omega_n\phi_{nn} \end{pmatrix}. \tag{14}$$

Therefore, if $det(I-C) \neq 0$, one can get the unique approximate solution.

4. Nunmerical Examples

In this part, the proposed algorithm will be employed to solve two Fredholm integral equations of the second kind. To show the efficiency of the algorithm we proposed the numerical results will be compared with the exact solution.

Example 1. Consider the following Fredholm integral equation of the second kind

$$y(x) = -\int_0^1 x e^t y(t) dt + e^{-x}, \tag{15}$$

where $x \in [0,1]$, and the exact solution is $y(x) = e^{-x} - \frac{x}{2}$.

Figure 1 shows the numerical and analytical solution of Example 1. One can find that the numerical solution and the exact solution are in good agreement. **Figure 2** shows the errors at different nodes in the interval $[0,1]$. The error bound is about 10^{-9}, when $n = 4$.

When we choose different collocation nodes, *i.e.* $n = 2,3,4,5,6$. **Table 1** shows that the errors at different points in the interval $[0,1]$ for Example 1.

One can find that the more nodes we use, the better accuracy we get.

Example 2. Consider the following Fredholm integral equation of the second kind

$$y(x) = \frac{2}{e-1}\int_0^1 e^x y(t)\,dt - e^x, \tag{16}$$

where $x \in [0,1]$, and the exact solution is $y(x) = e^x$.

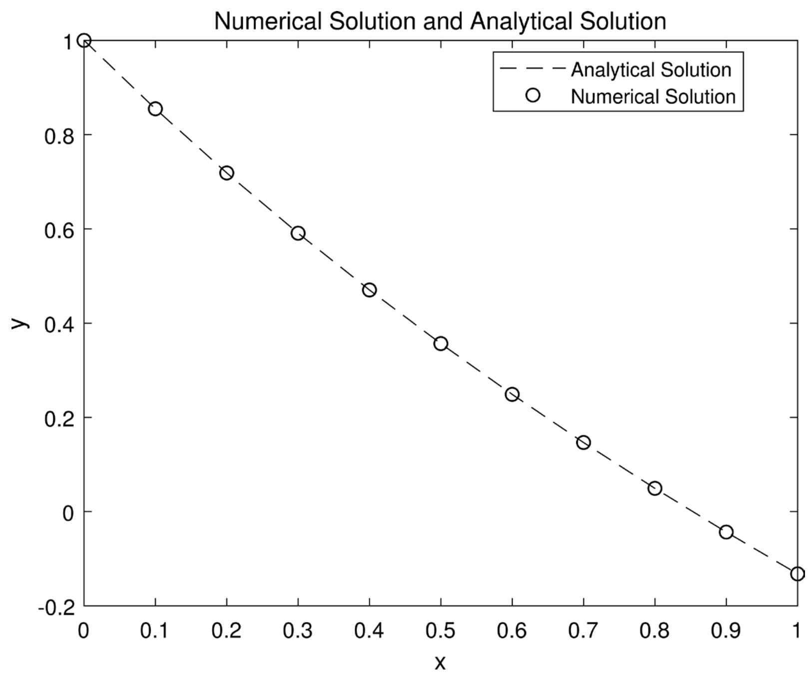

Figure 1. The exact solution and numerical solution (n = 4).

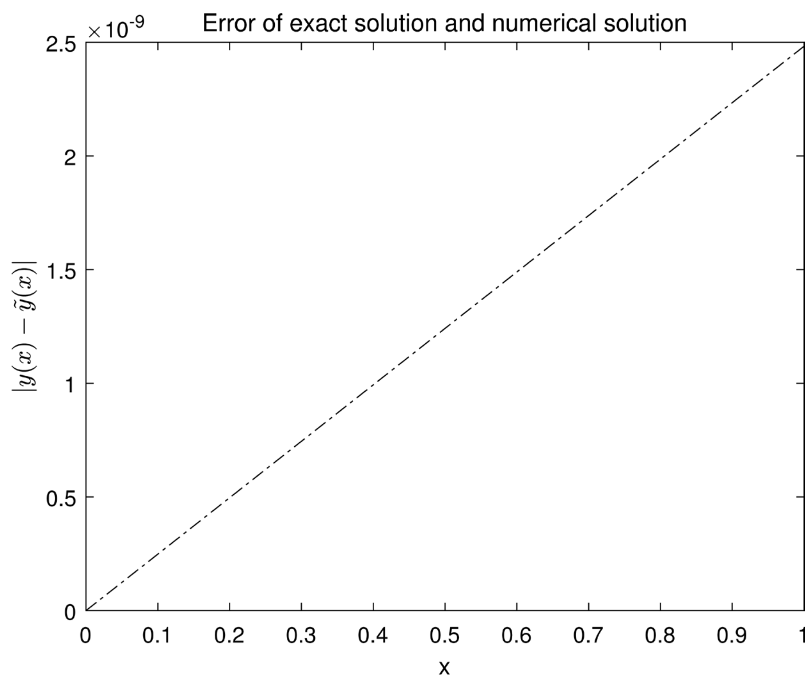

Figure 2. The error of the exact solution and the numerical solution (n = 4).

Comparison of the numerical and analytical solutions of Example 2 is illustrated in Figure 3. Figure 4 reveals the absolute error of the numerical and analytical solutions for Example 2. Figure 1 and Figure 2 show a good performance of the algorithm we proposed for solving the Fredholm intergral euation of the second kind.

From Table 2, it can be seen that the numerical results with the method we proposed are in excellent agreement with the exact solution. When the collocation nodes $n = 6$, the absolute error is 10^{-16}, which means a high precision of the method we used.

Table 1. The errors of Example 1.

x	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
0.1	−6.54E−05	−1.79E−07	−2.48E−10	−2.06E−13	−2.22E−16
0.2	−1.31E−04	−3.58E−07	−4.97E−10	−4.12E−13	−2.22E−16
0.3	−1.96E−04	−5.37E−07	−7.45E−10	−6.19E−13	−3.33E−16
0.4	−2.62E−04	−7.17E−07	−9.93E−10	−8.25E−13	−5.00E−16
0.5	−3.27E−04	−8.96E−07	−1.24E−09	−1.03E−12	−6.66E−16
0.6	−3.93E−04	−1.07E−06	−1.49E−09	−1.24E−12	−7.22E−16
0.7	−4.58E−04	−1.25E−06	−1.74E−09	−1.44E−12	−9.44E−16
0.8	−5.23E−04	−1.43E−06	−1.99E−09	−1.65E−12	−1.05E−15
0.9	−5.89E−04	−1.61E−06	−2.23E−09	−1.86E−12	−1.17E−15
1	−6.54E−04	−1.79E−06	−2.48E−09	−2.06E−12	−1.28E−15

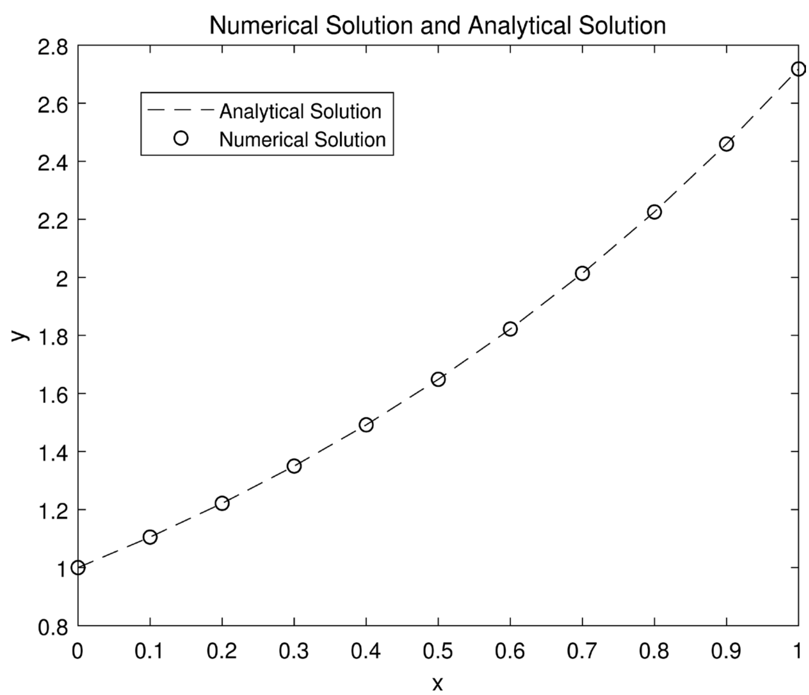

Figure 3. The exact solution and numerical solution ($n = 5$).

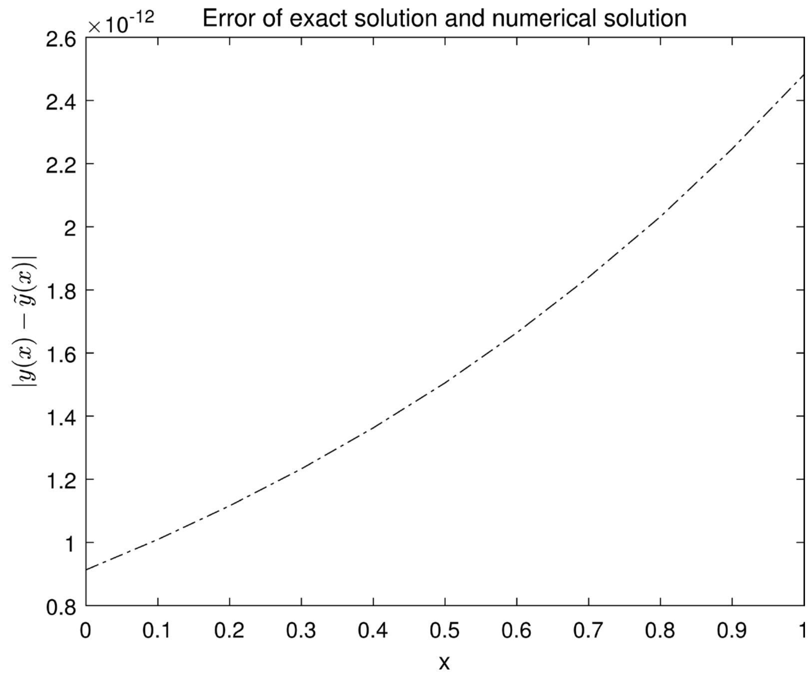

Figure 4. The error of the exact solution and the numerical solution ($n = 5$).

Table 2. The errors of Example 2.

x	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
0.1	7.45E−04	1.41E−06	1.5E−09	1.01E−12	4.44E−16
0.2	8.23E−04	1.56E−06	1.66E−09	1.12E−12	4.44E−16
0.3	9.10E−04	1.73E−06	1.83E−09	1.23E−12	4.44E−16
0.4	1.01E−03	1.91E−06	2.03E−09	1.36E−12	4.44E−16
0.5	1.11E−03	2.11E−06	2.24E−09	1.51E−12	4.44E−16
0.6	1.23E−03	2.33E−06	2.47E−09	1.66E−12	4.44E−16
0.7	1.36E−03	2.58E−06	2.73E−09	1.84E−12	8.88E−16
0.8	1.50E−03	2.85E−06	3.02E−09	2.03E−12	8.88E−16
0.9	1.66E−03	3.15E−06	3.34E−09	2.25E−12	8.88E−16
1	1.83E−03	3.48E−06	3.69E−09	2.48E−12	8.88E−16

5. Conclusion

In this work, we study the Fredholm integral equation of the second kind. For the integral term of the equation, we use Gauss-Lobatto quadrature formula to cope with it. We give the numerical algorithm and truncation error for the algorithm we used. Furthermore, we give the existence condition of the algorithm for Fredholm integral equation we discussed. Finally, two numerical examples are shown to verify the efficiency and accuracy of the algorithm.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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