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A Compact Finite Volume Scheme for the Multi-Term Time Fractional Sub-Diffusion Equation

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Abstract

In this paper, we introduce high-order finite volume methods for the multi-term time fractional sub-diffusion equation. The time fractional derivatives are described in Caputo's sense. By using some operators, we obtain the compact finite volume scheme have high order accuracy. We use a compact operator to deal with spatial direction; then we can get the compact finite volume scheme. It is proved that the finite volume scheme is unconditionally stable and convergent in L_∞-norm. The convergence order is $O\left(\tau^{2-\alpha}+h^4\right)$. Finally, two numerical examples are given to confirm the theoretical results. Some tables listed also can explain the stability and convergence of the scheme.

Keywords

Multi-Term Time Fractional Sub-Diffusion Equation, High-Order Compact Finite Volume Scheme, Stable, Convergent

1. Introduction

During the past several decades, the study of fractional partial differential equations has attracted many scholars' attention [1] [2] [3] [4]. Fractional order partial differential equations can provide mathematical tools to describe many phenomena, such as engineering [5] [6], chemistry, physics and so on. Fractional order partial differential equations are different from classical partial differential equations. The fractional integrals and derivatives satisfy the nonlocal properties. Although many important works about theoretical analysis have been car-

ried on, the analytic solutions cannot be obtained exactly in most fractional partial differential equations. Even if their solutions can be found, they are in the form of series, which are difficult to evaluate. We can refer to some recent related works [7] [8] [9]. So the numerical investigation of the fractional partial differential equations based on convergence and stability analyses has been an important topic in recent years [10]-[22]. Liu *et al.* have discussed the stability and convergence of fractional partial differential equations by using a new energy method [10] [11] [12] [13] [14]. Yuste and Acedo found a finite difference method that can solve the time fractional diffusion equation by using the forward-Euler scheme, and they discussed the stability and convergence of the scheme [15] [16].

For the time-fractional partial differential equations, there also have been lots of works. Henry and Langlands presented an implicit finite difference scheme by using the $L1$ scheme to approximate the time-fractional derivative, and they analyzed the stability and convergence by Fourier analysis [17]. Sun and Wu have derived the error estimate of the $L1$ formula which can approximate the Caputo derivative and derived a fully discrete scheme for the diffusion-wave equation [18]. Gao and Sun have derived the $L1$ approximation for the time-fractional derivative, and they constructed a compact finite difference scheme for the sub-diffusion equation [19]. Zhuang *et al.* constructed a Crank-Nicolson-type difference scheme for sub-diffusion equations which have a variable time, and they proved that this scheme is stable and convergent with the discrete $H1$ norm [13] [20]. Also, the maximum norm error estimate has been obtained [21]. In addition, more researchers found other numerical schemes such as the finite element method [22] and others. Tang has discussed convergence and superconvergence of fully discrete finite element for time fractional optimal control problems [23]. Wang *et al.* have derived the local discontinuous Galerkin method for the Time-Fractional KdV equation [24]. It is noted that only one item time fractional order is included in the study of time fractional sub-diffusion equation. In fractional physics, especially diffusion movement, the concept of brown movement is extended because of the generalization of the Gauss probability function. The scope of nuclear magnetic resonance is expanding by added resolving power, so one item time fractional order cannot explain this kind of problem.

The finite volume method is also called the control volume method. Let's take a brief look at the idea of this method. First, we mesh the space and there are non-repetitive control volumes near each grid point. Second, we integrate the equations separately on each control volume. And then, we approximate the first-order partial derivatives with the function values of nodes. This method is usually used to solve integer order equations, and it can also be used to solve the fractional equation. The finite volume method has advantages of integral conservation [25].

Consider the following one-dimensional multi-term time fractional sub-diffusion equation with homogeneous source term on the interval $[0, L]$

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) + {}_0^C D_t^\beta u(x,t) = \dfrac{\partial^2 u}{\partial x^2} + f(x,t), & (x,t) \in (0,L)\times(0,T], \\ u(0,t) = \mu(t), u(L,t) = \nu(t), & t \in [0,T], \\ u(x,0) = \phi(x), & x \in [0,L], \end{cases} \tag{1}$$

where $0 < \beta < \alpha < 1$ and the operator ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative of order α defined by

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\xi)}{(t-\xi)^\alpha} \mathrm{d}\xi.$$

$\Gamma(\cdot)$ is the gamma function, and $\mu(t), \nu(t), \phi(x)$ and $f(x,t)$ are the known functions, $\mu(0) = \phi(0)$, $\nu(0) = \phi(L)$.

For the numerical approximation, take two positive integers *M*, *N* and let $h = \frac{L}{M}, \tau = \frac{T}{N}$. Define $x_i = ih (0 \le i \le M)$, $t_n = n\tau (0 \le n \le N)$, $\Omega_h = \{x_i \mid 0 \le i \le M\}$, $\Omega_\tau = \{t_n \mid 0 \le i \le N\}$, then the computational domain $[0,L]\times[0,T]$ is covered by $\Omega_h \times \Omega_\tau$. Define primal partition I_h, grid element $I_i = [x_i, x_{i+1}] (0 \le i \le M-1)$. Define dual partition I_h^*, $I_0^* = \left[0, x_{\frac{1}{2}}\right]$, $I_i^* = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] (1 \le i \le M-1)$, $I_M^* = \left[x_{M-\frac{1}{2}}, L\right]$. Suppose $u = \{u_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ is a grid function on $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. For every grid function *u*, we define the following notations:

$$\delta_x u_{i-\frac{1}{2}}^n = \frac{1}{h}\left(u_i^n - u_{i-1}^n\right).$$

The main goal of this paper is to construct a high-order compact finite scheme and establish the corresponding error estimate. The remainder of the article is arranged as follows. In Section 2, the compact finite volume scheme is derived. In Section 3, the existence and uniqueness, stability and convergence of the finite volume scheme are proved. In Section 4, two numerical examples are given to demonstrate the theoretical results. Finally, we obtain a brief conclusion.

2. The Derivation of the Compact Finite Volume Scheme

We need to follow some lemmas in the derivation of the compact finite volume scheme.

Lemma 1. *If* $g(x) \in C^3[0,L]$, *and* $x_{i+\frac{1}{2}} = \left(i + \frac{1}{2}\right)h$, $0 \le i \le M-1$. *Then*

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(x)\mathrm{d}x = \frac{h}{24}\left[g(x_{i-1}) + 22g(x_i) + g(x_{i+1})\right] + Ch^4,$$

where $C = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{g^{(3)}(\theta)}{6} \xi\left(\xi^2 - 1\right)\mathrm{d}\xi$, $\theta \in \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right)$.

Proof. We use Lagrange interpolation to approximate $g(x)$ at the points $(x_{i-1}, g(x_{i-1})), (x_i, g(x_i))$ and $(x_{i+1}, g(x_{i+1}))$. We can obtain that

$$g(x) = L_2(x) + R_2(x),$$

where

$$L_2(x) = \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} g(x_{i-1}) + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})} g(x_i) + \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)} g(x_{i+1})$$

and

$$R_2(x) = \frac{g^{(3)}(\theta)}{6}(x-x_{i-1})(x-x_i)(x-x_{i+1}), \quad \theta \in \left(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right).$$

Integrating $g(x)$ by parts, we can have

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g(x)\mathrm{d}x = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(L_2(x) + R_2(x)\right)\mathrm{d}x.$$

To obtain the approximation order, we use substitution to simplify integral terms. For example, let $\xi = \frac{x-ih}{h}$, so the integral terms can be simplified into

$$L_2(\xi) = \frac{1}{2}\xi(\xi-1)g(x_{i-1}) - (\xi^2-1)g(x_i) + \frac{1}{2}\xi(\xi+1)g(x_{i+1})$$

and

$$R_2(\xi) = \frac{g^{(3)}(\theta)h^3}{6}\xi(\xi^2-1).$$

Such that

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} L_2(x)\mathrm{d}x = h\int_{-\frac{1}{2}}^{\frac{1}{2}} L_2(\xi)\mathrm{d}\xi = \frac{h}{24}\left(g(x_{i-1}) + 22g(x_i) + g(x_{i+1})\right)$$

and

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} R_2(x)\mathrm{d}x = h\int_{-\frac{1}{2}}^{\frac{1}{2}} R_2(\xi)\mathrm{d}\xi = h^4\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{g^{(3)}(\theta)}{6}\xi(\xi^2-1)\mathrm{d}\xi. \qquad \square$$

Define grid function space $U_h = \{g \mid g = (g_0, g_1, \cdots, g_M)\}$. For every $g \in U_h$, define the integral operator as follows:

$$(Sg)_i = \frac{h}{24}(g_{i-1} + 22g_i + g_{i+1}), \quad (1 \le i \le M-1).$$

with, $(Sg)_0 = g_0$, $(Sg)_M = g_M$.

Lemma 2. *If* $g(x) \in C^5[0, L]$, *and* $x_{i+\frac{1}{2}} = \left(i+\frac{1}{2}\right)h$, $0 \le i \le M-1$. *Then*

$$\frac{1}{24}\left[g'\left(x_{i-\frac{3}{2}}\right) + 22g'\left(x_{i-\frac{1}{2}}\right) + g'\left(x_{i+\frac{1}{2}}\right)\right] = \frac{1}{h}\left[g(x_i) - g(x_{i-1})\right] + Ch^4,$$

where

$$C=\int_0^1\left\{\frac{1}{144}\left[g^{(5)}\left(x_{i-\frac{1}{2}}-\lambda h\right)+g^{(5)}\left(x_{i-\frac{1}{2}}+\lambda h\right)\right](1-\lambda)^3 -\frac{1}{48}\left[g^{(5)}\left(x_{i-\frac{1}{2}}-\frac{\lambda}{2}h\right)+g^{(5)}\left(x_{i-\frac{1}{2}}+\frac{\lambda}{2}h\right)\right](1-\lambda)^4\right\}\mathrm{d}\lambda.$$

Especially,

$$3g'\left(x_{\frac{1}{2}}\right)-5g'\left(x_{\frac{3}{2}}\right)+4g'\left(x_{\frac{5}{2}}\right)-g'\left(x_{\frac{7}{2}}\right)=\frac{1}{h}\left[g\left(x_1\right)-g\left(x_0\right)\right]+O\left(h^4\right),$$

Proof. Based on the Taylor expansion, we can obtain the following equations

$$g\left(x_i\right)=g\left(x_{i-\frac{1}{2}}\right)+\frac{h}{2}g'\left(x_{i-\frac{1}{2}}\right)+\frac{h^2}{8}g''\left(x_{i-\frac{1}{2}}\right)+\frac{h^3}{48}g^{(3)}\left(x_{i-\frac{1}{2}}\right) +\frac{h^4}{384}g^{(4)}\left(x_{i-\frac{1}{2}}\right)+\frac{1}{24}\int_{x_{i-\frac{1}{2}}}^{x_i}g^{(5)}(s)\left(x_i-s\right)^4\mathrm{d}s, \tag{2}$$

$$g\left(x_{i-1}\right)=g\left(x_{i-\frac{1}{2}}\right)-\frac{h}{2}g'\left(x_{i-\frac{1}{2}}\right)+\frac{h^2}{8}g''\left(x_{i-\frac{1}{2}}\right)-\frac{h^3}{48}g^{(3)}\left(x_{i-\frac{1}{2}}\right) +\frac{h^4}{384}g^{(4)}\left(x_{i-\frac{1}{2}}\right)+\frac{1}{24}\int_{x_{i-\frac{1}{2}}}^{x_{i-1}}g^{(5)}(s)\left(x_{i-1}-s\right)^4\mathrm{d}s, \tag{3}$$

$$g'\left(x_{i-\frac{3}{2}}\right)=g'\left(x_{i-\frac{1}{2}}\right)-hg''\left(x_{i-\frac{1}{2}}\right)+\frac{h^2}{2}g^{(3)}\left(x_{i-\frac{1}{2}}\right)-\frac{h^3}{6}g^{(4)}\left(x_{i-\frac{1}{2}}\right) +\frac{1}{6}\int_{x_{i-\frac{1}{2}}}^{x_{i-\frac{3}{2}}}g^{(5)}(s)\left(x_{i-\frac{3}{2}}-s\right)^3\mathrm{d}s, \tag{4}$$

$$g'\left(x_{i+\frac{1}{2}}\right)=g'\left(x_{i-\frac{1}{2}}\right)+hg''\left(x_{i-\frac{1}{2}}\right)+\frac{h^2}{2}g^{(3)}\left(x_{i-\frac{1}{2}}\right)+\frac{h^3}{6}g^{(4)}\left(x_{i-\frac{1}{2}}\right) +\frac{1}{6}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}g^{(5)}(s)\left(x_{i+\frac{1}{2}}-s\right)^3\mathrm{d}s. \tag{5}$$

To obtain the approximation order, we use substitution to simplify integral remainder. For example, let $s=\left(i-\frac{1}{2}\right)h+\frac{\lambda}{2}h$, so the integral remainder of (2) can be simplified into

$$\frac{h^5}{48}\int_0^1 g^{(5)}\left(x_{i-\frac{1}{2}}+\frac{\lambda}{2}h\right)(1-\lambda)^4\,\mathrm{d}\lambda.$$

We simplify the integral remainder of (2)-(5) by using the same way, then

$$\frac{1}{24}\left[g'\left(x_{i-\frac{3}{2}}\right)+22g'\left(x_{i-\frac{1}{2}}\right)+g'\left(x_{i+\frac{1}{2}}\right)\right]-\frac{1}{h}\left[g\left(x_i\right)-g\left(x_{i-1}\right)\right] =\frac{h^4}{144}\int_0^1\left[g^{(5)}\left(x_{i-\frac{1}{2}}-\lambda h\right)+g^{(5)}\left(x_{i-\frac{1}{2}}+\lambda h\right)\right](1-\lambda)^3\,\mathrm{d}\lambda$$

$$-\frac{h^4}{48}\int_0^1\left[g^{(5)}\left(x_{i-\frac{1}{2}}-\frac{\lambda}{2}h\right)+g^{(5)}\left(x_{i-\frac{1}{2}}+\frac{\lambda}{2}h\right)\right](1-\lambda)^4\,\mathrm{d}\lambda$$

$$=h^4\int_0^1\left\{\frac{1}{144}\left[g^{(5)}\left(x_{i-\frac{1}{2}}-\lambda h\right)+g^{(5)}\left(x_{i-\frac{1}{2}}+\lambda h\right)\right](1-\lambda)^3\right.$$

$$\left.-\frac{1}{48}\left[g^{(5)}\left(x_{i-\frac{1}{2}}-\frac{\lambda}{2}h\right)+g^{(5)}\left(x_{i-\frac{1}{2}}+\frac{\lambda}{2}h\right)\right](1-\lambda)^4\right\}\mathrm{d}\lambda. \qquad \square$$

Define fractional index grid function space $\bar{U}_h=\left\{g\mid g=\left(g_{\frac{1}{2}},g_{\frac{3}{2}},\cdots,g_{M-\frac{1}{2}}\right)\right\}$.

For every $g_{i-\frac{1}{2}}\in\bar{U}_h$, define compact operator as follows:

$$\left(Ag\right)_{i-\frac{1}{2}}=\frac{1}{24}\left(g_{i-\frac{3}{2}}+22g_{i-\frac{1}{2}}+g_{i+\frac{1}{2}}\right),\quad \left(2\le i\le M-1\right).$$

with, $\left(Ag\right)_0=g_0$, $\left(Ag\right)_M=g_M$.

Lemma 3. [26] *If* $0<\alpha<1$, $g\in C^2\left[0,t_n\right]$. *Then,*

$${}_0^C D_t^\alpha g\left(t_n\right)=\frac{\tau^{-\alpha}}{\Gamma\left(2-\alpha\right)}\left[a_0^\alpha g\left(t_n\right)-\sum_{k=1}^{n-1}\left(a_{n-k-1}^\alpha-a_{n-k}^\alpha\right)g\left(t_k\right)-a_{n-1}^\alpha g\left(0\right)\right]+R_{t_n}^\alpha$$

and

$$\left|R_{t_n}^\alpha\right|\le\frac{1}{\Gamma\left(2-\alpha\right)}\left[\frac{1}{4}+\frac{\alpha}{\left(1-\alpha\right)\left(2-\alpha\right)}\right]\max_{0\le t\le t_n}\left|g''\left(t\right)\right|\tau^{2-\alpha},$$

where $a_k^\alpha=\left(k+1\right)^{1-\alpha}-k^{1-\alpha}$.

Define the $L1$ approximation operator as follows

$$D_\tau^\alpha g_i^n=\frac{\tau^{-\alpha}}{\Gamma\left(2-\alpha\right)}\left[a_0^\alpha g\left(x(i),t_n\right)-\sum_{k=1}^{n-1}\left(a_{n-k-1}^\alpha-a_{n-k}^\alpha\right)g\left(x(i),t_k\right)-a_{n-1}^\alpha g\left(x(i),0\right)\right],$$

$$0\le i\le M,\quad 0\le n\le N,$$

where the definition of a_k^α is as same as the Lemma 3.

Let us now construct a compact finite volume scheme for problem (1). On $\Omega_h\times\Omega_\tau$, we now define the grid functions

$$U_i^n=u\left(x_i,t_n\right),\quad f_i^n=f\left(x_i,t_n\right),\quad 0\le i\le M,\quad 0\le n\le N.$$

Suppose $u\left(x,t\right)\in C_{x,t}^{5,2}\left(\left[0,L\right]\times\left[0,T\right]\right)$, which symbols of $u\in C^5\left(x\right)$ and $u\in C^2\left(t\right)$ meanwhile. We consider the Equation (1) at the point $\left(x,t_n\right)$, and we can have

$${}_0^C D_t^\alpha u\left(x,t_n\right)+{}_0^C D_t^\beta u\left(x,t_n\right)=\frac{\partial^2 u\left(x,t_n\right)}{\partial x^2}+f\left(x,t_n\right),\ x\in\left(0,L\right),\ 1\le n\le N. \quad (6)$$

Integrating (6) in intervals, we can have

$$\begin{aligned}&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left({}_0^C D_t^\alpha u\left(x,t_n\right)+{}_0^C D_t^\beta u\left(x,t_n\right)\right)\mathrm{d}x\\&=\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left(\frac{\partial^2 u\left(x,t_n\right)}{\partial x^2}+f\left(x,t_n\right)\right)\mathrm{d}x,\ 1\le i\le M-1,\ 1\le n\le N.\end{aligned}\quad (7)$$

We can obtain

$$\begin{aligned}&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left({}_0^C D_t^{\alpha}u(x,t_n)+{}_0^C D_t^{\beta}u(x,t_n)\right)\mathrm{d}x\\&=\frac{\partial u\left(x_{i+\frac{1}{2}},t_n\right)}{\partial x}-\frac{\partial u\left(x_{i-\frac{1}{2}},t_n\right)}{\partial x}+\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}f(x,t_n)\mathrm{d}x,\ 1\le i\le M-1,\ 1\le n\le N.\end{aligned}\tag{8}$$

By using Lemma 2, we can get

$$\begin{aligned}&A\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left({}_0^C D_t^{\alpha}u(x,t_n)+{}_0^C D_t^{\beta}u(x,t_n)\right)\mathrm{d}x\\&=\delta_x U^n_{i+\frac{1}{2}}-\delta_x U^n_{i-\frac{1}{2}}+A\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}f(x,t_n)\mathrm{d}x+r_i^n,\ 1\le i\le M-1,\ 1\le n\le N,\end{aligned}\tag{9}$$

where $r_i^n=Ch^4$ and the definition of operator A is as same as Lemma 2.

Using numerical integration formula to deal with spatial integral and $L1$ interpolation to discretize the time fractional derivative, then we have from Lemma 1 and Lemma 3.

$$\begin{aligned}&\frac{1}{24}S\left(D_\tau^{\alpha}U_{i-1}^n+D_\tau^{\beta}U_{i-1}^n\right)+\frac{22}{24}S\left(D_\tau^{\alpha}U_i^n+D_\tau^{\beta}U_i^n\right)+\frac{1}{24}S\left(D_\tau^{\alpha}U_{i+1}^n+D_\tau^{\beta}U_{i+1}^n\right)\\&=\delta_x U^n_{i+\frac{1}{2}}-\delta_x U^n_{i-\frac{1}{2}}+\frac{1}{24}Sf_{i-1}^n+\frac{22}{24}Sf_i^n+\frac{1}{24}Sf_{i+1}^n+R_i^n,\ 1\le n\le N,\end{aligned}\tag{10}$$

where $R_i^n=C_1\left(\tau^{2-\alpha}\right)+C_2\left(h^4\right)$ and C_1,C_2 are constants which are independent of h,τ. The definition of operator S is as same as Lemma 3.

Notice that $u(0,t)=\mu(t),u(L,t)=\nu(t)$ and $u(x,0)=\phi(x)$, so we can have

$$U_i^0=\phi(x_i),\quad 1\le i\le M-1,\tag{11}$$

$$U_0^n=\mu(t_n),\quad U_M^n=\nu(t_n),\quad 0\le n\le N.\tag{12}$$

Therefore, we leave out the infinitesimal, then

$$\begin{aligned}&\frac{1}{24}S\left(D_\tau^{\alpha}u_{i-1}^n+D_\tau^{\beta}u_{i-1}^n\right)+\frac{22}{24}S\left(D_\tau^{\alpha}u_i^n+D_\tau^{\beta}u_i^n\right)+\frac{1}{24}S\left(D_\tau^{\alpha}u_{i+1}^n+D_\tau^{\beta}u_{i+1}^n\right)\\&=\delta_x u^n_{i+\frac{1}{2}}-\delta_x u^n_{i-\frac{1}{2}}+\frac{1}{24}Sf_{i-1}^n+\frac{22}{24}Sf_i^n+\frac{1}{24}Sf_{i+1}^n,\quad 1\le n\le N,\end{aligned}\tag{13}$$

$$u_i^0=\phi(x_i),\quad 1\le i\le M-1,\tag{14}$$

$$u_0^n=\mu(t_n),\quad u_M^n=\nu(t_n),\quad 0\le n\le N.\tag{15}$$

These above equations are the compact finite volume scheme of question (1).

3. Analysis of the Compact Finite Volume Scheme

In this section, we will prove the existence and uniqueness, stability and convergence.

First, we introduce the norms in the space γ. Let $g=(g_0,g_1,\cdots,g_M)\in\gamma$. Denote

$$\|g\|_\infty = \max_{0<i<M} |g_i|.$$

Now, we introduce some important lemmas which can be used in the following verifications.

Lemma 4. *Suppose* $\alpha \in (0,1)$, *and* a_l^α *is defined by Lemma 3,* $l = 0,1,\cdots$, *then*

1) $1 = a_0^\alpha > a_1^\alpha > a_2^\alpha > \cdots > a_l^\alpha$; $a_l^\alpha \to 0$, *when* $l \to \infty$;
2) $(1-\alpha)l^{-\alpha} < a_{l-1}^\alpha < (1-\alpha)(l-1)^{-\alpha}$, $l \geq 1$.

Lemma 5. *Let* $p = (p_0, p_1, \cdots, p_M) \in \gamma$, $q = (q_0, q_1, \cdots, q_M) \in \gamma$, *then*

$$\|p+q\|_\infty \leq \|p\|_\infty + \|q\|_\infty.$$

Theorem 1. *The solution of the compact finite volume scheme* (13)-(15) *is existent and unique.*

Proof. Suppose $u^n = (u_0^n, u_1^n, \cdots, u_M^n) \in \gamma$. The numerical solution of u^0 can be obtained by (14). If the numerical solutions of $u^0, u^1, \cdots, u^{n-1}$ are existing and unique, we can obtain non-homogeneous linear equations about u^n from (13) and (15). So if we prove the existence and uniqueness of u^n, we only need to prove that homogeneous linear equations only have a zero solution. Define $S_\alpha = \tau^\alpha \Gamma(2-\alpha)$, $S_\beta = \tau^\beta \Gamma(2-\beta)$.

$$\begin{aligned} &\left(\frac{S}{24S_\alpha} + \frac{S}{24S_\beta}\right)u_{i-1}^n + \left(\frac{22S}{24S_\alpha} + \frac{22S}{24S_\beta}\right)u_i^n + \left(\frac{S}{24S_\alpha} + \frac{S}{24S_\beta}\right)u_{i+1}^n \\ &= \delta_x u_{i+\frac{1}{2}}^n - \delta_x u_{i-\frac{1}{2}}^n, \quad 1 \leq i \leq M-1, \end{aligned} \tag{16}$$

$$u_0^n = u_M^n = 0. \tag{17}$$

Now, we prove that (13) and (15) only have a zero solution. Equation (16) can be rewritten into

$$\begin{aligned} &\left(\frac{S}{24S_\alpha} + \frac{S}{24S_\beta}\right)u_{i-1}^n + \left(\frac{22S}{24S_\alpha} + \frac{22S}{24S_\beta} + \frac{2}{h}\right)u_i^n + \left(\frac{S}{24S_\alpha} + \frac{S}{24S_\beta}\right)u_{i+1}^n \\ &= \frac{1}{h}\left(u_{i-1}^n + u_{i+1}^n\right), \quad 1 \leq i \leq M-1. \end{aligned}$$

According to Lemma 5, equation mentioned above also can be transformed into

$$\left(\frac{S}{S_\alpha} + \frac{S}{S_\beta} + \frac{2}{h}\right)\|u^n\|_\infty = \frac{2}{h}\|u^n\|_\infty.$$

So $\|u^n\|_\infty = 0$, and we can obtain $u^n = 0$.

According to inductive principle, Equations (13)-(15) have a unique solution. □

Theorem 2. *Suppose the finite volume scheme* (13)-(15) *has a solution. We record it as* $\{v_i^n \mid 0 \leq i \leq M, 0 \leq n \leq N\}$, *then*

$$\|v^k\|_\infty \leq \|v^0\|_\infty + c\max_{1\leq m\leq k}\|f^m\|_\infty, 1 \leq k \leq N, \tag{18}$$

where $c = \frac{1}{2}\max\{T^\alpha\Gamma(1-\alpha), T^\beta\Gamma(1-\beta)\}$, $\|f^m\|_\infty = \max_{1\leq i\leq M-1}|f_i^m|$.

Proof. Equation (13) can be written into

$$\begin{aligned}
&\frac{S}{24S_\alpha}\left[a_0^\alpha v_{i-1}^n-\sum_{k=1}^{n-1}\left(a_{n-k-1}^\alpha-a_{n-k}^\alpha\right)v_{i-1}^k-a_{n-1}^\alpha v_{i-1}^0\right]\\
&+\frac{S}{24S_\beta}\left[a_0^\beta v_{i-1}^n-\sum_{k=1}^{n-1}\left(a_{n-k-1}^\beta-a_{n-k}^\beta\right)v_{i-1}^k-a_{n-1}^\beta v_{i-1}^0\right]\\
&+\frac{22S}{24S_\alpha}\left[a_0^\alpha v_i^n-\sum_{k=1}^{n-1}\left(a_{n-k-1}^\alpha-a_{n-k}^\alpha\right)v_i^k-a_{n-1}^\alpha v_i^0\right]\\
&+\frac{22S}{24S_\beta}\left[a_0^\beta v_i^n-\sum_{k=1}^{n-1}\left(a_{n-k-1}^\beta-a_{n-k}^\beta\right)v_i^k-a_{n-1}^\beta v_i^0\right]\\
&+\frac{S}{24S_\alpha}\left[a_0^\alpha v_{i+1}^n-\sum_{k=1}^{n-1}\left(a_{n-k-1}^\alpha-a_{n-k}^\alpha\right)v_{i+1}^k-a_{n-1}^\alpha v_{i+1}^0\right]\\
&+\frac{S}{24S_\beta}\left[a_0^\beta v_{i+1}^n-\sum_{k=1}^{n-1}\left(a_{n-k-1}^\beta-a_{n-k}^\beta\right)v_{i+1}^k-a_{n-1}^\beta v_{i+1}^0\right]\\
&=\delta_x v_{i+\frac{1}{2}}^n-\delta_x v_{i-\frac{1}{2}}^n+\frac{1}{24}Sf_{i-1}^n+\frac{22}{24}Sf_i^n+\frac{1}{24}Sf_{i+1}^n,\\
&1\le i\le M-1,\quad 1\le n\le N.
\end{aligned}$$

Equation mentioned above also can be written into

$$\begin{aligned}
&\left(\frac{Sa_0^\alpha}{24S_\alpha}+\frac{Sa_0^\beta}{24S_\beta}\right)v_{i-1}^n+\left(\frac{22Sa_0^\alpha}{24S_\alpha}+\frac{22Sa_0^\beta}{24S_\beta}+\frac{2}{h}\right)v_i^n+\left(\frac{Sa_0^\alpha}{24S_\alpha}+\frac{Sa_0^\beta}{24S_\beta}\right)v_{i+1}^n\\
&=\frac{S}{24S_\alpha}\left[\sum_{k=1}^{n-1}\left(a_{n-k-1}^\alpha-a_{n-k}^\alpha\right)v_{i-1}^k+a_{n-1}^\alpha v_{i-1}^0\right]\\
&+\frac{S}{24S_\beta}\left[\sum_{k=1}^{n-1}\left(a_{n-k-1}^\beta-a_{n-k}^\beta\right)v_{i-1}^k+a_{n-1}^\beta v_{i-1}^0\right]\\
&+\frac{22S}{24S_\alpha}\left[\sum_{k=1}^{n-1}\left(a_{n-k-1}^\alpha-a_{n-k}^\alpha\right)v_i^k+a_{n-1}^\alpha v_i^0\right]\\
&+\frac{22S}{24S_\beta}\left[\sum_{k=1}^{n-1}\left(a_{n-k-1}^\beta-a_{n-k}^\beta\right)v_i^k+a_{n-1}^\beta v_i^0\right]\\
&+\frac{S}{24S_\alpha}\left[\sum_{k=1}^{n-1}\left(a_{n-k-1}^\alpha-a_{n-k}^\alpha\right)v_{i+1}^k+a_{n-1}^\alpha v_{i+1}^0\right]\\
&+\frac{S}{24S_\beta}\left[\sum_{k=1}^{n-1}\left(a_{n-k-1}^\beta-a_{n-k}^\beta\right)v_{i+1}^k+a_{n-1}^\beta v_{i+1}^0\right]\\
&+\frac{1}{h}\left(v_{i-1}^n+v_{i+1}^n\right)+\frac{1}{24}Sf_{i-1}^n+\frac{22}{24}Sf_i^n+\frac{1}{24}Sf_{i+1}^n,\\
&1\le i\le M-1,\quad 1\le n\le N.
\end{aligned}$$

And then, we add absolute values to both sides of the equation. By using Lemma 5, we can obtain

$$\begin{aligned}
&\left(S\frac{a_0^\alpha}{S_\alpha}+S\frac{a_0^\beta}{S_\beta}+\frac{2}{h}\right)\left\|v^n\right\|_\infty\\
&\le S\frac{1}{S_\alpha}\left[\sum_{k=1}^{n-1}\left(a_{n-k-1}^\alpha-a_{n-k}^\alpha\right)\left\|v^k\right\|_\infty+a_{n-1}^\alpha\left\|v^0\right\|_\infty\right]
\end{aligned}$$

$$
\begin{aligned}
&+S\frac{1}{S_\beta}\left[\sum_{k=1}^{n-1}\left(a_{n-k-1}^{\beta}-a_{n-k}^{\beta}\right)\left\|v^{k}\right\|_{\infty}+a_{n-1}^{\beta}\left\|v^{0}\right\|_{\infty}\right]+\frac{2}{h}\left\|v^{n}\right\|_{\infty}+S\left\|f^{n}\right\|_{\infty}\\
&=\sum_{k=1}^{n-1}\left[S\frac{1}{S_\alpha}\left(a_{n-k-1}^{\alpha}-a_{n-k}^{\alpha}\right)+S\frac{1}{S_\beta}\left(a_{n-k-1}^{\beta}-a_{n-k}^{\beta}\right)\right]\left\|v^{k}\right\|_{\infty}\\
&+\left[S\frac{a_{n-1}^{\alpha}}{S_\alpha}+S\frac{a_{n-1}^{\beta}}{S_\beta}\right]\left\|v^{0}\right\|_{\infty}+\frac{2}{h}\left\|v^{n}\right\|_{\infty}+S\frac{a_{n-1}^{\alpha}}{S_\alpha}\cdot\frac{S_\alpha}{2a_{n-1}^{\alpha}}\left\|f^{n}\right\|_{\infty}\\
&+S\frac{a_{n-1}^{\beta}}{S_\beta}\cdot\frac{S_\beta}{2a_{n-1}^{\beta}}\left\|f^{n}\right\|_{\infty}.
\end{aligned}
\tag{19}
$$

By using Lemma 4, we can obtain

$$
\frac{S_\alpha}{2a_{n-1}^{\alpha}}\le\frac{\tau^{\alpha}\Gamma(2-\alpha)}{2h(1-\alpha)n^{-\alpha}}=\frac{t_n^{\alpha}\Gamma(1-\alpha)}{2h},
\tag{20}
$$

$$
\frac{S_\beta}{2a_{n-1}^{\beta}}\le\frac{\tau^{\beta}\Gamma(2-\beta)}{2h(1-\beta)n^{-\beta}}=\frac{t_n^{\beta}\Gamma(1-\beta)}{2h}.
\tag{21}
$$

Bring (20) and (21) into (19), then

$$
\begin{aligned}
&\left(A\frac{a_0^{\alpha}}{S_\alpha}+A\frac{a_0^{\beta}}{S_\beta}\right)\left\|v^{n}\right\|_{\infty}\\
&\le\sum_{k=1}^{n-1}\left[S\frac{1}{S_\alpha}\left(a_{n-k-1}^{\alpha}-a_{n-k}^{\alpha}\right)+S\frac{1}{S_\beta}\left(a_{n-k-1}^{\beta}-a_{n-k}^{\beta}\right)\right]\left\|v^{k}\right\|_{\infty}\\
&+\left(S\frac{a_{n-1}^{\alpha}}{S_\alpha}+S\frac{a_{n-1}^{\beta}}{S_\beta}\right)\left(\left\|v^{0}\right\|_{\infty}+c\left\|f^{n}\right\|_{\infty}\right),\quad 1\le n\le N,
\end{aligned}
\tag{22}
$$

where $c=\frac{1}{2}\max\left\{T^{\alpha}\Gamma(1-\alpha),T^{\beta}\Gamma(1-\beta)\right\}$.

Next, we use mathematical induction to prove (18).

According to (22), we can obtain when $n=1$,

$$
\left(S\frac{a_0^{\alpha}}{S_\alpha}+S\frac{a_0^{\beta}}{S_\beta}\right)\left\|v^{1}\right\|_{\infty}\le\left(S\frac{a_0^{\alpha}}{S_\alpha}+S\frac{a_0^{\beta}}{S_\beta}\right)\left(\left\|v^{0}\right\|_{\infty}+c\left\|f^{1}\right\|_{\infty}\right).
$$

It also can be written as

$$
\left\|v^{1}\right\|_{\infty}=\left\|v^{0}\right\|_{\infty}+c\left\|f^{1}\right\|_{\infty}.
$$

Therefore when $k=1$, Equation (18) satisfies conditions.

Suppose Equation (22) also satisfies conditions when $k=1,2,\cdots,n-1$. According to (22), we can obtain

$$
\begin{aligned}
&\left(S\frac{a_0^{\alpha}}{S_\alpha}+S\frac{a_0^{\beta}}{S_\beta}\right)\left\|v^{n}\right\|_{\infty}\\
&\le\sum_{k=1}^{n-1}\left[S\frac{1}{S_\alpha}\left(a_{n-k-1}^{\alpha}-a_{n-k}^{\alpha}\right)+S\frac{1}{S_\beta}\left(a_{n-k-1}^{\beta}-a_{n-k}^{\beta}\right)\right]\left(\left\|v^{0}\right\|_{\infty}+c\max_{1\le m\le k}\left\|f^{m}\right\|_{\infty}\right)\\
&+\left(S\frac{a_{n-1}^{\alpha}}{S_\alpha}+S\frac{a_{n-1}^{\beta}}{S_\beta}\right)\left(\left\|v^{0}\right\|_{\infty}+c\left\|f^{n}\right\|_{\infty}\right)
\end{aligned}
$$

$$\le \left\{ \sum_{k=1}^{n-1} \left[S\frac{1}{S_\alpha}\left(a_{n-k-1}^{\alpha} - a_{n-k}^{\alpha}\right) + S\frac{1}{S_\beta}\left(a_{n-k-1}^{\beta} - a_{n-k}^{\beta}\right) \right] + \left(S\frac{a_{n-1}^{\alpha}}{S_\alpha} + S\frac{a_{n-1}^{\beta}}{S_\beta} \right) \right\} \cdot \left(\left\| u^0 \right\|_\infty + c \max_{1\le m\le n} \left\| f^m \right\|_\infty \right)$$

$$= \left(S\frac{a_0^{\alpha}}{S_\alpha} + S\frac{a_0^{\beta}}{S_\beta} \right) \left(\left\| v^0 \right\|_\infty + c \left\| f^n \right\|_\infty \right).$$

It also can be written as

$$\left\| v^n \right\|_\infty \le \left\| v^0 \right\|_\infty + c \left\| f^n \right\|_\infty .$$

So when $k = n$, Equation (18) satisfies conditions.

According to the inductive principle, theorem 2 holds. □

Theorem 3. *The solution of the compact finite volume scheme* (13)-(15) *is convergent. Suppose* $\left\{ U_i^n \mid 0 \le i \le M, 0 \le n \le N \right\}$ *is the solution of equation* (1), $\left\{ u_i^n \mid 0 \le i \le M, 0 \le n \le N \right\}$ *is the solution of the compact finite volume scheme (*13)-(17)*. Define*

$$e_i^n = U_i^n - u_i^n, \quad 0 \le i \le M, \quad 0 \le n \le N,$$

then

$$\left\| e^n \right\|_\infty \le c\left(\tau^{2-\alpha} + h^4 \right), \quad 1 \le n \le N. \tag{23}$$

Proof. We subtract (18)-(22) from (13)-(15), then we can obtain error equations.

$$\begin{aligned} & S\frac{1}{S_\alpha}\left[a_0^{\alpha} e_i^n - \sum_{k=1}^{n-1} \left(a_{n-k-1}^{\alpha} - a_{n-k}^{\alpha} \right) e_i^k - a_{n-1}^{\alpha} e_i^0 \right] \\ & + S\frac{1}{S_\beta}\left[a_0^{\beta} e_i^n - \sum_{k=1}^{n-1} \left(a_{n-k-1}^{\beta} - a_{n-k}^{\beta} \right) e_i^k - a_{n-1}^{\beta} e_i^0 \right] \\ & = \delta_x e^n_{i+\frac{1}{2}} - \delta_x e^n_{i-\frac{1}{2}} + r_i^n, \quad 1 \le i \le M-1, \quad 1 \le n \le N, \end{aligned} \tag{24}$$

$$e_i^0 = 0, \quad 1 \le i \le M-1, \tag{25}$$

$$e_0^n = 0, \quad e_M^n = 0, \quad 0 \le n \le N. \tag{26}$$

By using Theorem 2, we can obtain

$$\left\| e^n \right\|_\infty \le \left\| e^0 \right\|_\infty + c \max_{1\le k\le n} \left\| r^k \right\|_\infty , \quad 1 \le k \le N,$$

where $c = \frac{1}{2}\max\left\{ T^{\alpha}\Gamma(1-\alpha), T^{\beta}\Gamma(1-\beta) \right\}$.

According to Lemma 5, it's easy to obtain

$$\left\| r^k \right\|_\infty \le c_1\left(\tau^{2-\alpha} + h^4 \right).$$

Such that, the result

$$\left\| e^n \right\|_\infty \le c\left(\tau^{2-\alpha} + h^4 \right), \quad 1 \le n \le N,$$

is proved.

4. Numerical Experiments

In this section, we report on some numerical results to show the convergence orders and effectiveness of our finite volume scheme.

Example 1. Let $L=1$, $T=1$. In order to obtain the order of convergence of the finite volume scheme, we refer to the exact solution of the problem (1) is

$$u(x,t)=t^2\sin(\pi x).$$

We can obtain the corresponding source term $f(x,t)$ and the initial and boundary conditions with $\alpha=0.5$, $\beta=0.2$, which are respectively

$$f(x,t)=\frac{\Gamma(3)\sin(\pi x)}{\Gamma(3-\alpha)}t^{2-\alpha}+\frac{\Gamma(3)\sin(\pi x)}{\Gamma(3-\beta)}t^{2-\beta}+\pi^2t^2\sin(\pi x)$$

and

$$\mu(t)=0,\quad \nu(t)=0,\quad \phi(x)=0.$$

Denote the maximum errors

$$E_\infty(h,\tau)=\max_{1\le i\le M}\left|U_i^N-u_i^N\right|.$$

Table 1 and Table 2 show the maximum errors and the convergence order of the finite volume scheme. Suppose

$$E_\infty(h,\tau)=O\left(h^p+\tau^q\right).$$

If τ is small enough, then $E_\infty(h,\tau)\approx O\left(h^p\right)$. Consequently,

$$\frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}\approx 2^p,\quad \log 2\frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}\approx p.$$

Table 1. Error behavior with Dirichlet boundary condition for $h=\frac{1}{1000}$.

τ	$E_\infty(h,\tau)$	$Order1\approx$
2^{-3}	2.4782e−003	-
2^{-4}	7.9090e−004	1.6461
2^{-5}	2.6505e−004	1.5773
2^{-6}	9.1099e−005	1.5407
2^{-7}	3.1935e−005	1.5123

Table 2. Error behavior with Dirichlet boundary condition for $\tau=\frac{1}{1000}$.

h	$E_\infty(h,\tau)$	$Mass_h$	$Order2\approx$
2^{-3}	4.8386e−004	6.3470e−001	-
2^{-4}	3.0735e−005	6.3614e−001	3.9767
2^{-5}	1.9317e−006	6.3646e−001	3.9920
2^{-6}	1.2167e−007	6.3649e−001	3.9892
2^{-7}	7.8016e−009	6.3655e−001	3.9626

So p is the convergence order with respect to the spatial step-size. Similarly, we can obtain

$$q \approx \log 2 \frac{E_\infty(h, 2\tau)}{E_\infty(h, \tau)}$$

for small enough h. Denote

$$Order1 = \log 2 \frac{E_\infty(h, 2\tau)}{E_\infty(h, \tau)}, \quad Order2 = \frac{E_\infty(2h, \tau)}{E_\infty(h, \tau)}.$$

In order to verify the integral conservation of the scheme, we define $Mass_h = \sum_{i=0}^{M} u_i^N h$. Under different meshes, $Mass_h$ keep about the same size, which we can explain the integral conservation.

In Table 1 and Table 2, we compare the exact solution and the numerical solution. In order to test the convergence order of our scheme in temporal direction, we fix sufficiently small spatial step size $h = \frac{1}{1000}$ and vary the temporal step sizes. Table 1 list the numerical results for different temporal step sizes. In order to test the convergence order of our finite volume scheme in spatial direction, we fix sufficiently small temporal step sizes $\tau = \frac{1}{1000}$ and vary the spatial step sizes. Table 2 list the numerical results for different spatial step sizes. Figure 1 show the effect of the numerical solution and exact solution at fixed $h = \frac{1}{1000}$ and $\tau = \frac{1}{64}$. Figure 2 show the effect of the numerical solution and exact solution at

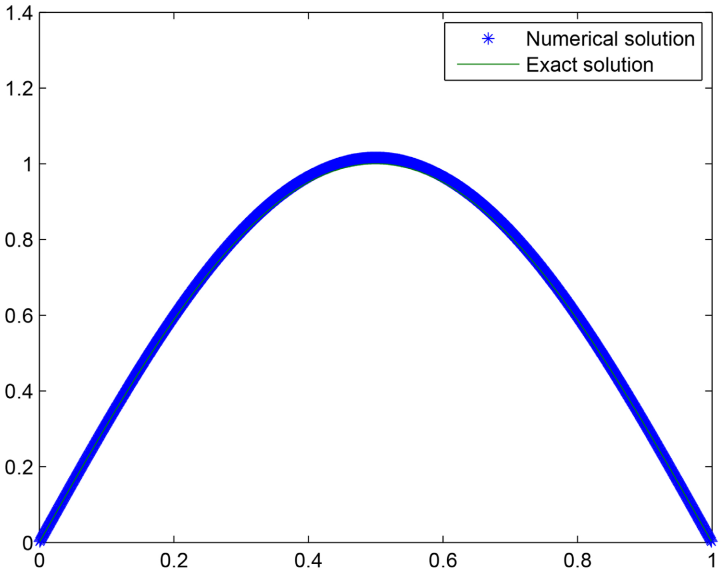

Figure 1. The effect of numerical solution and exact solution at fixed $h = \frac{1}{1000}$ and $\tau = \frac{1}{64}$.

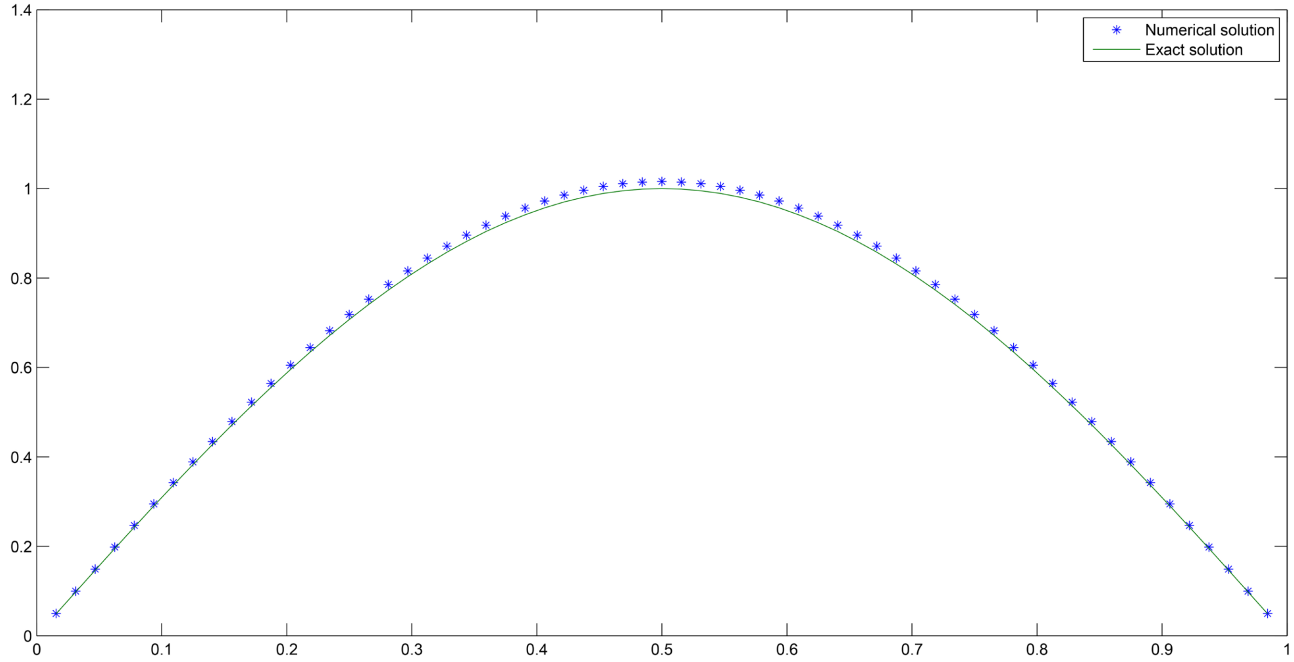

Figure 2. The effect of numerical solution and exact solution at fixed $\tau = \frac{1}{1000}$ and $h = \frac{1}{64}$.

fixed $\tau = \frac{1}{1000}$ and $h = \frac{1}{64}$. In **Figure 3**, the red line represents temporal error order and the blue line represents spatial error order.

In order to observe error orders more intuitively, we plot a figure about error orders which the slope represents the error order. We can observe that the temporal error order is about $2-\alpha$ and the spatial error order is about h^4.

Example 2. Let $L=1$, $T=1$. As before, we refer to the exact solution of the problem (1) is

$$u(x,t) = t^2 x(1-x).$$

It is also not difficult to obtain the corresponding source term $f(x,t)$ and the initial and boundary conditions with $\alpha = 0.9$, $\beta = 0.1$, which are respectively

$$f(x,t) = \frac{\Gamma(3)x(1-x)}{\Gamma(3-\alpha)}t^{2-\alpha} + \frac{\Gamma(3)x(1-x)}{\Gamma(3-\beta)}t^{2-\beta} + 2t^2$$

and

$$\mu(t) = 0, \quad \nu(t) = 0, \quad \phi(x) = 0.$$

In order to test the order of convergence of the finite volume scheme, we compute the maximum norm errors of the numerical solution, which is defined as same as Example 1. Denote

$$Order1 = \log 2 \frac{E_\infty(h, 2\tau)}{E_\infty(h,\tau)}, \quad Order2 = \frac{E_\infty(2h,\tau)}{E_\infty(h,\tau)}.$$

In **Table 3**, we compute the problem for a longer time by fixing

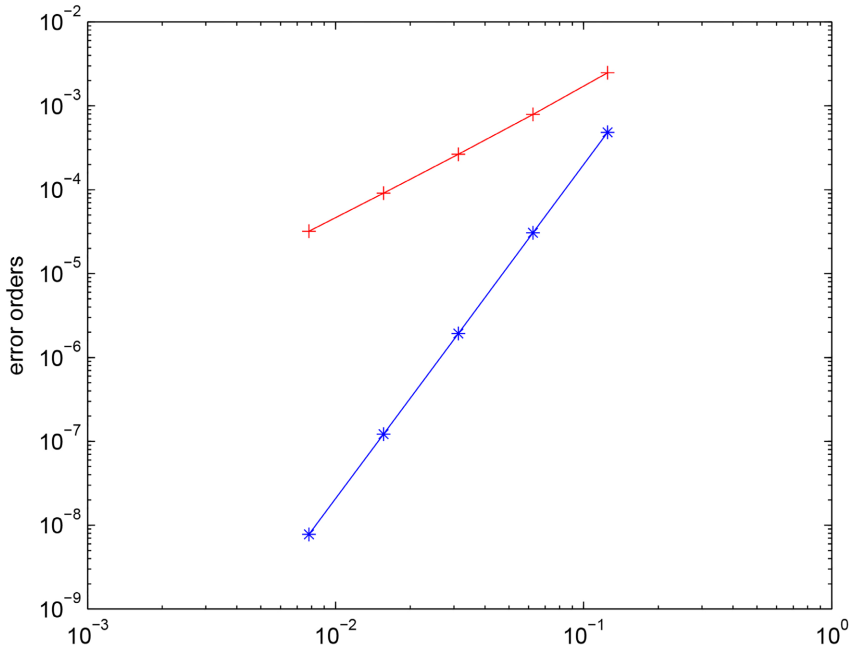

Figure 3. The red line represents temporal error order, the blue line represents spatial error order.

Table 3. Error behavior with Dirichlet boundary condition for $h=\frac{1}{1000}$.

τ	$E_{\infty}(h,\tau)$	$Order1\approx$
2^{-3}	9.5335e−003	-
2^{-4}	4.1078e−003	1.2148
2^{-5}	1.8437e−003	1.1560
2^{-6}	8.4351e−004	1.1277
2^{-7}	3.8995e−004	1.1132

$N=8,16,32,64,128$, and still choosing $h=\frac{1}{1000}$. In **Table 4**, we compute the problem for a longer space by fixing $M=8,16,32,64,128$, and still choosing $\tau=\frac{1}{1000}$. **Figure 4** shows the effect of the numerical solution and exact solution at fixed $h=\frac{1}{1000}$ and $\tau=\frac{1}{64}$. **Figure 5** shows the effect of the numerical solution and exact solution at fixed $\tau=\frac{1}{1000}$ and $h=\frac{1}{64}$. In **Figure 6**, the red line represents temporal error order and the blue line represents spatial error order.

In Example 1, we plot a figure about error orders which slope represents the error order. We can observe that the temporal error order is about $2-\alpha$ and the spatial error order is about h^4.

According to these tables, we can obtain that the compact finite volume scheme is convergent with different spatial step sizes and time step sizes. Under

Table 4. Error behavior with Dirichlet boundary condition for $\tau = \frac{1}{1000}$.

h	$E_\infty(h,\tau)$	$Mass_h$	$Order2 \approx$
2^{-3}	4.7018e−004	1.6404e−001	-
2^{-4}	3.3230e−005	1.6509e−001	3.8227
2^{-5}	2.2894e−006	1.6646e−001	3.8596
2^{-6}	1.4756e−007	1.6660e−001	3.9557
2^{-7}	9.2892e−009	1.6662e−001	3.9893

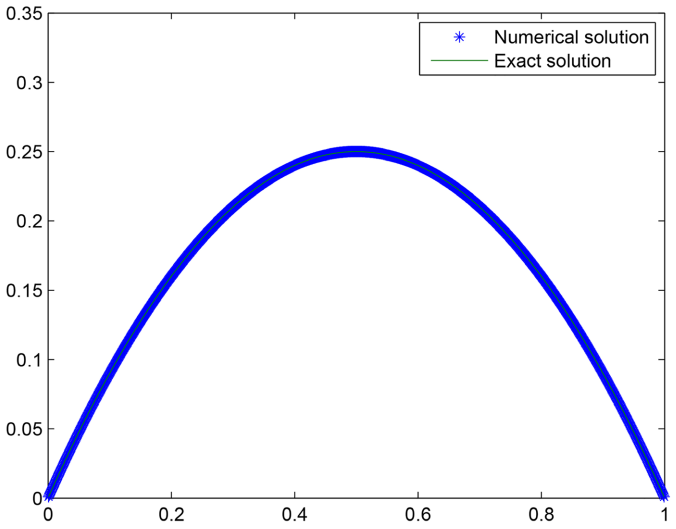

Figure 4. The effect of numerical solution and exact solution at fixed $h = \frac{1}{1000}$ and $\tau = \frac{1}{64}$.

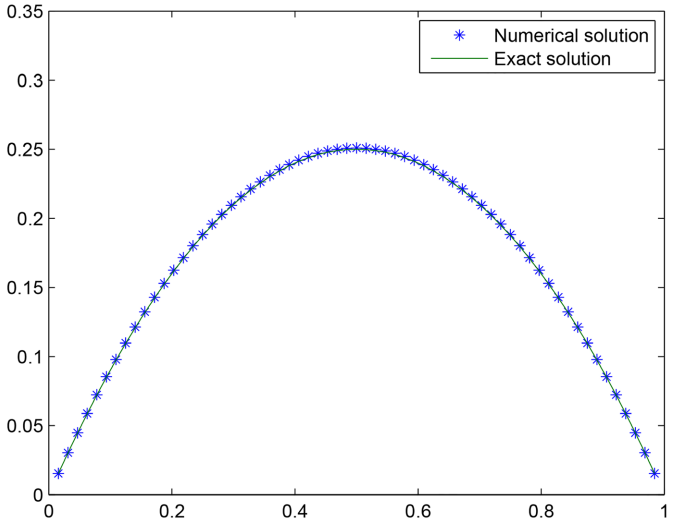

Figure 5. The effect of numerical solution and exact solution at fixed $\tau = \frac{1}{1000}$ and $h = \frac{1}{64}$.

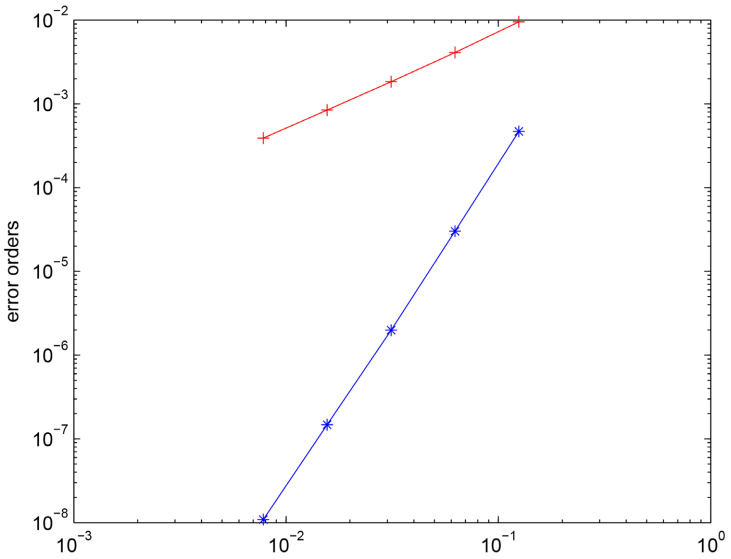

Figure 6. The red line represents temporal error order, the blue line represents spatial error order.

different space steps, $Mass_h$ keep about the same size, which we can get a conclusion that the scheme is conserved.

5. Conclusion

In this article, we constructed a compact finite volume scheme for the multi-term time fractional sub-diffusion equation. Indeed, we use some important operators to deal with the multi-term time fractional sub-diffusion equation. By using a compact operator, we obtain a high order accuracy scheme. We proved the existence and uniqueness, stability and L_∞ convergence of our scheme. Two numerical results show that the scheme is conserved and convergent with the order $O\left(\tau^{2-\alpha}+h^4\right)$. Some tables and figures also can prove the compact finite volume scheme is stable and convergent.

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Authors' Contributions

Baojin Su carried out the main part of this article. All authors read and approved the final manuscript.

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Conflicts of Interest

The authors declare that there is no Conflict of interests regarding the publication of this paper.

References

[1] Oustaloup, A. (1991) Commande Robuste d'Ordre Non Entier. Hermès, Paris.

[2] Podlubny, I. (1999) Fractional Differential Equations. Academic Press, San Diego.

[3] Oldham, K.B. and Spanier, J. (1974) Theory and Application of Differentiation and Integration to Arbitrary Order. Academic Press, Cambridge.

[4] Samko, S.G, Kilbas, A.A. and Marichev, O.I. (1993) Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Philadelphia.

[5] Machado, A.T. (1997) Analysis and Design of Fractional-Order Digital Control Systems. Gordon and Breach Science Publishers, Philadelphia.

[6] Baleanu, D., Defterli, O. and Agrawal, O.P. (2009) A Central Difference Numerical Scheme for Fractional Optimal Control Problems. *Journal of Vibration and Control*, **15**, 583-597. https://doi.org/10.1177/1077546308088565

[7] Das, S. (2011) Functional Fractional Calculus. Springer, Berlin. https://doi.org/10.1007/978-3-642-20545-3

[8] Luchko, Y. and Mainardi, F. (2013) Some Properties of the Fundamental Solution to the Signalling Problem for the Fractional Diffusion-Wave Equation. *Open Physics*, **11**, 666-675. https://doi.org/10.2478/s11534-013-0247-8

[9] Tomovski, I. and Sandev, T. (2013) Exact Solutions for Fractional Diffusion Equation in a Bounded Domain with Different Boundary Conditions. *Nonlinear Dynamics*, **71**, 671-683. https://doi.org/10.1007/s11071-012-0710-x

[10] Liu, F., Anh, V. and Turner, I. (2004) Numerical Solution of the Space Fractional Fokker-Planck Equation. *Journal of Computational and Applied Mathematics*, **166**, 209-219. https://doi.org/10.1016/j.cam.2003.09.028

[11] Tadjeran, C. and Meerschaert, M.M. (2006) Hans-Peter Scheffler, a Second-Order Accurate Numerical Approximation for the Fractional Diffusion Equation. *Journal of Computational Physics*, **213**, 205-213. https://doi.org/10.1016/j.jcp.2005.08.008

[12] Liu, F.W., Zhuang, P., Anh, V., Turner, I. and Burrage, K. (2007) Stability and Convergence of the Difference Methods for the Space-Time Fractional Advection-Diffusion Equation. *Applied Mathematics and Computation*, **191**, 12-20. https://doi.org/10.1016/j.amc.2006.08.162

[13] Zhuang, P., Liu, F.W., Anh, V. and Turner, I. (2008) New Numerical Methods for the Time Fractial Sub-Diffusion Equation. *SIAM Journal on Numerical Analysis*, **46**, 1079-1095. https://doi.org/10.1137/060673114

[14] Liu, F.W., Yang, C. and Burrage, K. (2009) Numerical Method and Analytical Technique of the Modified Anomalous Subdiffusion Equation with a Nonlinear Source Term. *Journal of Computational and Applied Mathematics*, **231**, 160-176. https://doi.org/10.1016/j.cam.2009.02.013

[15] Yuste, S.B. and Acedo, L. (2005) An Explicit Finite Difference Method and a New Von Neumann-Type Stability Analysis for Fractional Diffusion Equations. *SIAM Journal on Numerical Analysis*, **42**, 1862-1874. https://doi.org/10.1137/030602666

[16] Yuste, S.B. (2006) Weighted Average Finite Difference Methods for Fractional Diffusion Equations. *Journal of Computational Physics*, **216**, 264-274. https://doi.org/10.1016/j.jcp.2005.12.006

[17] Langlands, T.A.M. and Henry, B.I. (2005) The Accuracy and Stability of an Implicit Solution Method for the Fractional Diffusion Equation. *Journal of Computational Physics*, **205**, 719-736. https://doi.org/10.1016/j.jcp.2004.11.025

[18] Sun, Z.Z. and Wu, X.N. (2006) A Fully Discrete Difference Scheme for a Diffusion-Wave System. *Applied Numerical Mathematics*, **56**, 193-209. https://doi.org/10.1016/j.apnum.2005.03.003

[19] Gao, G. and Sun, Z.Z. (2011) A Compact Finite Difference Scheme for the Fractional Sub-Diffusion Equations. *Journal of Computational Physics*, **230**, 586-595. https://doi.org/10.1016/j.jcp.2010.10.007

[20] Zhuang, P., Liu, F.W., Anh, V. and Turner, I. (2009) Stability and Convergence of an Implicit Numerical Method for the Non-Linear Fractional Reaction-Subdiffusion Process. *IMA Journal of Applied Mathematics*, **74**, 645-667. https://doi.org/10.1093/imamat/hxp015

[21] Zhang, Y., Sun, Z.Z. and Wu, H.W. (2011) Error Estimates of Crank-Nicolson-Type Difference Schemes for the Sub-Diffusion Equation. *SIAM Journal on Numerical Analysis*, **49**, 2302-2322. https://doi.org/10.1137/100812707

[22] Ervin, V.J. and Roop, J.P. (2006) Variational Formulation for the Stationary Fractional Advection Dispersion Equation. *Numerical Methods for Partial Differential Equations*, **22**, 558-576. https://doi.org/10.1002/num.20112

[23] Tang, Y. (2021) Convergence and Superconvergence of Fully Discrete Finite Element for Time Fractional Optimal Control Problems. *American Journal of Computational Mathematics*, **11**, 53-63. https://doi.org/10.4236/ajcm.2021.111005

[24] Wang, H., Xu, X., Dou, J., *et al.* (2022) Local Discontinuous Galerkin Method for the Time-Fractional KdV Equation with the Caputo-Fabrizio Fractional Derivative. *Journal of Applied Mathematics and Physics*, **10**, 1918-1935. https://doi.org/10.4236/jamp.2022.106132

[25] Rui, H.X. (2008) A Conservative Characteristic Finite Volume Element Method for Solution of the Advection-Diffusion Equation. *Computer Methods in Applied Mechanics and Engineering*, **197**, 3862-3869. https://doi.org/10.1016/j.cma.2008.03.013

[26] Ji, C.C. and Sun, Z.Z. (2015) A High-Order Compact Finite Difference Scheme for the Fractional Sub-Diffusion Equation. *Journal of Scientific Computing*, **64**, 959-985. https://doi.org/10.1007/s10915-014-9956-4