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New Oscillation Criteria for Second Order Half-Linear Neutral Type Dynamic Equations on Time Scales

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Abstract

In this paper, we will study the oscillatory properties of the second order half-linear dynamic equations with distributed deviating arguments on time scales. We obtain several new sufficient conditions for the oscillation of all solutions of this equation. Our results not only unify the oscillation of second order nonlinear differential and difference equations but also can be applied to different types of time scales with $\sup \mathbb{T} = \infty$. Our results improve and extend some known results in the literature. Examples which dwell upon the importance of our results are also included.

Keywords

Dynamic Equation, Oscillatory Solutions, Neutral, Deviating Arguments

1. Introduction

In this paper, we are concerned with the oscillatory behavior of solutions of second-order half-linear neutral type dynamic equation with distributed deviating arguments of the form

$$\left(r(t)\psi(x(t))\left|\left(x(t)+p(t)x(t-\tau)\right)^{\Delta}\right|^{\alpha-1}\left(x(t)+p(t)x(t-\tau)\right)^{\Delta}\right)^{\Delta} + \int_a^b q(t,\xi) f\left(x\left(g(t,\xi)\right)\right)\Delta\xi = 0 \tag{1.1}$$

where τ is nonnegative integers. By a solution of (1.1), we mean a nontrivial real-valued function $x \in C_{rd}^1\left[\mathbb{T}_x,\infty\right), \mathbb{T}_x \geq t_0$ which satisfies Equation (1.1) on $\left[\mathbb{T}_x,\infty\right)$, where C_{rd} is the space of rd-continuous functions. A solution $x(t)$ of Equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and non-oscillatory otherwise. Equation (1.1) is called oscillatory if all its solutions are oscillatory. Throughout this paper, we will assume the following hypotheses:

(A_1) $p(t)$ is positive, $0 \le p(t) \le p \le +\infty$, where p is a constant;

(A_2) $r(t) > 0$, $\psi : \mathbb{R} \to (0,\infty)$;

(A_3) $q \in C_{rd}\left([0,\infty)_{\mathbb{T}} \times [a,b],[0,\infty)\right)$ and $g \in C_{rd}\left([0,\infty)_{\mathbb{T}} \times [a,b],[0,\infty)\right)$ satisfies

$$t \ge g(t,\xi) \text{ for } \xi \in [a,b] \text{ and } \lim_{t\to\infty} \min g(t,\xi) = \infty;$$

(A_4) $f \in C(\mathbb{R},\mathbb{R})$ such that $xf(x) > 0$ for $x \ne 0$ and $f(u)/u^{\alpha} \ge K > 0$.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For any $t \in \mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \ \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively. The graininess function $\mu : \mathbb{T} \to [t_0,\infty)$ is defined by $\mu := \sigma(t) - t$.

If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{T}$, then f is continuous at t. Furthermore, we assume that $g : \mathbb{T} \to R$ is Δ-differentiable. The following formulas are useful:

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t);$$

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t) = f(t) g^{\Delta}(t) + f^{\Delta}(t) g(\sigma(t));$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) + f(t) g^{\Delta}(t)}{g(t) g(\sigma(t))};$$

$$\int_a^b f^{\Delta}(t) \Delta t = F(b) - F(a).$$

where $a,b \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = t, \mu(t) = 0, f^{\Delta}(t) = f'(t), \int_a^b f^{\Delta}(t) \Delta t = \int_a^b f^{\Delta}(t) \mathrm{d}t,$$

and (1.1) becomes the second-order half-linear differential equation with distributed deviating arguments:

$$\begin{aligned} &\left(r(t)\psi(x(t)) \left| \left(x(t) + p(t)x(t-\tau)\right)' \right|^{\alpha-1} \left(x(t) + p(t)x(t-\tau)\right)' \right)' \\ &+ \int_a^b q(s,\xi) f\left(x\left(g(s,\xi)\right)\right) \mathrm{d}\xi = 0. \end{aligned} \tag{1.2}$$

If $\mathbb{T} = \mathbb{N}$, we have

$$\sigma(t) = t+1, \mu(t) = 1, f^{\Delta}(t) = \Delta f(t), \int_a^b f^{\Delta}(t) \Delta t = \sum_{t=a}^{b-1} f^{\Delta}(t),$$

and (1.1) becomes the second-order half-linear difference equation with distributed deviating arguments:

$$\begin{aligned} &\Delta\left(a_n \psi(x_n) \left| \Delta\left(x_n + p_n x_{n-\tau}\right) \right|^{\alpha-1} \Delta\left(x_n + p_n x_{n-\tau}\right) \right) \\ &+ \sum_{\xi=a}^{b} q(n,\xi) f\left(x\left(g(n,\xi)\right)\right) = 0, \end{aligned} \tag{1.3}$$

In recent years, there has been an increasing interest in the study of the oscillatory behavior of solutions of dynamic equations. We refer to the papers [1]-[16] and the references cited therein.

In [1] Bohner *et al.* proved several theorems provided sufficient conditions for oscillation of all solutions of the second order Emden-Fowler dynamic equations of the form

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta}+q(t)x^{\gamma}(\sigma(t))=0.$$

They studied both the cases

$$\int_{t_0}^{\infty}\frac{\Delta s}{p(s)}=\infty \text{ and } \int_{t_0}^{\infty}\frac{\Delta s}{p(s)}<\infty.$$

In [2] Baoguo *et al.* discussed the oscillatory behavior of second-order linear dynamic equations:

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta}+p(t)x^{\sigma}(t)=0.$$

In [3] Grace *et al.* discussed the oscillation criteria of second order nonlinear dynamic equations:

$$\left(a(t)\left(x^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}+q(t)x^{\beta}(t)=0.$$

In [4], by a Riccati transformation technique, Tripathy, obtained some oscillation results for nonlinear neutral second-order dynamic equations of the form

$$\left(r(t)\left(\left(y(t)+p(t)y(t-\tau)\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+q(t)\left|y(t-\delta)\right|^{\gamma}\operatorname{sgn} y(t-\delta)=0.$$

In [5], Chen *et al.* studied the oscillatory and asymptotic properties of second-order nonlinear neutral dynamic equations of the form

$$\left(r(t)\left|\left(x(t)+p(t)x(\tau(t))\right)^{\Delta}\right|^{\alpha-1}\left(x(t)+p(t)x(\tau(t))\right)^{\Delta}\right)^{\Delta}+q(t)\left|x(t)\right|^{\beta-1}x(t)=0.$$

They studied both the cases

$$\int_{t_0}^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}\Delta s=\infty \text{ and } \int_{t_0}^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\alpha}}\Delta s<\infty. \tag{1.4}$$

In [6] by a generalized Riccati transformation technique, Chen studied the oscillatory of second-order dynamic equations

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\alpha-1}x^{\Delta}(t)\right)^{\Delta}+q(t)\left|x(t)\right|^{\beta-1}x(t)=0,$$

when α,β are constants.

In [7] by a generalized Riccati transformation technique, Zhang *et al.* obtained some new oscillation results for second-order neutral delay dynamic equation of the form

$$\left(r(t)\left(x(t)+p(t)x(\tau(t))\right)^{\Delta}\right)^{\Delta}+q(t)x(\delta(t))=0.$$

In [8] under condition (1.4), Li *et al.* considered nonlinear second order neutral dynamic equations of the form

$$\left(r(t)\left(\left(y(t)+p(t)y(t-\tau)\right)^{\Delta}\right)^{\alpha}\right)^{\Delta}+q(t)y^{\alpha}(t-\delta)=0.$$

In [9] Li *et al.* studied the oscillatory for second-order half-linear delay damped dynamic equations on time scales of the form

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\alpha-1}x^{\Delta}(t)\right)^{\Delta}+b(t)\left|x^{\Delta}(t)\right|^{\alpha-1}x^{\Delta}(t)+p(t)\left|x(\delta(t))\right|^{\alpha-1}x(\delta(t))=0.$$

In [10] under condition (1.4) and by generalized Riccati transformation technique and the integral averaging, Zhang *et al.* obtained some new oscillation criteria of second-order nonlinear delay dynamic equations on time scales of the form

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+q(t)f\left(x(\tau(t))\right)=0.$$

In this paper, we will consider both the case when

$$\int_{t_0}^{\infty}\left(r(s)\psi(x(s))\right)^{\frac{-1}{\alpha}}\Delta s=\infty, \tag{1.5}$$

holds and the case when

$$\int_{t_0}^{\infty}\left(r(s)\psi(x(s))\right)^{\frac{-1}{\alpha}}\Delta s<\infty, \tag{1.6}$$

holds. For more details, see [13] [14] [15] [16]. When $\mathbb{T}=\mathbb{N}$, we refer the reader to [17] [18] [19] [20] and the references cited therein.

The details of the proofs of results for non-oscillatory solutions will be carried out only for eventually positive solutions, since the arguments are similar for eventually negative solutions.

The paper is organized as follows. In Section 2, we will state and prove the main oscillation theorems and we provide some examples to illustrate the main results.

2. Main Results

In this section, we establish some new oscillation criteria for the Equation (1.1). We begin with some useful lemmas, which will be used later.

Lemma 2.1. Let $x(t)$ be a non-oscillatory solution of Equation (1.1). Then there exists a $t\geq t_0$ such that

$$z(t)\geq 0,\quad z^{\Delta}(t)\geq 0\quad\text{and}\quad\left(r(t)\psi(x(t))\left|z^{\Delta}(t)\right|^{\alpha-1}z^{\Delta}(t)\right)^{\Delta}\leq 0\quad\text{for}\quad t\geq t_0. \tag{2.1}$$

Proof. Let $x(t)$ is eventually positive solution of equation(1.1), we may assume that $x(t)>0$, $x(t-\tau)>0$, and $x(g(t,\xi))>0$ for $t\geq t_0$, $\xi\in[a,b]$. Set $z(t)=x(t)+p(t)x(t-\tau)$. By, assumption (A_1), we have $z(t)>0$, and from Equation (1.1), we get

$$\left(r(t)\psi(x(t))\left|z^{\Delta}(t)\right|^{\alpha-1}z^{\Delta}(t)\right)^{\Delta}=-\int_a^b q(t,\xi)f\left(x\left(g(t,\xi)\right)\right)\Delta\xi\le 0. \quad (2.2)$$

Therefore, $r(t)\psi(x(t))\left|z^{\Delta}(t)\right|^{\alpha-1}z^{\Delta}(t)$ is non-increasing function. Now we have two possible cases for $z^{\Delta}(t)$ either $z^{\Delta}(t)<0$ eventually or $z^{\Delta}(t)>0$ eventually. Suppose that $z^{\Delta}(t)<0$ for $t\ge t_0$. Then from (2.2), there is an integer t_1 such that $z^{\Delta}(t_1)<0$ and

$$r(t)\psi(x(t))\left(z^{\Delta}(t)\right)^{\alpha}\le a(t_1)\psi(x(t_1))\left(z^{\Delta}(t_1)\right)^{\alpha},\ \text{for } t\ge t_1. \quad (2.3)$$

Dividing by $r(t)\psi(x(t))$ and integrating the last inequality from t_1 tot, we obtain

$$z(t)\le z(t_1)+\left(r(t_1)\psi(x(t_1))\right)^{\frac{1}{\alpha}}z^{\Delta}(t_1)\int_{t_1}^{t}\frac{1}{\left(r(s)\psi(x(s))\right)^{\frac{1}{\alpha}}}\Delta s \ \text{ for } t\ge t_1.$$

This implies that $z(t)\to-\infty$ as $t\to\infty$, by (1.5), which is a contradiction the fact that $z(t)$ is positive. Then $z^{\Delta}(t)>0$. This completes the proof of Lemma 2.1. □

Lemma 2.2. Assume that $\alpha\ge 1$, $x_1,x_2\in[0,\infty)$. Then

$$x_1^{\alpha}+x_2^{\alpha}\ge\frac{1}{2^{\alpha-1}}(x_1+x_2)^{\alpha}.$$

Proof. The proof can be found in [11].

Lemma 2.3. Assume that $0<\alpha\le 1$, $x_1,x_2\in[0,\infty)$. Then

$$x_1^{\alpha}+x_2^{\alpha}\ge(x_1+x_2)^{\alpha}. \quad (2.4)$$

Proof. The proof can be found in [12].

Throughout this subsection we assume that there exists a double functions $\{H(t,s)\,|\,t\ge s\ge 0\}$ and $h(t,s)$ such that

1) $H(t,t)=0$ for $t\ge 0$,

2) $H(t,s)>0$ for $t>s>0$,

3) H has a nonpositive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable, and satisfies

$$h(t,s)=-\frac{H^{\Delta_s}(t,s)}{\sqrt{H(t,s)}}.$$

In the following results, we shall use the following notation

$$R(t):=\frac{1}{\left(r(G(t))\psi\left(x(G(t))\right)\right)^{\frac{1}{\alpha}}\mu^{\alpha-1}(G(t))},\ \Theta(t):=\beta(t)\frac{R(t)}{\left(\beta^{\sigma}(t)\right)^{1+\frac{1}{\alpha}}},$$

$$\varphi(t):=2^{1-\alpha}\frac{\beta(t)\Gamma(t)}{\left(\beta^{\sigma}(t)\right)^{2}},\ r_+:=\max\{0,r\},\ \vartheta(t,s):=\left(\frac{\left(\beta^{\Delta}(t)\right)_+}{\beta^{\sigma}(t)}-\frac{h(t,s)}{\sqrt{H(t,s)}}\right),$$

$$\phi(t):=\frac{\left(\beta^{\sigma}(t)\right)^{\alpha+1}}{(1+\alpha)^{1+\alpha}}\frac{\left(\frac{\eta(t)H(t,s)}{\beta^{\sigma}(t)}-h(t,s)\sqrt{H(t,s)}\right)^{1+\alpha}}{\left(H(t,s)\beta(t)R(t)\right)^{\alpha}},$$

$$\Psi(t) := K\beta(t)\int_a^b q(t,\xi)\left(1-p\left(g(t,\xi)\right)\right)^{\alpha}\Delta\xi - \left(\beta(t)a(t)\psi\left(x(t)\right)A(t)\right)^{\Delta}$$
$$+\beta(t)R(t)\left(r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)A^{\sigma}(t)\right)^{1+\frac{1}{\alpha}},$$
$$\eta(t) := \beta^{\Delta}(t)+\alpha\beta(t)R(t)\left(1+\frac{1}{\alpha}\right)\left(r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)A^{\sigma}(t)\right)^{\frac{1}{\alpha}}.$$

Next, we state and prove the main theorems.

Theorem 2.1. Let $\alpha \geq 1$ and (1.5) holds. Further, assume that there exists a positive non-decreasing rd-continuous Δ-differentiable function $\beta(t)$, such that for any $t_1 \in N$, there exists an integer $t_2 > t_1$, with

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+p^{\alpha}\right)\frac{\left(\beta^{\Delta}(s)\right)^{\alpha+1}}{\left(\beta(s)R(s)\right)^{\alpha}}\right)\Delta s = \infty, \quad (2.5)$$

where $Q(t,\xi) = \min\left\{q(t,\xi),\left(q(t,\xi)-\tau\right)\right\}$. Then every solution of Equation (1.1) is oscillatory.

Proof. Assume that Equation (1.1) has a non-oscillatory solution, say $x(t) > 0$, $x(t-\tau) > 0$ and $x\left(g(t,\xi)\right) > 0$ for all $t \geq t_0$. From Equation (1.1), Lemma 2.2 and condition (A_4) there exists $t_2 \geq t_1$ such that for $t \geq t_2$, we get

$$\left[\left(r(t)\psi\left(x(t)\right)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + p^{\alpha}\left[\left(r(t-\tau)\psi\left(x(t-\tau)\right)\left(z^{\Delta}(t-\tau)\right)^{\alpha}\right)^{\Delta}\right]\right]$$
$$+\frac{K}{2^{\alpha-1}}\int_a^b Q(t,\xi)z^{\alpha}\left(g(t,\xi)\right)\Delta\xi \leq 0. \quad (2.6)$$

Further, it is clear from (A_3)

$$g(t,\xi) \geq \min\left\{g(t,a),g(t,b)\right\} \equiv G(t),\ \xi \in [a,b].$$

Thus

$$\left[\left(r(t)\psi\left(x(t)\right)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + p^{\alpha}\left[\left(r(t-\tau)\psi\left(x(t-\tau)\right)\left(z^{\Delta}(t-\tau)\right)^{\alpha}\right)^{\Delta}\right]\right]$$
$$+\frac{Kz^{\alpha}\left(G(t)\right)}{2^{\alpha-1}}\int_a^b Q(t,\xi)\Delta\xi \leq 0. \quad (2.7)$$

Define

$$\omega(t) := \beta(t)\frac{r(t)\psi\left(x(t)\right)\left(z^{\Delta}(t)\right)^{\alpha}}{z^{\alpha}\left(G(t)\right)}. \quad (2.8)$$

Then $\omega(t) > 0$. From (2.8), we have

$$\omega^{\Delta}(t) = \beta^{\Delta}(t)\frac{r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}}{z^{\alpha}\left(G^{\sigma}(t)\right)} + \beta(t)\frac{\left(r(t)\psi\left(x(t)\right)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}}{z^{\alpha}\left(G(t)\right)}$$
$$-\beta(t)\frac{r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}\left(z^{\alpha}\left(G(t)\right)\right)^{\Delta}}{z^{\alpha}\left(G^{\sigma}(t)\right)z^{\alpha}\left(G(t)\right)}. \quad (2.9)$$

Since $z^{\Delta}(t) > 0$, and By using the inequality

$$x^{\alpha}-y^{\alpha}\ge \alpha y^{\alpha-1}(x-y) \text{ for } x\ne y>0 \text{ and } \alpha\ge 1,$$

we have

$$\left(z^{\alpha}(G(t))\right)^{\Delta}=\frac{z^{\alpha}(G(\sigma(t)))-z^{\alpha}(G(t))}{\mu^{\alpha}(G(t))}\ge \frac{\alpha z^{\alpha-1}(G(t))}{\mu^{\alpha-1}(G(t))}\left(z^{\Delta}(G(t))\right),\alpha\ge 1. \quad (2.10)$$

Substitute from (2.10) in (2.9), we have

$$\begin{aligned}\omega^{\Delta}(t)\le{}& \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)+\beta(t)\frac{\left(r(t)\psi(x(t))\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}}{z^{\alpha}(G(t))}\\ &-\alpha\beta(t)\frac{r^{\sigma}(t)\psi(x^{\sigma}(t))\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}z^{\alpha-1}(G(t))\left(z^{\Delta}(G(t))\right)}{z^{2\alpha}(G^{\sigma}(t))\mu^{\alpha-1}(G(t))}.\end{aligned} \quad (2.11)$$

By Lemma (2.1), since $r(t)\psi(x(t))\left|z^{\Delta}(t)\right|^{\alpha-1}z^{\Delta}(t)=r(t)\psi(x(t))\left(z^{\Delta}(t)\right)^{\alpha}$ is decreasing function then $r(t)\psi(x(t))\left(z^{\Delta}(t)\right)^{\alpha}\le r(G(t))\psi(x(G(t)))\left(z^{\Delta}(G(t))\right)^{\alpha}$. Then it follows that

$$\frac{z^{\Delta}(G(t))}{z^{\Delta}(t)}\ge\left(\frac{r(t)\psi(x(t))}{r(G(t))\psi(x(G(t)))}\right)^{\frac{1}{\alpha}}. \quad (2.12)$$

It follows from (2.11) and (2.12) that

$$\begin{aligned}\omega^{\Delta}(t)\le{}& \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)+\beta(t)\frac{\left(r(t)\psi(x(t))\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}}{z^{\alpha}(G(t))}\\ &-\alpha\beta(t)\frac{R(t)}{\left(\beta^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}}\left(\omega^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}.\end{aligned} \quad (2.13)$$

Similarly, define another function $v(t)$ by

$$v(t):=\beta(t)\frac{a(t-\tau)\psi(x(t-\tau))\left(z^{\Delta}(t-\tau)\right)^{\alpha}}{z^{\alpha}(G(t))}. \quad (2.14)$$

Then $v(t)>0$. From (2.14), we have

$$\begin{aligned}v^{\Delta}(t)={}& \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}v^{\sigma}(t)+\beta(t)\frac{\left(r(t-\tau)\psi(x(t-\tau))\left(z^{\Delta}(t-\tau)\right)^{\alpha}\right)^{\Delta}}{z^{\alpha}(G(t))}\\ &-\beta(t)\frac{r^{\sigma}(t-\tau)\psi(x^{\sigma}(t-\tau))\left(\left(z^{\sigma}(t-\tau)\right)^{\Delta}\right)^{\alpha}\left(z^{\alpha}(G(t))\right)^{\Delta}}{z^{\alpha}(G^{\sigma}(t))z^{\alpha}(G(t))}.\end{aligned} \quad (2.15)$$

From (2.10), (2.14), (2.15) and (2.12), we have

$$\begin{aligned}v^{\Delta}(t)\le{}& \beta(t)\frac{\left(r(t-\tau)\psi(x(t-\tau))\left(z^{\Delta}(t-\tau)\right)^{\alpha}\right)^{\Delta}}{z^{\alpha}(G(t))}+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}v^{\sigma}(t)\\ &-\alpha\frac{\beta(t)R(t)}{\left(\beta^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}}\left(v^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}.\end{aligned} \quad (2.16)$$

From (2.13) and (2.16), we obtain

$$\begin{aligned}
&\omega^{\Delta}(t)+p^{\alpha}v^{\Delta}(t) \\
&\le \beta(t)\frac{\left[\left(r(t)\psi(x(t))\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}+p^{\alpha}\left(r(t-\tau)\psi(x(t-\tau))\left(z^{\Delta}(t-\tau)\right)^{\alpha}\right)^{\Delta}\right]}{z^{\alpha}(G(t))} \\
&\quad+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)-\alpha\beta(t)\frac{R(t)}{\left(\beta^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}}\left(\omega^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}} \qquad (2.17)\\
&\quad+p^{\alpha}\left[\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}v^{\sigma}(t)-\alpha\beta(t)\frac{R(t)}{\left(\beta^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}}\left(v^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}\right].
\end{aligned}$$

From (2.7) and (2.17), we have

$$\begin{aligned}
&\omega^{\Delta}(t)+p^{\alpha}v^{\Delta}(t) \\
&\le -\beta(t)\frac{K}{2^{\alpha-1}}\int_a^b Q(t,\xi)\Delta\xi+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)-\alpha\beta(t)\frac{R(t)}{\left(\beta^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}}\left(\omega^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}} \qquad (2.18)\\
&\quad+p^{\alpha}\left[\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}v^{\sigma}(t)-\alpha\beta(t)\frac{R(t)}{\left(\beta^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}}\left(v^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}\right].
\end{aligned}$$

Using (2.18) and the inequality

$$Bu-Au^{\frac{\alpha+1}{\alpha}}\le\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}},\ A>0, \qquad (2.19)$$

we have

$$\begin{aligned}
\omega^{\Delta}(t)+p^{\alpha}v^{\Delta}(t)&\le-\beta(t)\frac{K}{2^{\alpha-1}}\int_a^b Q(t,\xi)\Delta\xi+\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\beta^{\Delta}(t)\right)^{\alpha+1}}{\left(\beta(t)R(t)\right)^{\alpha}} \\
&\quad+\frac{p^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\left(\beta^{\Delta}(t)\right)^{\alpha+1}}{\left(\beta(t)R(t)\right)^{\alpha}}.
\end{aligned}$$

Integrating the last inequality from t_2 to t, we obtain

$$\begin{aligned}
&\int_{t_2}^{t}\left(\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+p^{\alpha}\right)\frac{\left(\beta^{\Delta}(s)\right)^{\alpha+1}}{\left(\beta(s)R(s)\right)^{\alpha}}\right)\Delta s \\
&\le\omega(t_2)+p^{\alpha}v(t_2).
\end{aligned}$$

which yields

$$\int_{t_2}^{t}\left(\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+p^{\alpha}\right)\frac{\left(\beta^{\Delta}(s)\right)^{\alpha+1}}{\left(\beta(s)R(s)\right)^{\alpha}}\right)\Delta s\le c_1,$$

where $c_1>0$ is a finite constant. But, this contradicts (2.5). This completes the proof of Theorem 2.1. □

Corollary 2.1. If $\mathbb{T}=\mathbb{N}$, then (2.5) becomes

$$\limsup_{n\to\infty}\sum_{s=0}^{m-1}\left(\frac{K\rho_s}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{s,\xi}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+p^{\alpha}\right)\frac{(\Delta\rho_s)^{\alpha+1}}{(\rho_s R_s)^{\alpha}}\right)=\infty. \qquad (\overline{2.5})$$

Then every solution of Equation (1.2) is oscillatory.

Example 2.1. Consider the nonlinear delay dynamic equation

$$\Delta\left(\frac{n}{n+1}\Delta x_n\right)+\sum_{\xi=0}^{1}\frac{\lambda\xi}{n^2}x_n\left(\xi+x_n^2\right)=0, n\geq 1,$$

where $a_n=\frac{n}{n+1}$, $\psi(x_n)=1$, $p_n=0$, $\alpha=1$, $q(n,\xi)=\frac{\lambda\xi}{n^2}$ If we take $\rho_n=n$, $K=1$ then we have $R_l=\frac{n+1}{n}$,

$$\sum_{l=n_0}^{n}\left(\frac{K\rho_l}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{l,\xi}-\frac{\left(1+p^{\alpha}\right)\left((l+1)-l\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}(lR_l)^{\alpha}}\right)$$

$$=\sum_{l=1}^{n}\left(\frac{\lambda l}{l^2}-\frac{l}{4l(l+1)}\right)=\sum_{l=1}^{n}\left(\frac{\lambda}{l}-\frac{1}{4(l+1)}\right)\geq\sum_{l=1}^{n}\left(\frac{4\lambda-1}{4l}\right)\to\infty$$

as $n\to\infty$ if $\lambda>\frac{1}{4}$ Thus Corollary 2.1 asserts that every solution of (3.1) is oscillatory when $\lambda>\frac{1}{4}$.

Theorem 2.2. Let $0<\alpha\leq 1$ and (1.5) holds. Further, assume that there exists a positive non-decreasing function $\beta(t)$, such that for any t_1, there exists an integer $t_2>t_1$, with

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(K\beta(s)\int_{a}^{b}Q(s,\xi)\Delta\xi-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+p^{\alpha}\right)\frac{\left(\beta^{\Delta}(s)\right)^{\alpha+1}}{\left(\beta(s)R(s)\right)^{\alpha}}\right)\Delta s=\infty.$$

Then Equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.1 and hence the details are omitted.

Theorem 2.3. Assume that $\alpha\geq 1$ and (1.5) holds. Let $\beta(t)$ be a positive rd-continuous Δ-differentiable function. Furthermore, we assume that there exists a double function $\{H(t,s)\,|\,t\geq s\geq 0\}$. If

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\int_{0}^{t}\left(H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_{\xi=a}^{b}Q(s,\xi)\Delta\xi\right.$$
$$\left.-\frac{\left(1+p^{\alpha}\right)}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta^{\alpha}(s)}\right)\Delta s=\infty. \qquad (2.20)$$

Then every solution of Equation (1.1) is oscillatory.

Proof. Proceeding as in Theorem 2.1 we assume that Equation (1.1) has a non-oscillatory solution, say $x(t)>0$, $x(t-\tau)>0$ and $x(g(t,\xi))>0$ for all $t\geq t_0$. From the proof of Theorem 2.1, we find that (2.18) holds for all $t\geq t_2$.

From (2.18), we have

$$\begin{aligned}&\beta(t)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi\\&\le -\omega^{\Delta}(t)-p^{\alpha}v^{\Delta}(t)+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)-\alpha\Theta(t)\left(\omega^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}\\&\quad+p^{\alpha}\left[\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}v^{\sigma}(t)-\alpha\Theta(t)\left(v^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}\right].\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\int_{t_2}^{t}H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi\Delta s\\&\le-\int_{t_2}^{t}H(t,s)\omega^{\Delta}(s)\Delta s-p^{\alpha}\int_{t_2}^{t}H(t,s)v^{\Delta}(s)\Delta s\\&\quad+\int_{t_2}^{t}H(t,s)\frac{\left(\beta^{\Delta}(s)\right)_+}{\beta^{\sigma}(s)}\omega^{\sigma}(s)\Delta s-\alpha\int_{t_2}^{t}H(t,s)\Theta(s)\left(\omega^{\sigma}(s)\right)^{\frac{\alpha+1}{\alpha}}\Delta s\\&\quad+p^{\alpha}\int_{t_2}^{t}H(t,s)\frac{\left(\beta^{\Delta}(s)\right)_+}{\beta^{\sigma}(s)}v^{\sigma}(s)\Delta s-\alpha p^{\alpha}\int_{t_2}^{t}H(t,s)\Theta(s)\left(v^{\sigma}(s)\right)^{\frac{\alpha+1}{\alpha}}\Delta s,\end{aligned}$$

which yields after integrating by parts

$$\begin{aligned}&\int_{t_2}^{t}H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi\Delta s\\&\le H(t,t_2)\omega(t_2)+\int_{t_2}^{t}H(t,s)\vartheta(t,s)\omega^{\sigma}(s)\Delta s\\&\quad-\alpha\int_{t_2}^{t}H(t,s)\Theta(s)\left(\omega^{\sigma}(s)\right)^{\frac{\alpha+1}{\alpha}}\Delta s+p^{\alpha}H(t,t_2)v(t_2)\\&\quad+p^{\alpha}\int_{t_2}^{t}H(t,s)\vartheta(t,s)v^{\sigma}(s)\Delta s-\alpha p^{\alpha}\int_{t_2}^{t}H(t,s)\Theta(s)\left(v^{\sigma}(s)\right)^{\frac{\alpha+1}{\alpha}}\Delta s.\end{aligned}$$

From (2.19), we have

$$\begin{aligned}&\int_{t_2}^{t}H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi\Delta s\\&\le H(t,t_2)\omega(t_2)+\int_{t_2}^{t}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta^{\alpha}(s)}\Delta s\\&\quad+p^{\alpha}H(t,t_2)v(t_2)+p^{\alpha}\int_{t_2}^{t}\frac{1}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta^{\alpha}(s)}\Delta s.\end{aligned}$$

Then,

$$\begin{aligned}&\int_{t_2}^{t}\left(H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi-\left(1+p^{\alpha}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta^{\alpha}(s)}\right)\Delta s\\&\le H(t,t_2)\omega(t_2)+p^{\alpha}H(t,t_2)v(t_2),\end{aligned}$$

which implies

$$\int_{t_2}^{t}\left(H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi-\left(1+p^{\alpha}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta^{\alpha}(s)}\right)\Delta s$$
$$\le H(t,0)\left|\omega(t_2)\right|+p^{\alpha}H(t,0)\left|v(t_2)\right|.$$

Hence,

$$\int_{0}^{t}\left(H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi-\left(1+p^{\alpha}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta^{\alpha}(s)}\right)\Delta s$$
$$\le H(t,0)\left\{\int_0^t\left|\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi\right|\Delta s+\left|\omega(t_2)\right|+p^{\alpha}\left|v(t_2)\right|\right\}.$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\int_0^t\left(H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi\right.$$
$$\left.-\left(1+p^{\alpha}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta^{\alpha}(s)}\right)\Delta s$$
$$\le\int_0^t\left|\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi\right|\Delta s+\left|\omega(t_2)\right|+p^{\alpha}\left|v(t_2)\right|<\infty,$$

which is contrary to (2.20). This completes the proof of Theorem 2.3. □

Corollary 2.2. If $\mathbb{T}=\mathbb{N}$, then (2.20) becomes

$$\limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-\left(1+p^{\alpha}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\vartheta_{m,n}^{\alpha+1}H_{m,n}}{\Theta_n^{\alpha}}\right)=\infty. \quad (\overline{2.20})$$

Then every solution of Equation (1.2) is oscillatory.

Corollary 2.3. If $\mathbb{T}=\mathbb{R}$, then (2.20) becomes

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\int_0^t\left(H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\mathrm{d}\xi\right.$$
$$\left.-\frac{\left(1+p^{\alpha}\right)}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta_1^{\alpha}(s)}\right)\mathrm{d}s=\infty, \quad (\overline{\overline{2.20}})$$

where

$$\Theta_1(t):=\frac{J(t)G'(t)}{(\beta(t))^{\frac{1}{\alpha}}},\ J(t):=\left(r(G(t))\psi(x(G(t)))\right)^{-\frac{1}{\alpha}}.$$

Then every solution of Equation (1.3) is oscillatory.

Example 2.2. Consider the differential equation

$$\left(t\frac{4t^2+2\cos^2(\ln t)}{t^2+2\cos^2(\ln t)}x'(t)\right)'+\int_0^1\left(\frac{\xi}{t}\right)x[t+\xi]\mathrm{d}\xi=0\quad\text{for } t\ge t_0=1,$$

If we take $\beta(t)=1$ and $H(t,s)=(t-s)^2$, then we have

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t\left(H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\mathrm{d}\xi\right.$$

$$-\left(1+p^{\alpha}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta_1^{\alpha}(s)}\Bigg)\mathrm{d}s$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_1^t\left\{\frac{(t-s)^2}{2s}-2s\frac{4s^2+2\cos^2(\ln s)}{s^2+2\cos^2(\ln s)}\right\}\mathrm{d}s$$

$$\geq\limsup_{t\to\infty}\frac{1}{(t-1)^2}\int_1^t\left\{\frac{(t-s)^2}{2s}-2s\right\}\mathrm{d}s$$

$$=\limsup_{t\to\infty}\frac{1}{(t-1)^2}\left(t-\frac{7}{4}t^2+\frac{1}{2}t^2\ln t+\frac{3}{4}\right)=\infty.$$

Hence, this equation is oscillatory by Corollary 2.4.

Theorem 2.4. Let $0<\alpha\leq 1$ and (1.5) holds. Further, assume that there exists a positive rd-continuous Δ-differentiable function $\beta(t)$, such that for any t_1, there exists an integer $t_2>t_1$, with

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\int_0^t\Bigg(\beta(s)H(t,s)K\int_{\xi=a}^{b}Q(s,\xi)\Delta\xi$$

$$-\left(1+p^{\alpha}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\vartheta^{\alpha+1}(t,s)H(t,s)}{\Theta^{\alpha}(s)}\Bigg)\Delta s=\infty.$$

Then Equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.3 and hence the details are omitted.

Corollary 2.4. If $\mathbb{T}=\mathbb{R}$, then the condition of Theorem 2.4 becomes

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\int_0^t\Bigg(H(t,s)\beta(s)K\int_a^b Q(s,\xi)\mathrm{d}\xi$$

$$-\frac{\left(1+p^{\alpha}\right)}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\Theta_1^{\alpha}(s)}\Bigg)\mathrm{d}s=\infty.$$

Then every solution of Equation (1.3) is oscillatory.

Example 2.3. Consider the differential Equation

$$\left(x(t)+\frac{1}{t+2}x(t-1)\right)''+\int_0^1\frac{\gamma(t-\xi+2)}{t^2(t-\xi+1)}x(t-\xi)\mathrm{d}\xi=0,$$

where $\alpha=1$, $a(t)=\psi(x)=1$, $p(t)=\dfrac{1}{t+2}$, $q(t,\xi)=\dfrac{\gamma(t-\xi+2)}{t^2(t-\xi+1)}$, $g(t,\xi)=t-\xi$ and $f(x)=x$. If we take $p=1$, $H(t,s)=(t-s)^2$ and $\beta(t)\equiv 1$, then

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^t(t-s)^2\left(\beta(s)K\int_a^b Q(s,\xi)\mathrm{d}\xi-\left(1+p^{\alpha}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\vartheta^{\alpha+1}(t,s)}{\Theta_1^{\alpha}(s)}\right)\mathrm{d}s$$

$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)^2}\int_{t_0}^t(t-s)^2\left(\frac{\gamma}{s}-\frac{1}{4}\left(\frac{2}{t-s}\right)^2\right)\mathrm{d}s=\infty.$$

Hence, by Corollary 2.4, this equation is oscillatory.

Theorem 2.5. Let $\alpha \ge 1$ and (1.5) holds. Further, assume that there exists a positive rd-continuous Δ-differentiable function $\beta(t)$, such that for any t_1, there exists an integer $t_2 > t_1$, with

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(\beta(l)\frac{K}{2^{\alpha-1}}\int_{a}^{b}Q(l,\xi)\Delta\xi-\frac{(1+p^{\alpha})}{2^{3-\alpha}}\frac{(\beta^{\Delta}(l))^{2}}{\beta(l)\Gamma(l)}\right)\Delta l=\infty, \tag{2.21}$$

where $\Gamma(l) := \dfrac{1}{r(G(l))\psi(x(G(l)))}$.

Then every solution of Equation (1.1) oscillatory.

Proof. Assume that Equation (1.1) has a non-oscillatory solution, say $x(t)>0$, $x(t-\tau)>0$ and $x(g(t,\xi))>0$ for all $t \ge t_0$. By Lemma 2.1, we have (2.1) and from Theorem 2.1, we have (2.7). Define $\omega(t)$ and $v(t)$ by (2.8) and (2.14) respectively. Proceeding as in the proof of Theorem 2.1, we obtain (2.9) and (2.15). By using the inequality $x^{\alpha}-y^{\alpha} \ge 2^{1-\alpha}(x-y)^{\alpha}$ for $x \ge y > 0$ and $\alpha \ge 1$, we have

$$\left(z^{\alpha}(G(t))\right)^{\Delta}=\frac{z^{\alpha}(G_{n+1})-z^{\alpha}(G_{n})}{\mu^{\alpha}(G_{n})}\ge 2^{1-\alpha}\left(z^{\Delta}(G(t))\right)^{\alpha},\alpha\ge 1. \tag{2.22}$$

Substitute from (2.22) in (2.9), we have

$$\begin{aligned}\omega^{\Delta}(t)\le{}&\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)+\beta(t)\frac{\left(r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right)^{\Delta}}{z^{\alpha}(G(t))}\\&-\frac{2^{1-\alpha}\beta(t)r^{\sigma}(t)\psi(x^{\sigma}(t))\left((z^{\sigma}(t))^{\Delta}\right)^{\alpha}\left(z^{\Delta}(G(t))\right)^{\alpha}}{z^{2\alpha}(G^{\sigma}(t))}.\end{aligned} \tag{2.23}$$

From (2.12), we have

$$\begin{aligned}\omega^{\Delta}(t)\le{}&\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)+\beta(t)\frac{\left(r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right)^{\Delta}}{z^{\alpha}(G(t))}\\&-2^{1-\alpha}\frac{\beta(t)\Gamma(t)}{(\beta^{\sigma}(t))^{2}}(\omega^{\sigma}(t))^{2}.\end{aligned} \tag{2.24}$$

On the other hand, from (2.15), we have

$$\begin{aligned}v^{\Delta}(t)\le{}&\beta(t)\frac{\left(r(t-\tau)\psi(x(t-\tau))(z^{\Delta}(t-\tau))^{\alpha}\right)^{\Delta}}{z^{\alpha}(G(t))}+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}v^{\sigma}(t)\\&-2^{1-\alpha}\frac{\beta(t)\Gamma(t)}{(\beta^{\sigma}(t))^{2}}(v^{\sigma}(t))^{2}.\end{aligned} \tag{2.25}$$

From (2.24) and (2.25), we obtain

$$\begin{aligned}&\omega^{\Delta}(t)+p^{\alpha}v^{\Delta}(t)\\&\le\beta(t)\frac{\left[\left(r(t)\psi(x(t))(z^{\Delta}(t))^{\alpha}\right)^{\Delta}+p^{\alpha}\left(r(t-\tau)\psi(x(t-\tau))(z^{\Delta}(t-\tau))^{\alpha}\right)^{\Delta}\right]}{z^{\alpha}(G(t))}\end{aligned}$$

$$+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)-2^{1-\alpha}\frac{\beta(t)\Gamma(t)}{(\beta^{\sigma}(t))^{2}}(\omega^{\sigma}(t))^{2}$$
$$+p^{\alpha}\left[\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}v^{\sigma}(t)-2^{1-\alpha}\frac{\beta(t)\Gamma(t)}{(\beta^{\sigma}(t))^{2}}(v^{\sigma}(t))^{2}\right]. \tag{2.26}$$

From (2.7) and (2.26), we have

$$\omega^{\Delta}(t)+p^{\alpha}v^{\Delta}(t)$$
$$\leq -\beta(t)\frac{K}{2^{\alpha-1}}\int_{a}^{b}Q(t,\xi)\Delta\xi+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)-2^{1-\alpha}\frac{\beta(t)\Gamma(t)}{(\beta^{\sigma}(t))^{2}}(\omega^{\sigma}(t))^{2} \tag{2.27}$$
$$+p^{\alpha}\left[\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}v^{\sigma}(t)-2^{1-\alpha}\frac{\beta(t)\Gamma(t)}{(\beta^{\sigma}(t))^{2}}(v^{\sigma}(t))^{2}\right].$$

Using the inequality $Bu-Au^{2}\leq\frac{B^{2}}{4A}, A>0$ in (2.27), we have

$$\omega^{\Delta}(t)+p^{\alpha}v^{\Delta}(t)\leq -\beta(t)\frac{K}{2^{\alpha-1}}\int_{a}^{b}Q(t,\xi)\Delta\xi+\frac{1}{2^{3-\alpha}}(1+p^{\alpha})\frac{(\beta^{\Delta}(t))^{2}}{\beta(t)\Gamma(t)}. \tag{2.28}$$

Integrating (2.28) from t_2 to t, we obtain

$$\int_{t_2}^{t}\left(\beta(s)\frac{K}{2^{\alpha-1}}\int_{a}^{b}Q(s,\xi)\Delta\xi-\frac{1}{2^{3-\alpha}}(1+p^{\alpha})\frac{(\beta^{\Delta}(s))^{2}}{\beta(s)\Gamma(s)}\right)\Delta s\leq\omega(t_2)+p^{\alpha}v(t_2),$$

which yields

$$\int_{t_2}^{t}\left(\beta(s)\frac{K}{2^{\alpha-1}}\int_{a}^{b}Q(s,\xi)\Delta\xi-\frac{(1+p^{\alpha})}{2^{3-\alpha}}\frac{(\beta^{\Delta}(s))^{2}}{\beta(s)\Gamma(s)}\right)\Delta s\leq c_1,$$

where $c_1>0$ is a finite constant. Taking lim sup in the above inequality, we obtain a contradiction with (2.21). This completes the proof of Theorem 2.5. □

Corollary 2.5. If $\mathbb{T}=\mathbb{N}$, then (2.21) becomes

$$\limsup_{m\to\infty}\sum_{l=n_0}^{m-1}\left(\frac{K\beta_l}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{l,\xi}-\frac{(1+p^{\alpha})}{2^{3-\alpha}}\frac{(\Delta\beta_l)^{2}}{\beta_l\Gamma_l}\right)=\infty. \tag{$\overline{2.21}$}$$

Then every solution of Equation (1.2) oscillatory.

Example 2.4. Consider the nonlinear neutral dynamic equation

$$\Delta\left(\frac{1}{n^{3}}(\Delta(x_n+px_{n-1}))^{3}\right)+\sum_{\xi=0}^{1}\frac{\lambda\xi}{n^{3}}x_{n-1}^{3}=0,\ n\geq 1,$$

where $a_n=\frac{1}{n^{3}}$, $\psi(x_n)=1$, $p_n=p>0$, $\alpha=3$, $q(n,\xi)=\frac{\lambda\xi}{n^{3}}$. If we take $\beta_n=n^{2}$, $K=1$, then, we have $J_l=l^{3}$,

$$\limsup_{n\to\infty}\sum_{l=n_0}^{n-1}\left(\frac{K\beta_l}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{l,\xi}-\frac{(1+p^{\alpha})}{2^{3-\alpha}}\frac{(\Delta\beta_l)^{2}}{\beta_l\Gamma_l}\right)$$

$$= \limsup_{n\to\infty} \sum_{l=1}^{n-1} \left(\frac{\lambda}{4l} - \frac{4(1+p^3)}{(l-1)^3} \right) = \infty$$

if $\lambda > 0$. By Corollary 2.8 every solution of this equation is oscillatory when $\lambda > 0$.

Theorem 2.6. Let $0 < \alpha \le 1$ and (1.5) holds. Further, assume that there exists a positive rd-continuous Δ-differentiable function $\beta(t)$, such that for any t_1, there exists an integer $t_2 > t_1$, with

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\beta(l) K \int_a^b Q(l,\xi)\Delta\xi - \frac{(1+p^\alpha)}{2^{3-\alpha}} \frac{(\beta^\Delta(l))^2}{\beta(l)\Gamma(l)} \right) \Delta l = \infty.$$

Then Equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.5 and hence the details are omitted.

Theorem 2.7. Assume that $\alpha \ge 1$ and (1.5) holds. Let $\beta(t)$ be a positive rd-continuous Δ-differentiable function. Furthermore, we assume that there exists a double function $\{H(t,s) \mid t \ge s \ge 0\}$. If

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,0)} \int_0^t \left(H(t,s)\beta(s)\frac{K}{2^{\alpha-1}} \int_a^b Q(s,\xi)\Delta\xi \right. \\ &\left. -(1+p^\alpha)\frac{\vartheta^2(t,s)H(t,s)}{4\varphi(s)} \right) \Delta s = \infty. \end{aligned} \tag{2.29}$$

Then every solution of Equation (1.1) is oscillatory.

Proof. Proceeding as in Theorem 2.5 we assume that Equation (1.1) has a non-oscillatory solution, say $x(t) > 0$, $x(t-\tau) > 0$ and $x(g(t,\xi)) > 0$ for all $t \ge t_0$. From the proof of Theorem 2.5, we find that (2.27) holds for all $t \ge t_2$. From (2.27), we have

$$\begin{aligned} \beta(t)\frac{K}{2^{\alpha-1}} \int_a^b Q(t,\xi)\Delta\xi \le& -\omega^\Delta(t) - p^\alpha v^\Delta(t) + \frac{\beta^\Delta(t)}{\beta^\sigma(t)}\omega^\sigma(t) - \varphi(t)(\omega^\sigma(t))^2 \\ &+ p^\alpha \left[\frac{\beta^\Delta(t)}{\beta^\sigma(t)} v^\sigma(t) - \varphi(t)(v^\sigma(t))^2 \right]. \end{aligned} \tag{2.30}$$

Therefore, we have

$$\begin{aligned} &\int_{t_2}^t H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_a^b Q(s,\xi)\Delta\xi\Delta s \\ &\le -\int_{t_2}^t H(t,s)\omega^\Delta(s)\Delta s - p^\alpha \int_{t_2}^t H(t,s)v^\Delta(s)\Delta s \\ &+ \int_{t_2}^t H(t,s)\frac{(\beta^\Delta(s))_+}{\beta^\sigma(s)}\omega^\sigma(s)\Delta s - \int_{t_2}^t H(t,s)\varphi(s)(\omega^\sigma(s))^2 \Delta s \\ &+ p^\alpha \int_{t_2}^t H(t,s)\frac{(\beta^\Delta(s))_+}{\beta^\sigma(s)} v^\sigma(s)\Delta s - p^\alpha \int_{t_2}^t H(t,s)\varphi(s)(v^\sigma(s))^2 \Delta s, \end{aligned}$$

which yields after summing by parts

$$\begin{aligned}
&\int_{t_2}^{t} H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_{a}^{b} Q(s,\xi)\Delta\xi\Delta s \\
&\le H(t,t_2)\omega(t_2)+\int_{t_2}^{t} H(t,s)\vartheta(t,s)\omega^{\sigma}(s)\Delta s \\
&\quad -\int_{t_2}^{t} H(t,s)\varphi(s)\left(\omega^{\sigma}(s)\right)^2\Delta s + p^{\alpha}H(t,t_2)v(t_2) \\
&\quad + p^{\alpha}\int_{t_2}^{t} H(t,s)\vartheta(t,s)v^{\sigma}(s)\Delta s - p^{\alpha}\int_{t_2}^{t} H(t,s)\varphi(s)\left(v^{\sigma}(s)\right)^2\Delta s.
\end{aligned}$$

Using the inequality $Bu - Au^2 \le \frac{B^2}{4A}, A>0$, we have

$$\begin{aligned}
&\int_{t_2}^{t} H(t,s)\beta(s)\frac{K}{2^{\alpha-1}}\int_{a}^{b} Q(s,\xi)\Delta\xi\Delta s \\
&\le H(t,t_2)\omega(t_2)+\int_{t_2}^{t}\frac{\vartheta^2(t,s)H(t,s)}{4\varphi(s)}\Delta s + p^{\alpha}H(t,t_2)v(t_2) \\
&\quad + p^{\alpha}\int_{t_2}^{t}\frac{\vartheta^2(t,s)H(t,s)}{4\varphi(s)}\Delta s.
\end{aligned} \tag{2.31}$$

The rest of the proof is similar that of Theorem 2.3 and hence the details are omitted. This completes the proof of Theorem 2.7. □

Corollary 2.6. If $\mathbb{T}=\mathbb{N}$, then (2.29) becomes

$$\limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left(H_{m,n}\rho_n\frac{K}{2^{\alpha-1}}\sum_{\xi=a}^{b}Q_{n,\xi}-\left(1+p^{\alpha}\right)\frac{\vartheta_{m,n}^2H_{m,n}}{4\varphi_n}\right)=\infty. \tag{$\overline{2.29}$}$$

Then every solution of Equation (1.2) is oscillatory.

Theorem 2.8. Let $0<\alpha\le 1$ and (1.5) holds. Further, assume that there exists a positive rd-continuous Δ-differentiable function $\beta(t)$, such that for any t_1, there exists an integer $t_2>t_1$, with

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\int_{0}^{t}\left(KH(t,s)\beta(s)\int_{a}^{b}Q(s,\xi)\Delta\xi-\left(1+p^{\alpha}\right)\frac{\vartheta^2(t,s)H(t,s)}{4\varphi(s)}\right)\Delta s=\infty.$$

Then Equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.7 and hence the details are omitted.

Theorem 2.9. Let (1.5) holds. Assume that there exists a positive non-decreasing rd-continuous Δ-differentiable function $\beta(t)$ such that for any t_1, there exists an integer $t_2>t_1$, with

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\int_{0}^{t}\left(H(t,s)\Psi(s)-\phi(s)\right)\Delta s=\infty, \tag{2.32}$$

Then every solution of Equation (1.1) is oscillatory.

Proof. Assume that Equation (1.1) has a non-oscillatory solution, say $x(t)>0$, $x(t-\tau)>0$ and $x(g(t,\xi))>0$ for all $t\ge t_0$. From Equation (1.1), From (2.1) and the fact that $x(t)\le z(t)$, we see that

$$x\left(g(t,\xi)-\tau\right) \le z\left(g(t,\xi)-\tau\right) \le z\left(g(t,\xi)\right), t \ge t_2, \xi \in [a,b] \tag{2.33}$$

Further, it is clear form (A_3) that

$$g(t,\xi) \ge \min\{g(t,a), g(t,b)\} \equiv G(t), \xi \in [a,b].$$

which in view of (2.1) leads to

$$z\left(g(t,\xi)\right) \ge z\left(G(t)\right), \text{ for } t \ge t_3 \ge t_2, \xi \in [a,b].$$

Using the above inequality together with (2.1), (2.33), (A_3) and (A_4) in Equation (1.1) for $t \ge t_3$, we get

$$0 \ge \left(r(t)\psi\left(x(t)\right)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + Kz^{\alpha}\left(G(t)\right)\int_a^b q(t,\xi)\left(1-p\left(g(t,\xi)\right)\right)^{\alpha}\Delta\xi. \tag{2.34}$$

Define the function $\omega(t)$ by the generalized Riccati substitution

$$\omega(t) := \beta(t)r(t)\psi\left(x(t)\right)\left[\frac{\left(z^{\Delta}(t)\right)^{\alpha}}{z^{\alpha}\left(G(t)\right)} + A(t)\right], t \ge t_3. \tag{2.35}$$

It follows that

$$\omega^{\Delta}(t) = \left(\beta(t)a(t)\psi\left(x(t)\right)A(t)\right)^{\Delta} + a^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}\left[\frac{\beta(t)}{z^{\alpha}\left(G(t)\right)}\right]^{\Delta} + \frac{\beta(t)\left(a(t)\psi\left(x(t)\right)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}}{z^{\alpha}\left(G(t)\right)}.$$

From (2.34) = and (2.35), we have

$$\omega^{\Delta}(t) \le -K\beta(t)\int_a^b q(t,\xi)\left(1-p\left(g(t,\xi)\right)\right)^{\alpha}\Delta\xi + \left(\beta(t)a(t)\psi\left(x(t)\right)A(t)\right)^{\Delta} + \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t) - \beta(t)\frac{a^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}\left(z^{\alpha}\left(G(t)\right)\right)^{\Delta}}{z^{\alpha}\left(G^{\sigma}(t)\right)z^{\alpha}\left(G(t)\right)}. \tag{2.36}$$

First: we consider the case when $\alpha \ge 1$. By using the inequality

$$x^{\alpha} - y^{\alpha} \ge \alpha y^{\alpha-1}(x-y) \text{ for } x \ne y > 0 \text{ and } \alpha \ge 1,$$

we have

$$\left(z^{\alpha}\left(G(t)\right)\right)^{\Delta} = \frac{z^{\alpha}\left(G^{\sigma}(t)\right) - z^{\alpha}\left(G(t)\right)}{\mu^{\alpha}\left(G(t)\right)} \ge \frac{\alpha z^{\alpha-1}\left(G(t)\right)}{\mu^{\alpha-1}\left(G(t)\right)}z^{\Delta}\left(G(t)\right), \alpha \ge 1.$$

Substituting in (2.36), we have

$$\omega^{\Delta}(t) \le -K\beta(t)\int_a^b q(t,\xi)\left(1-p\left(g(t,\xi)\right)\right)^{\alpha}\Delta\xi + \left(\beta(t)a(t)\psi\left(x(t)\right)A(t)\right)^{\Delta} + \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t) - \alpha\beta(t)\frac{r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}z^{\alpha-1}\left(G(t)\right)z^{\Delta}\left(G(t)\right)}{z^{2\alpha}\left(G^{\sigma}(t)\right)\mu^{\alpha-1}\left(G(t)\right)}. \tag{2.37}$$

From (2.12) and (2.37), we find

$$\omega^{\Delta}(t) \leq -K\beta(t)\int_a^b q(t,\xi)\left(1-p\left(g(t,\xi)\right)\right)^{\alpha}\Delta\xi$$
$$+\left(\beta(t)a(t)\psi\left(x(t)\right)A(t)\right)^{\Delta}+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t) \tag{2.38}$$
$$-\alpha\beta(t)R(t)\left(\frac{a^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}}{z^{\alpha}\left(G^{\sigma}(t)\right)}\right)^{\frac{\alpha+1}{\alpha}}.$$

Second: we consider the case when $0<\alpha<1$. By using the inequality

$$x^{\alpha}-y^{\alpha} \geq \alpha x^{\alpha-1}(x-y) \text{ for } x \neq y>0 \text{ and } 0<\alpha<1,$$

We may write

$$\left(z^{\alpha}\left(G(t)\right)\right)^{\Delta}=\frac{z^{\alpha}\left(G^{\sigma}(t)\right)-z^{\alpha}\left(G(t)\right)}{\mu^{\alpha}\left(G(t)\right)} \geq \frac{\alpha z^{\alpha-1}\left(G^{\sigma}(t)\right)}{\mu^{\alpha-1}\left(G(t)\right)}z^{\Delta}\left(G(t)\right),\ 0<\alpha<1.$$

Substituting in (2.36), we have

$$\omega^{\Delta}(t) \leq -K\beta(t)\int_a^b q(t,\xi)\left(1-p\left(g(t,\xi)\right)\right)^{\alpha}\Delta\xi$$
$$+\left(\beta(t)a(t)\psi\left(x(t)\right)A(t)\right)^{\Delta}+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)$$
$$-\alpha\beta(t)\frac{r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}z^{\alpha-1}\left(G^{\sigma}(t)\right)z^{\Delta}\left(G(t)\right)}{z^{2\alpha}\left(G^{\sigma}(t)\right)\mu^{\alpha-1}\left(G(t)\right)}.$$

From (2.12) and by Lemma (2.1), since $r(t)\psi\left(x(t)\right)\left(z^{\Delta}(t)\right)^{\alpha}$ is decreasing function, we have

$$-\alpha\beta(t)\frac{r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}z^{\alpha-1}\left(G^{\sigma}(t)\right)z^{\Delta}\left(G(t)\right)}{z^{2\alpha}\left(G^{\sigma}(t)\right)}$$
$$\leq -\alpha\beta(t)R(t)\left(\frac{r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}}{z^{\alpha}\left(G^{\sigma}(t)\right)}\right)^{\frac{\alpha+1}{\alpha}}. \tag{2.39}$$

Thus, we again obtain (2.38). However, from (2.35) we see that

$$\left(\frac{r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)\left(\left(z^{\sigma}(t)\right)^{\Delta}\right)^{\alpha}}{z^{\alpha}\left(G^{\sigma}(t)\right)}\right)^{\frac{\alpha+1}{\alpha}}=\left(\frac{\omega^{\sigma}(t)}{\beta^{\sigma}(t)}-r^{\sigma}(t)\psi\left(x^{\sigma}(t)\right)A^{\sigma}(t)\right)^{1+\frac{1}{\alpha}}. \tag{2.40}$$

Then, by using the inequality

$$(v-u)^{1+\frac{1}{\alpha}} \geq v^{1+\frac{1}{\alpha}}+\frac{1}{\alpha}u^{1+\frac{1}{\alpha}}-\left(1+\frac{1}{\alpha}\right)u^{\frac{1}{\alpha}}v,\ \alpha=\frac{odd}{odd}\geq 1,$$

we may write Equation (2.40) as follows

$$\left(\frac{\omega^\sigma(t)}{\beta^\sigma(t)}-r^\sigma(t)\psi\left(x^\sigma(t)\right)A^\sigma(t)\right)^{1+\frac{1}{\alpha}}$$

$$\geq\left(\frac{\omega^\sigma(t)}{\beta^\sigma(t)}\right)^{1+\frac{1}{\alpha}}+\frac{\left(r^\sigma(t)\psi\left(x^\sigma(t)\right)A^\sigma(t)\right)^{1+\frac{1}{\alpha}}}{\alpha}$$

$$-\frac{\left(1+\frac{1}{\alpha}\right)\left(r^\sigma(t)\psi\left(x^\sigma(t)\right)A^\sigma(t)\right)^{\frac{1}{\alpha}}}{\beta^\sigma(t)}\omega^\sigma(t).$$

Substituting back in (2.38), we have

$$\begin{aligned}\omega^\Delta(t)\leq&-K\beta(t)\int_a^b q(t,\xi)\left(1-p\left(g(t,\xi)\right)\right)^\alpha\Delta\xi\\&+\left(\beta(t)a(t)\psi\left(x(t)\right)A(t)\right)^\Delta-\beta(t)R(t)\left(r^\sigma(t)\psi\left(x^\sigma(t)\right)A^\sigma(t)\right)^{1+\frac{1}{\alpha}}\\&+\left(\beta^\Delta(t)+\alpha\beta(t)R(t)\left(1+\frac{1}{\alpha}\right)\left(r^\sigma(t)\psi\left(x^\sigma(t)\right)A^\sigma(t)\right)^{\frac{1}{\alpha}}\right)\frac{\omega^\sigma(t)}{\beta^\sigma(t)}\\&-\left(\frac{\alpha\beta(t)R(t)}{\left(\beta^\sigma(t)\right)^{1+\frac{1}{\alpha}}}\right)\left(\omega^\sigma(t)\right)^{1+\frac{1}{\alpha}}.\end{aligned}\tag{2.41}$$

Thus,

$$\Psi(t)\leq-\omega^\Delta(t)+\frac{\eta(t)}{\beta^\sigma(t)}\omega^\sigma(t)-\left(\frac{\alpha\beta(t)R(t)}{\left(\beta^\sigma(t)\right)^{1+\frac{1}{\alpha}}}\right)\left(\omega^\sigma(t)\right)^{1+\frac{1}{\alpha}}.$$

Therefore, we have

$$\begin{aligned}\int_{t_2}^t H(t,s)\Psi(s)\Delta s\leq&-\int_{t_2}^t H(t,s)\omega^\Delta(s)\Delta s+\int_{t_2}^t\frac{\eta(s)H(t,s)}{\beta^\sigma(s)}\omega^\sigma(s)\Delta s\\&-\int_{t_2}^t\left(\frac{\alpha H(t,s)\beta(t)R(t)}{\left(\beta^\sigma(t)\right)^{1+\frac{1}{\alpha}}}\right)\left(\omega^\sigma(t)\right)^{1+\frac{1}{\alpha}}\Delta s,\end{aligned}$$

which yields after summing by parts

$$\begin{aligned}&\int_{t_2}^t H(t,s)\Psi(s)\Delta s\\&\leq H(t,t_2)\omega(t_2)-\int_{t_2}^t H^{\Delta_s}(t,s)\omega^\sigma(s)\Delta s+\int_{t_2}^t\frac{\eta(s)H(t,s)}{\beta^\sigma(s)}\omega^\sigma(s)\Delta s\\&-\int_{t_2}^t\left(H(t,s)\frac{\alpha\beta(t)R(t)}{\left(\beta^\sigma(t)\right)^{1+\frac{1}{\alpha}}}\right)\left(\omega^\sigma(t)\right)^{1+\frac{1}{\alpha}}\Delta s.\end{aligned}$$

Hence

$$\int_{t_2}^t H(t,s)\Psi(s)\Delta s$$

$$\leq H(t,t_2)\omega(t_2)+\int_{t_2}^{t}\left(\frac{\eta(s)H(t,s)}{\beta^{\sigma}(s)}-h(t,s)\sqrt{H(t,s)}\right)\omega^{\sigma}(t)\Delta s$$
$$-\int_{t_2}^{t}\left(H(t,s)\frac{\alpha\beta(t)R(t)}{\left(\beta^{\sigma}(t)\right)^{1+\frac{1}{\alpha}}}\right)\left(\omega^{\sigma}(t)\right)^{1+\frac{1}{\alpha}}\Delta s.$$

From (2.19), $A=H(t,s)\frac{\alpha\beta(t)R(t)}{\left(\beta^{\sigma}(t)\right)^{1+\frac{1}{\alpha}}}$ and $B=\left(\frac{\eta(s)H(t,s)}{\beta^{\sigma}(s)}-h(t,s)\sqrt{H(t,s)}\right)$, we obtain

$$\int_{t_2}^{t}\left(H(t,s)\Psi(s)-\phi(s)\right)\Delta s\leq H(t,t_2)\omega(t_2)\leq H(t,0)\left|\omega(t_2)\right|,$$

which implies

$$\int_{0}^{t}\left(H(t,s)\Psi(s)-\phi(s)\right)\Delta s\leq H(t,0)\left\{\int_{0}^{t}\Psi(s)\Delta s+\left|\omega(t_2)\right|\right\}<\infty,$$

which is contrary to (2.32). This completes the proof of Theorem 2.9. □

Theorem 2.10. Let (1.6) and (2.5) hold. Assume that $\beta(t)$ be as defined as Theorem 2.1. If

$$\int_{t_0}^{\infty}\left(\frac{1}{r(u)\psi(x(u))}\int_{t_2}^{u}\int_{a}^{b}q(s,\xi)\left(1-p(g(s,\xi))\right)^{\alpha}\Delta\xi\Delta s\right)^{\frac{1}{\alpha}}\Delta u=\infty, \tag{2.42}$$

then every solution of Equation (1.1) either oscillates or tends to zero.

Proof. Assume that Equation (1.1) has a non-oscillatory solution. Without loss of generality, we may assume that $x(t)>0$, $x(\tau(t))>0$ and $x(g(t,\xi))>0$ for all $t\geq t_0$. Proceeding as in the proof of Lemma 2.1, we have (2.2) holds. Therefore, $r(t)\psi(x(t))\left|z^{\Delta}(t)\right|^{\alpha-1}z^{\Delta}(t)$ is non-increasing function. Now we have two possible cases for $z^{\Delta}(t)$ either $z^{\Delta}(t)<0$ eventually or $z^{\Delta}(t)>0$ eventually. If $z^{\Delta}(t)>0$, The proof is similar to that of Theorem 2.1 and hence is omitted. Suppose that $z^{\Delta}(t)<0$ for $t\geq t_0$. Since $z(t)$ is a positive decreasing solution of Equation (1.1), then $\lim_{t\to\infty}z(t)=b\geq 0$. Now we claim that $b=0$. If $b>0$ then $z^{\alpha}(G(t))\geq b^{\alpha}$ for $t\geq t_2\geq t_1$. Therefore from (A_4) and (1.1), we have

$$\left(r(t)\psi(x(t))\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}+Kb^{\alpha}\int_{a}^{b}q(t,\xi)\left(1-p(g(t,\xi))\right)^{\alpha}\Delta\xi\leq 0,\ t\geq t_2.$$

Integrating the above inequality from t_2 tot, we obtain

$$r(t)\psi(x(t))\left(z^{\Delta}(t)\right)^{\alpha}$$
$$\leq r(t_2)\psi(x(t_2))\left(z^{\Delta}(t_2)\right)^{\alpha}-A\int_{t_2}^{t}\int_{a}^{b}q(s,\xi)\left(1-p(g(s,\xi))\right)^{\alpha}\Delta\xi\Delta s$$
$$\leq -A\int_{t_2}^{t}\int_{a}^{b}q(s,\xi)\left(1-p(g(s,\xi))\right)^{\alpha}\Delta\xi\Delta s,$$

where $A=Kb^{\alpha}>0$. Dividing by $r(t)\psi(x(t))$ and integrating the last inequality from t_3 tot, we obtain

$$z(t) \le z(t_3) - A^{\frac{1}{\alpha}} \int_{t_3}^{t} \left(\frac{1}{r(u)\psi(x(u))} \int_{t_2}^{u} \int_{a}^{b} q(s,\xi)\big(1 - p(g(s,\xi))\big)^{\alpha} \Delta\xi \Delta s \right)^{\frac{1}{\alpha}} \Delta u.$$

Condition (2.42) implies that $z(t) \to -\infty$ as $t \to \infty$ which is contradiction with the fact that $z(t) > 0$. Then $b = 0$. *i.e.* $\lim_{t\to\infty} z(t) = 0$. Since $0 < x(t) \le z(t)$ then $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

3. Conclusion

We established some new sufficient conditions for the oscillation of all solutions of this equation. Our results not only unify the oscillation of second order nonlinear differential and difference equations but also can be applied to different types of time scales with $\sup \mathbb{T} = \infty$. Our results improved and expanded some known results, see e.g. the following results:

Remark 3.1. If $\psi(x(t)) \equiv 1$, $p(t) \equiv 0$, $\alpha \equiv 1$, $g(t,\xi) \equiv g(t)$, $q(t,\xi) \equiv q(t)$, $f(x(g(t))) \equiv x^{\gamma}(g(t))$, then we extended and improved Theorems in [1].

Remark 3.2. If $\psi(x(t)) \equiv 1$, $p(t) \equiv 0$, $g(t,\xi) \equiv t$, $q(t,\xi) \equiv q(t)$, $f(u) = u^{\beta}$, then we generalized the results in [3].

Remark 3.3. If $\psi(x(t)) \equiv 1$, $g(t,\xi) \equiv g(t)$, $q(t,\xi) \equiv q(t)$, $f(u) \equiv |y(u-\delta)|^{\gamma} \operatorname{sgn} y(u-\delta) K$, then we extended and improved Theorems in [4].

Remark 3.4. If $\psi(x(t)) \equiv 1$, $q(t,\xi) \equiv q(t)$, $f(u) \equiv |u(t)|^{\beta-1} u(t)$, then we reduced to Theorems in [5].

Remark 3.5. If $\psi(x(t)) \equiv 1$, $p(t) \equiv 0$, $g(t,\xi) \equiv g(t)$, $q(t,\xi) \equiv q(t)$, $f(u) \equiv |u(t)|^{\beta-1} u(t)$, then we reduced to a special case in [6].

Remark 3.6. If $\psi(x(t)) \equiv 1$, $\alpha = 1$, $g(t,\xi) \equiv t$, $q(t,\xi) \equiv q(t)$, $f(u) = u$, then we reduced to a special case in [7].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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