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Quantum Light and Coherent States in Conducting Media

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Abstract

We present a simple description of classical and quantum light propagating through homogeneous conducting linear media. With the choice of Coulomb gauge, we demonstrate that this description can be performed in terms of a damped harmonic oscillator which is governed by the Caldirola-Kanai Hamiltonian. By using the dynamical invariant method and the Fock states representation we solve the time-dependent Schrödinger equation associated with this Hamiltonian and write its solutions in terms of a special solution of the Milne-Pinney equation. We also construct coherent states for the quantized light and show that they are equivalent to the well-known squeezed states. Finally, we evaluate some important properties of the quantized light such as expectation values of the amplitude and momentum of each mode, their variances and the respective uncertainty principle.

Keywords

Light, Schrödinger Equation, Invariant Method, Fock States, Coherent and Squeezed States

1. Introduction

For a long time, the old and fascinating problem (from classical and quantum viewpoint) of the interaction of light with matter has received considerable attention of physicists. The story of the solution of this problem is a familiar one. Further, the solution of this problem has been of crucial importance for the development of our understanding of nature.

In order to obtain the basic concepts to study the classical and quantum behavior of light we must take into account Maxwell's equations. In the quantum case, the quantization of these equations is traditionally performed in free space or in empty cavities by associating a time-independent mechanical oscillator with each mode of the electromagnetic field [1] [2]. In the past few years, considerable attention has been devoted to the study of the properties (mainly quantum aspects) of light propagating through material media. This great interest is partially due to the advent of modern optical materials and partially by the growth of experiments on quantum optics process taking place within material media [3]-[9]. Several different approaches have been employed to treat with the propagation of light waves in conducting material media [10]-[18].

The main purpose of this work is to present a simpler and clearer approach to describe the classical and quantum behavior of light propagating through homogenous conducting linear media without charge sources. In order to do this, we choose the Coulomb gauge and demonstrate that this description can be performed by associating a damped harmonic oscillator, which is described by the Cadirola-Kanai Hamiltonian [19] [20] [21] [22], with each mode of the electromagnetic field. Further, by using the dynamical invariant method developed by Lewis and Riesenfeld [23] and the Fock states representation, we easily solve the Schrödinger equation associated with this Hamiltonian and write its solutions in terms of a particular solution of the Milne-Pinney equation [22] [24] [25] [26]. Yet, by employing these solutions we construct coherent states for the quantized light and show that they correspond to the well-known squeezed states. Finally, we use Fock states and coherent states to calculate some important quantum properties of quantum light such as expectation values of the amplitude and momentum of each mode, their variances and the respective uncertainty principle.

We organize this work as follows. In Sec. 2, by making use of the Coulomb gauge we first discuss the classical propagation of light in liner conducting media. We study the quantum behavior of light propagating in conducting media in Sec. 3. In this section, we use the invariant method and Fock states to solve the Schrödinger equation associated with the Caldirola-Kanai Hamiltonian and employ these solutions to derive some physical quantities of quantum light. In Sec. 4, we construct coherent and squeezed states for the quantized light and calculate the expectation values of the coordinate and momentum, their quantum variances and the corresponding uncertainty principle. In Sec. 5, we conclude our work with a short summary.

2. Classical Light Propagation in Conducting Media

In this section, we present a simple classical description of the propagation of light in linear conducting media. To do this, let us write the Maxwell's equations for the electromagnetic field in conducting linear media in the absence of charge distributions as [1] [2]

$$\nabla \cdot \boldsymbol{D} = 0, \tag{1}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3}$$

$$\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}, \tag{4}$$

where $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$ and $\boldsymbol{J} = \sigma \boldsymbol{E}$. Here, ε, σ and μ are respectively the electric permittivity, conductivity and magnetic permeability of the media. In general, the electric permittivity and the magnetic permeability are complex; however, we will restrict our discussion to materials where they are real [27] [28]. Now, in the Coulomb gauge [1] [2] the divergence of the vector potential $\boldsymbol{A}$ is zero and the scalar potential is null in the absence of sources. Consequently, both the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields are determined from the vector potential as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{and} \quad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} \tag{5}$$

It is worth remarking that in the Coulomb gauge the vector potential is purely transverse [1] [2]. Therefore, it is easy to verify that it satisfies the damped wave equation

$$\nabla^2 \boldsymbol{A} - \mu\sigma \frac{\partial \boldsymbol{A}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0. \tag{6}$$

Now, in order to obtain a solution of this equation we consider light waves in a certain volume of space. So, by the familiar procedure of separation of variables, we write the vector potential in terms of the mode $\boldsymbol{u}_l(\boldsymbol{r})$ and amplitude $q_l(t)$ functions of each cavity mode [1] [2] as

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_l \boldsymbol{u}_l(\boldsymbol{r}) q_l(t). \tag{7}$$

The substitution of this equation into wave Equation (6) yields

$$\nabla^2 \boldsymbol{u}_l(\boldsymbol{r}) + \frac{\omega_l^2}{c^2} \boldsymbol{u}_l(\boldsymbol{r}) = 0, \tag{8}$$

$$\frac{\partial^2 q_l}{\partial t^2} + \frac{\sigma}{\varepsilon}\frac{\partial q_l}{\partial t} + \omega_l^2 q_l = 0, \tag{9}$$

where ω_l is the natural frequency of the mode l and $c = 1/(\mu\varepsilon)^{1/2}$ is the velocity of light in the medium.

In the follows let us discuss the solutions of Equations (8) and (9). The solution of Equation (9) can be written in the form

$$q_l(t) = A_l \mathrm{e}^{-\sigma t/2\varepsilon} \sin(\Omega_l t + \delta_l), \tag{10}$$

where A_l and δ_l are constants to be determined by the initial conditions and Ω_l is given by

$$\Omega_l^2 = \omega_l^2 - \left(\frac{\sigma}{2\varepsilon}\right)^2. \tag{11}$$

Here we have only considered the oscillatory solutions, that is, $\Omega_l^2 > 0$. At this point, it is worth noticing that the equation of motion (9) can be directly obtained from the classical Hamiltonian

$$H_l(t) = \mathrm{e}^{-\sigma t/\varepsilon}\frac{p_l^2}{2\varepsilon} + \frac{1}{2}\mathrm{e}^{\sigma t/\varepsilon}\varepsilon\omega_l^2 q_l^2, \quad (12)$$

where the coordinate q_l and momentum p_l are canonically conjugate variables. This Hamiltonian is the well-known Caldirola-Kanai Hamiltonian, which has been used in the literature to study time-dependent systems in various areas of physics [29] [30] [31] [32] [33]. Hence, the total Hamiltonian of the electromagnetic field is a sum of individual Hamiltonians corresponding to each mode, that is, $H = \sum_l H_l$.

In the following discussion we focus our attention on the solution of Equation (8). Considering that the electromagnetic field is contained in a certain cubic volume V of side L of nonrefracting media, the mode functions are required to satisfy the transversality condition $\nabla \cdot \boldsymbol{u}_l(\boldsymbol{r}) = 0$ and to form a complete orthonormal set. Furthermore, assuming periodic boundary conditions on the surface, the mode function $\boldsymbol{u}_l(\boldsymbol{r})$ can be written in terms of plane waves as [1] [2]

$$\boldsymbol{u}_{l\nu}(\boldsymbol{r}) = L^{-3/2}\mathrm{e}^{\pm i\boldsymbol{k}_l\cdot\boldsymbol{r}}\hat{e}_{l\nu}, \quad (13)$$

where $L = V^{1/3}$ is the size of the cube, $|\boldsymbol{k}_l| = \omega_l/c$ is the wave vector, and $\hat{e}_{l\nu}$ are unit vectors in the directions of polarization ($\nu = 1,2$), which must be perpendicular to the wave vector because of the transversality condition. Now, as the mode $\boldsymbol{u}_l(\boldsymbol{r})$ and amplitude $q_l(t)$ functions are completely determined, we can obtain the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ (see Equation (7)) by using Equations (10) and (13). Hence, using Equations (7) and (13) we can write, for each mode l, the electric and magnetic fields (see Equation (5)), as

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\mathrm{e}^{-\sigma t/\varepsilon}}{\varepsilon L^{3/2}}\sum_l\sum_{\nu=1,2}\hat{e}_{l\nu}\mathrm{e}^{\pm i\boldsymbol{k}_l\cdot\boldsymbol{r}}p_l(t), \quad (14)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{i}{cL^{3/2}}\sum_l\sum_{\nu=1,2}\omega_l(\boldsymbol{k}_l\times\hat{e}_{l\nu})\mathrm{e}^{\pm i\boldsymbol{k}_l\cdot\boldsymbol{r}}q_l(t), \quad (15)$$

where we have used that $p_l(t) = \varepsilon\mathrm{e}^{\sigma t/\varepsilon}\dot{q}_l$.

Therefore, the above results give us a complete classical description of the propagation of light in conducting linear media since the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields are completely specified. Here it is worth noticing that in the previous description we have associated a damped harmonic oscillator to the each mode of the electromagnetic field. Let us also observe that in the absence of the dissipation, that is, $\sigma = 0$ the Hamiltonian (12) reduces to that of the standard harmonic oscillator with the permittivity playing the role of the mass of the mechanical oscillator. As a consequence, all of our previous results coincide with those of the propagation of light in empty cavities.

3. Quantum Light Propagation in Conducting Media

In order to obtain a quantum description of light propagating in a conducting linear media we need to quantize the electromagnetic field. Now as the spatial mode functions $\boldsymbol{u}_l(\boldsymbol{r})$ are completely determined, the amplitude of each normal mode in Equation (7) needed to specify a particular field configuration is $q_l(t)$ [1]. Thus, for each canonical operator q_l the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields operators may be derived from the potential vector $\boldsymbol{A}$ by using Equation (5). So, let us move our attention to the canonical operator $q_l(t)$ in order to obtain the vector potential. For this purpose, let us solve the Schrödinger equation associated with the Hamiltonian (12)

$$H_l|\Psi,t\rangle = i\hbar\frac{\partial}{\partial t}|\Psi,t\rangle, \tag{16}$$

where the coordinate $q_l(t)$ and the momentum p_l are now canonically conjugate operators satisfying the relation $[q_l, p_l] = i\hbar$ with $p_l = -i\hbar\,\partial/\partial q$. We can obtain the solutions of this equation with the aid of the dynamical invariant method developed by Lewis and Riesenfeld [23]. According to this method, we must look for a nontrivial Hermitian operator $I_l(t)$ which satisfies the equation

$$\frac{\mathrm{d}I_l}{\mathrm{d}t} = \frac{1}{i\hbar}[I_l, H_l] + \frac{\partial I_l}{\partial t} = 0. \tag{17}$$

Then, the solutions of the Schrödinger Equation (16) can be written in terms of orthonormalized eigenstates $|\phi_{n_l},t\rangle$ of $I_l(t)$ (constant of motion)

$$I_l(t)|\phi_{n_l},t\rangle = \lambda_{n_l}|\phi_{n_l},t\rangle, \tag{18}$$

and the phase functions $\beta_{n_l}(t)$ as

$$|\psi_{n_l},t\rangle = \mathrm{e}^{i\beta_{n_l}(t)}|\phi_{n_l},t\rangle. \tag{19}$$

Here, the λ_{n_l} are time-independent eigenvalues and the phase functions $\beta_{n_l}(t)$ are derived of the equation

$$\hbar\frac{\mathrm{d}\beta_{n_l}(t)}{\mathrm{d}t} = \left\langle\phi_{n_l},t\left|i\hbar\frac{\partial}{\partial t} - H_l(t)\right|\phi_{n_l},t\right\rangle. \tag{20}$$

with the orthonormality condition $\langle\phi_{n_l'},t|\phi_{n_l},t\rangle = \delta_{n_l'n_l}$. In what follows, let us consider a quadratic invariant that satisfies Equation (17). Here, we assume an invariant in the form

$$I_l(t) = \frac{1}{2}\left[\left(\frac{q_l}{\rho_l}\right)^2 + \left(\rho_l p_l - \wedge(t)\dot{\rho}_l q_l\right)^2\right], \tag{21}$$

where $\rho_l(t)$ is a time-dependent real function satisfying the Milne-Pinney equation [24] [25]

$$\ddot{\rho}_l(t) + \frac{\sigma}{\varepsilon}\dot{\rho}_l(t) + \omega_l^2\rho_l = \frac{1}{\wedge^2\rho_l^3}, \tag{22}$$

where the dots represent time derivatives and $\wedge(t)$ is given by

$$\wedge(t) = \varepsilon\mathrm{e}^{\sigma t/\varepsilon}. \tag{23}$$

We must now find the eigenstates of the invariant $I_l(t)$. To this end, we will use the Fock representation since, as is well-known, the quantum behavior of some quantum systems, in particular quantum harmonic oscillator-type systems, is most obvious in Fock ststes, which are states with specific numbers of energy quanta. Then, let us introduce annihilation and creation-type operators $a_l(t)$ and $a_l^\dagger(t)$ defined by [16] [23]

$$a_l(t)=\left(\frac{1}{2\hbar}\right)^{1/2}\left[\frac{q_l}{\rho_l}+i\left(\rho_l p_l-\wedge(t)\dot{\rho}_l q_l\right)\right], \tag{24}$$

$$a_l^\dagger(t)=\left(\frac{1}{2\hbar}\right)^{1/2}\left[\frac{q_l}{\rho_l}-i\left(\rho_l p_l-\wedge(t)\dot{\rho}_l q_l\right)\right], \tag{25}$$

with

$$\left[a_l(t),a_l^\dagger(t)\right]=1. \tag{26}$$

In terms of these operators, the invariant (21) can be factored as

$$I_l(t)=\hbar\left[a_l^\dagger(t)a_l(t)+\frac{1}{2}\right]. \tag{27}$$

From Equations (26) and (27) we see that the eigenvalue equation for $I_l(t)$ (see Equation (18)) can also be solved exactly, just as for harmonic oscillator in the time-independent case by using the Fock states $|n_l,t\rangle$. So, defining the Hermitian number operator by $N_l=a_l^\dagger a_l$ so that $N_l|n_l,t\rangle=n_l|n_l,t\rangle$, we find that

$$I_l(t)=\hbar\left(N_l+\frac{1}{2}\right). \tag{28}$$

$$I_l(t)|n_l,t\rangle=\hbar\left(n_l+\frac{1}{2}\right)|n_l,t\rangle, \tag{29}$$

$$a_l(t)|n_l,t\rangle=n_l^{1/2}|n_l-1,t\rangle, \tag{30}$$

$$a_l^\dagger|n_l,t\rangle=\left(n_l+1\right)^{1/2}|n_l+1,t\rangle. \tag{31}$$

From Equation (28) we see that the eigenstates of $I_l(t)$ are also eigenstates of N_l and vice versa.

The next step is to find the phase functions given by Equation (20). By making the change $|\phi_n,t\rangle\to|n_l,t\rangle$ and after performing some basic calculations, we get that

$$\beta_{n_l}(t)=-\left(n_l+\frac{1}{2}\right)\int_0^t\frac{1}{\wedge(\tau)\rho_l^2(\tau)}\mathrm{d}\tau. \tag{32}$$

We now consider a particular solution of the Milne-Pinney Equation (22) given by

$$\rho_l(t)=\frac{\mathrm{e}^{-\sigma t/2\varepsilon}}{\left(\varepsilon\Omega_l\right)^{1/2}}. \tag{33}$$

By inserting this equation into (32) we get that,

$$\beta_{n_l}(t)=-\Omega_l\left(n_l+\frac{1}{2}\right)t. \tag{34}$$

Therefore, the solutions of the Schrödinger Equation (16) can be written as

$$\left|\psi_{nl},t\right\rangle = \mathrm{e}^{i\beta_{nl}(t)}\left|n_l,t\right\rangle, \tag{35}$$

with $\beta_{nl}(t)$ given by (32). The general solution to the Schrödinger Equation (16) can be written as $\left|\Psi,t\right\rangle = \sum_{n_l} c_{nl}\left|\psi_{nl},t\right\rangle$, where the coefficients c_{nl} are constants.

In the following, let us move our attention for the operators $a_l(t)$ and $a_l^{\dagger}(t)$ given by Equations (24) and (25). From the expressions of these operators, we obtain that

$$q_l(t) = \left(\frac{\hbar}{2}\right)^{1/2}\rho_l\left[a_l(t)+a_l^{\dagger}(t)\right], \tag{36}$$

$$p_l(t) = i\left(\frac{\hbar}{2}\right)^{1/2}\left[\left(\frac{1}{\rho_l}-i\wedge\dot{\rho}_l\right)a_l^{\dagger}(t)-\left(\frac{1}{\rho_l}+i\wedge\dot{\rho}_l\right)a_l(t)\right]. \tag{37}$$

Thus, using Equations (7), (13), (23), (33) and (36) we can write the potential vector $\boldsymbol{A}$ in the form

$$\boldsymbol{A}(\boldsymbol{r},t) = \left(\frac{\hbar}{2\varepsilon}\right)^{1/2}\frac{\mathrm{e}^{-\sigma t/2\varepsilon}}{L^{3/2}}\sum_l\sum_{\nu=1,2}\frac{\hat{e}_{l\nu}}{(\Omega_l)^{1/2}}\left[\mathrm{e}^{i\boldsymbol{k}_l\cdot\boldsymbol{r}}a_{l\nu}(t)+\mathrm{e}^{-i\boldsymbol{k}_l\cdot\boldsymbol{r}}a_{l\nu}^{\dagger}(t)\right]. \tag{38}$$

In the above expression we have written the annihilation and creation operators $a_l(t)$ and $a_l^{\dagger}(t)$ in terms of the directions of polarization so that we now have that $\left[a_{l\nu}(t),a_{l\nu}^{\dagger}(t)\right]=1$. Then, by inserting the above equation into Equation (5) we obtain the electric e magnetic field operators as

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) = {} & \left(\frac{\hbar}{2\varepsilon}\right)^{1/2}\frac{\mathrm{e}^{-\sigma t/2\varepsilon}}{L^{3/2}}\sum_l\sum_{\nu=1,2}\frac{\hat{e}_{l\nu}}{(\Omega_l)^{1/2}}\left[\mathrm{e}^{i\boldsymbol{k}_l\cdot\boldsymbol{r}}\left(\frac{\sigma}{2\varepsilon}-i\Omega_l\right)a_{l\nu}(t)\right.\\ & \left.+\mathrm{e}^{-i\boldsymbol{k}_l\cdot\boldsymbol{r}}\left(\frac{\sigma}{2\varepsilon}+i\Omega_l\right)a_{l\nu}^{\dagger}(t)\right].\end{aligned} \tag{39}$$

and

$$\boldsymbol{B}(\boldsymbol{r},t) = i\left(\frac{\hbar}{2\varepsilon}\right)^{1/2}\frac{\mathrm{e}^{-\sigma t/2\varepsilon}}{cL^{3/2}}\sum_l\sum_{\nu=1,2}\frac{\omega_l(\boldsymbol{k}_l\times\hat{e}_{l\nu})}{(\Omega_l)^{1/2}}\left[\mathrm{e}^{i\boldsymbol{k}_l\cdot\boldsymbol{r}}a_{l\nu}(t)-\mathrm{e}^{-i\boldsymbol{k}_l\cdot\boldsymbol{r}}a_{l\nu}^{\dagger}(t)\right]. \tag{40}$$

The above field operators describe the quantum propagation of light in conducting linear media. We also see that both electric and magnetic fields decrease exponentially in time due the conductivity of the medium proportionally to $\exp\left[-\sigma t/(2\varepsilon)\right]$. Further, in the absence of the dissipation, that is, $\sigma=0$ these fields reduce to that in empty cavities [1].

In what follows, we use the Fock states to calculate the expectation values of the amplitude q_l, momentum p_l, their variances and the respective uncertainty principle. Hence, making use of Equations (30) and (31) and after a little of algebra, we find that

$$\left\langle I_l\right\rangle = \hbar\left(n_l+\frac{1}{2}\right), \tag{41}$$

$$\langle q_l \rangle = \langle p_l \rangle = 0, \tag{42}$$

$$\langle q_l^2 \rangle = \hbar \rho_l^2 \left(n_l + \frac{1}{2} \right), \tag{43}$$

$$\langle p_l^2 \rangle = \hbar \left[\frac{1}{\rho_l^2} + \left(\wedge \dot{\rho}_l \right)^2 \right] \left(n_l + \frac{1}{2} \right). \tag{44}$$

The quantum variances are given by

$$\left(\Delta q_l \right)^2 = \langle q_l^2 \rangle - \langle q_l \rangle^2 = \hbar \rho_l^2 \left(n_l + \frac{1}{2} \right), \tag{45}$$

$$\left(\Delta p_l \right)^2 = \langle p_l^2 \rangle - \langle p_l \rangle^2 = \hbar \left[\frac{1}{\rho_l^2} + \left(\wedge \dot{\rho}_l \right)^2 \right] \left(n_l + \frac{1}{2} \right), \tag{46}$$

By using the above expressions we obtain the uncertainty principle as

$$\left(\Delta q_l \right) \left(\Delta p_l \right) = \hbar \left[1 + \wedge^2 \rho_l^2 \dot{\rho}_l^2 \right]^{1/2} \left(n_l + \frac{1}{2} \right), \tag{47}$$

which, by making of Equation (23) and the particular solution (33), becomes

$$\left(\Delta q_l \right) \left(\Delta p_l \right) = \frac{\hbar \omega_l}{\Omega_l} \left(n_l + \frac{1}{2} \right). \tag{48}$$

Here, it worth mentioning that if we multiply both sides of Equation (41) by the frequency Ω_l we get

$$\Omega_l \langle I \rangle = \Omega_l \hbar \left(n_l + \frac{1}{2} \right). \tag{49}$$

whose right-hand side represents the energy eigenvalue of a harmonic oscillator with frequency Ω_l. Finally, let us observe that for $\sigma = 0$ the particular solution of Equation (33) becomes $\rho_l = 1 / \left(\varepsilon_l \omega_l \right)^{1/2}$ and the above results are reduced to that the time-independent harmonic oscillator, as it should be.

4. Coherent and Squeezed States of Quantum Light

It is well-known that, in addition to the Fock states, the coherent states provide another important set of states to investigate quantum properties of many physical systems. Yet, it is worth remarking that the coherent states were discovered at the early days of quantum mechanics by Schrödinger who was interested in finding quantum mechanical states that followed the motion of a classical particle in a given potential [34]. These states become popular during the 1960s for their usefulness in describing the radiation filed [35] [36]. This section will be devoted to construct coherent states for the quantized light propagating in linear conducting media. As will be seen later, these states are indeed equivalent to the squeezed states of the quantized light.

4.1. Coherent States of Light in Conducting Media

In Ref. [37] Hartley and Ray constructed coherent states for a mechanical oscillator with time-dependent frequency. Thus, following the same step of these authors we find that the coherent states for the quantum system described by the Hamiltonian (12) are given by

$$\left|\alpha_l,t\right\rangle = \exp\left(-\frac{\left|\alpha_l\right|^2}{2}\right)\sum_{n_l}\frac{(\alpha_l)^{n_l}}{(n_l!)^{1/2}}\exp\left[i\beta_{n_l}(t)\right]\left|n_l,t\right\rangle \tag{50}$$

where α_l is an arbitrary complex number. These states satisfy the eigenvalue equation

$$a_l\left|\alpha_l,t\right\rangle = \alpha_l(t)\left|\alpha_l,t\right\rangle, \tag{51}$$

with $\alpha_l(t)$ given by

$$\alpha_l(t) = \alpha_l \mathrm{e}^{-2i\Omega_l t}, \tag{52}$$

where we have used Equation (34) for $n_l = 0$.

Let us now calculate the expectation value of q_l in the coherent states $\left|\alpha_l,t\right\rangle$. A straightforward calculation yields

$$\left\langle q_l\right\rangle = \left(\frac{2\hbar\left|\alpha_l\right|^2}{\varepsilon\Omega_l}\right)^{1/2}\mathrm{e}^{-\sigma t/2\varepsilon}\sin\left(\Omega_l + \xi_l\right), \tag{53}$$

where ξ_l is the argument of the complex number α_l. By comparing this result with that of Equation (10) we see that the center of the coherent state wave packet follows the classical motion of a particle [34]. Thus, the above result agrees with the original idea of Schrödinger about the coherent states.

In what follows we evaluate the quantum variances in q_l and p_l in the state $\left|\alpha_l,t\right\rangle$. After some algebra we find that

$$\left\langle\Delta q_l\right\rangle^2 = \left\langle q_l^2\right\rangle - \left\langle q_l\right\rangle^2 = \frac{\hbar}{2}\rho_l^2, \tag{54}$$

$$\left\langle\Delta p_l\right\rangle^2 = \left\langle p_l^2\right\rangle - \left\langle p_l\right\rangle^2 = \frac{\hbar}{2}\left[\frac{1}{\rho_l^2} + \left(\wedge\dot{\rho}_l\right)^2\right]. \tag{55}$$

Thus, the uncertainty principle can be expressed as

$$\left(\Delta q_l\right)\left(\Delta p_l\right) = \frac{\hbar\omega_l}{2\Omega_l}, \tag{56}$$

where we have used the particular solution (33). By comparing Equations (48) and (56) we see that the uncertainty principle in the coherent states is exactly the same as the minimum value of that in the number states. It may be helpful at this point to note that these uncertainty principles do not depend on time and that their values become larger when the conductivity increases. We also observe that the uncertainty principle (56), in general, does not attain its minimum value. This occurs because the states $\left|\alpha_l,t\right\rangle$ are indeed equivalent to the squeezed states. This will be seen more clearly below. Further, it is worth noticing that when the conductivity is null, that is, $\sigma = 0$ the uncertainty principle attains its minimum value because in this case the states $\left|\alpha_l,t\right\rangle$ reduce to the coherent states of the ordinary mechanical harmonic oscillator model.

4.2. Squeezed States of Light in Conducting Media

In the following discussion, we are going to show that the states $|\alpha_l, t\rangle$ correspond to the squeezed states. In order to do so, let us consider the annihilation and creation operators b_l and $b_l^\dagger$ of the standard oscillator model given by

$$b_l = \left(\frac{1}{2\hbar\varepsilon\omega_l}\right)^{1/2}\left[\varepsilon\omega_l q_l + ip_l\right], \tag{57}$$

$$b_l^\dagger = \left(\frac{1}{2\hbar\varepsilon\omega_l}\right)^{1/2}\left[\varepsilon\omega_l q_l - ip_l\right]. \tag{58}$$

These operators are related to operators a_l and $a_l^\dagger$, which were defined previously, by the relations [33] [34] [35] [36]

$$b_l = u(t)a_l + v(t)a_l^\dagger, \tag{59}$$

$$b_l^\dagger = u^*(t)a_l^\dagger + v^*(t)a_l, \tag{60}$$

whose coefficients can be expressed as

$$u(t) = \left(\frac{1}{4\varepsilon\omega_l}\right)^{1/2}\left[\frac{1}{\rho_l} - i\wedge(t)\dot{\rho}_l + \varepsilon\omega_l\rho_l\right], \tag{61}$$

$$v(t) = \left(\frac{1}{4\varepsilon\omega_l}\right)^{1/2}\left[\frac{1}{\rho_l} - i\wedge(t)\dot{\rho}_l - \varepsilon\omega_l\rho_l\right]. \tag{62}$$

A straightforward calculation shows that the coefficients $u(t)$ and $v(t)$ fulfills the relation

$$|u(t)|^2 - |v(t)|^2 = 1. \tag{63}$$

Therefore, from Equations (51), (59) and (63), we see that the states $|\alpha_l, t\rangle$ are, by definition, equal to the well-known squeezed states [38] [39] [40] [41] [42]. Furthermore, in terms of the coefficients $u(t)$ and $v(t)$ the quantum variances in $q_l(t)$ and $p_l(t)$ for the squeezed states $|\alpha_l, t\rangle$ can be written as

$$(\Delta q_l)^2 = \frac{\hbar}{2\varepsilon\omega_l}|u - v|^2, \tag{64}$$

$$(\Delta p_l)^2 = \frac{\hbar\varepsilon\omega_l}{2}|u + v|^2, \tag{65}$$

so that the uncertainty principle is converted into

$$(\Delta q_l)(\Delta p_l) = \frac{\hbar}{2}|u - v||u + v|. \tag{66}$$

The uncertainty principle is minimized if $u = \gamma v$ for γ real [39] [41]. Further, note that the relation (66) is equivalent to Equation (56), as it should be.

5. Summary

In this work, we have presented a direct and simple approach to describe the propagation of classical and quantum light in a homogenous conducting linear media. We have used the Coulomb gauge and considered light waves confined in a cubical volume of side L filled with a conductive medium as well as light propagating under periodic boundary conditions. We have demonstrated that this propagation can be performed by associating a damped mechanical oscillator with each mode of the electromagnetic field. As a consequence, we have established a unification of the procedure to obtain the classical and quantum propagation of light in empty cavities (or free space) and cavities filled with a material medium. In the former case, it is usually performed by associating an ordinary harmonic oscillator with each mode of the electromagnetic field, and in the latter one it can be performed by the association of a damped harmonic oscillator. Further, using the invariant method, appropriated annihilation and creation-type operators and the Fock states we have easily solved the time-dependent Schrödinger for our problem and write its solutions in terms of a special solution of the Milne-Pinney equation. We have also constructed coherent states for the quantized light and have calculated the quantum variances of the amplitude $q_l(t)$ and momentum $p_l(t)$ as well as the uncertainty principle for each mode of the electromagnetic field in both states, namely, Fock and coherent states. We have seen that the uncertainty product in the coherent states is equal to the minimum value of that of the number states. In addition, we have seen that the uncertainty principle in the coherent states, in general, does not attain its minimum value. By employing a direct procedure we have shown that this latter result occurs because the coherent states constructed previously correspond to the squeezed states. Finally, we expected that the simple procedure developed in this work can be helpful to investigate subjects related to the interaction of light with material media.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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