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Structure of Essential Spectra and Discrete Spectrum of the Energy Operator of Six-Electron Systems in the Hubbard Model. First Quintet and First Singlet States

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Abstract

We consider the energy operator of six-electron systems in the Hubbard model and investigate the structure of essential spectra and discrete spectrum of the system in the first quintet and first singlet states in the ν-dimensional lattice.

Keywords

Hubbard Model, Essential Spectrum, Discrete Spectrum, Six Electron Systems, Octet State, Singlet State, Quintet State

1. Introduction

The Hubbard model first appeared in 1963 in the works [1] [2] [3]. The Hubbard model is a simple model of metal was proposed that has become a fundamental model in the theory of strongly correlated electron systems. In that model, a single nondegenerate electron band with a local Coulomb interaction is considered. The model Hamiltonian contains only two parameters: the parameter B of electron hopping from a lattice site to a neighboring site and the parameter U of the on-site Coulomb repulsion of two electrons. In the secondary quantization representation, the Hamiltonian can be written as

$$H = B\sum_{m,\tau,\gamma} a^{+}_{m,\gamma} a_{m+\tau,\gamma} + U\sum_{m} a^{+}_{m,\uparrow} a_{m,\uparrow} a^{+}_{m,\downarrow} a_{m,\downarrow}, \tag{1}$$

where B is the transfer integral between neighboring sites, $\tau = \pm e_j$, $j = 1, 2, \cdots, \nu$, here e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, U is the parameter of the on-site Coulomb interaction of two electrons, γ is the spin index, $\gamma = \uparrow$ or $\gamma = \downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2}$, and $a_{m,\gamma}^{+}$ and $a_{m,\gamma}$ denote Fermi operators of creation and annihilation of an electron with spin γ on a site $m \in Z^{\nu}$, here Z^{ν} ν-dimensional integer valued lattice, ν-lattice dimensionality.

The model proposed in [1] [2] [3] was called the Hubbard model after John Hubbard, who made a fundamental contribution to studying the statistical mechanics of that system, although the local form of Coulomb interaction was first introduced for an impurity model in a metal by Anderson [4]. We also recall that the Hubbard model is a particular case of the Shubin-Wonsowsky polaron model [5], which had appeared 30 years before [1] [2] [3]. In the Shubin-Wonsowsky model, along with the on-site Coulomb interaction, the interaction of electrons on neighboring sites is also taken into account. The simplicity and sufficiency of Hamiltonian (1) have made the Hubbard model very popular and effective for describing strongly correlated electron systems.

The Hubbard model well describes the behavior of particles in a periodic potential at sufficiently low temperatures such that all particles are in the lower Bloch band and long-range interactions can be neglected. If the interaction between particles on different sites is taken into account, then the model is often called the extended Hubbard model. In considering electrons in solids, the Hubbard model can be considered a sophisticated version of the model of strongly bound electrons, involving only the electron hopping term in the Hamiltonian. In the case of strong interactions, these two models can give essentially different results. The Hubbard model exactly predicts the existence of so-called Mott insulators, where conductance is absent due to strong repulsion between particles. The Hubbard model is based on the approximation of strongly coupled electrons. In the strong-coupling approximation, electrons initially occupy orbital's in atoms (lattice sites) and then hop over to other atoms, thus conducting the current. Mathematically, this is represented by the so-called hopping integral. This process can be considered the physical phenomenon underlying the occurrence of electron bands in crystal materials. But the interaction between electrons is not considered in more general band theories. In addition to the hopping integral, which explains the conductance of the material, the Hubbard model contains the so-called on-site repulsion, corresponding to the Coulomb repulsion between electrons. This leads to a competition between the hopping integral, which depends on the mutual position of lattice sites, and the on-site repulsion, which is independent of the atom positions. As a result, the Hubbard model explains the metal-insulator transition in oxides of some transition metals. When such a material is heated, the distance between nearest-neighbor sites increases, the hopping integral decreases, and on-site repulsion becomes dominant.

The Hubbard model is currently one of the most extensively studied multielectron models of metals [6] [7] [8]. Therefore, obtaining exact results for the spectrum and wave functions of the crystal described by the Hubbard model is of great interest. The spectrum and wave functions of the system of two electrons in a crystal described by the Hubbard Hamiltonian were studied in [6]. It is known that two-electron systems can be in two states, triplet and singlet [6] [7] [8]. The work [6] is considered the Hamiltonian of the form

$$H = A\sum_{m,\gamma} a_{m,\gamma}^{+} a_{m,\gamma} + B\sum_{m,\tau,\gamma} a_{m,\gamma}^{+} a_{m+\tau,\gamma} + U\sum_{m} a_{m,\uparrow}^{+} a_{m,\uparrow} a_{m,\downarrow}^{+} a_{m,\downarrow}. \quad (2)$$

Here A is the electron energy at a lattice site, B is the transfer integral between neighboring sites, $\tau = \pm e_j, j = 1, 2, \cdots, \nu$, where e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, U is the parameter of the on-site Coulomb interaction of two electrons, γ is the spin index, $\gamma = \uparrow$ or $\gamma = \downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2}$, and $a_{m,\gamma}^{+}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^{\nu}$. It was proved in [6] that the spectrum of the system Hamiltonian H^{t} in the triplet state is purely continuous and coincides with a segment $[m, M] = [2A - 4B\nu, 2A + 4B\nu]$, where ν is the lattice dimensionality, and the operator H^{s} of the system in the singlet state, in addition to the continuous spectrum $[m, M]$, has a unique antibound state for some values of the quasimomentum. For the antibound state, correlated motion of the electrons is realized under which the contribution of binary states is large. Because the system is closed, the energy must remain constant and large. This prevents the electrons from being separated by long distances. Next, an essential point is that bound states (sometimes called scattering-type states) do not form below the continuous spectrum. This can be easily understood because the interaction is repulsive. We note that a converse situation is realized for $U < 0$: below the continuous spectrum, there is a bound state (antibound states are absent) because the electrons are then attracted to one another.

For the first band, the spectrum is independent of the parameter U of the on-site Coulomb interaction of two electrons and corresponds to the energy of two noninteracting electrons, being exactly equal to the triplet band. The second band is determined by Coulomb interaction to a much greater degree: both the amplitudes and the energy of two electrons depend on U, and the band itself disappears as $U \to 0$ and increases without bound as $U \to \infty$. The second band largely corresponds to a one-particle state, namely, the motion of the doublet, *i.e.*, two-electron bound states.

The spectrum and wave functions of the system of three electrons in a crystal described by the Hubbard Hamiltonian were studied in [9]. In the three-electron systems are exists quartet state, and two type doublet states. The quartet state corresponds to the free motion of three electrons over the lattice with the basic functions $q_{m,n,p}^{3/2} = a_{m,\uparrow}^{+} a_{n,\uparrow}^{+} a_{p,\uparrow}^{+} \varphi_0$. In the work [9] is proved that the essential spectrum of the system in a quartet state consists of a single segment and the three-electron bound state or the three-electron antibound state is absent. The doublet state corresponds to the basic functions ${}^{2}d_{m,n,p}^{1/2} = a_{m,\uparrow}^{+} a_{n,\downarrow}^{+} a_{p,\uparrow}^{+} \varphi_0$, and

${}^{2}d^{1/2}_{m,n,p} = a^{+}_{m,\uparrow}a^{+}_{n,\uparrow}a^{+}_{p,\downarrow}\varphi_0$. If $\nu = 1$ and $U > 0$, then the essential spectrum of the system of first doublet state operator $\tilde{H}^{d}_{1}$ is exactly the union of three segments and the discrete spectrum of $\tilde{H}^{d}_{1}$ consists of a single point, *i.e.*, in the system exists unique antibound state. In the two-dimensional case, we have the analogous results. In the three-dimensional case, or the essential spectrum of the system in the first doublet state operator $\tilde{H}^{d}_{1}$ is the union of three segments and the discrete spectrum of operator $\tilde{H}^{d}_{1}$ consists of a single point, *i.e.*, in the system exists only one antibound state, or the essential spectrum of the system in the first doublet state operator $\tilde{H}^{d}_{1}$ is the union of two segments and the discrete spectrum of the operator $\tilde{H}^{d}_{1}$ is empty, or the essential spectrum of the system in the first doublet state operator $\tilde{H}^{d}_{1}$ is consists of a single segment, and discrete spectrum is empty, *i.e.*, in the system the antibound state is absent. In the one-dimensional case, the essential spectrum of the operator $\tilde{H}^{d}_{2}$ of second doublet state is the union of three segments, and the discrete spectrum of operator $\tilde{H}^{d}_{2}$ consists of no more than one point. In the two-dimensional case, we have analogous results. In the three-dimensional case, or the essential spectrum of the system in the second doublet state operator $\tilde{H}^{d}_{2}$ is the union of three segments and the discrete spectrum of operator $\tilde{H}^{d}_{2}$ consists of no more than one point, *i.e.*, in the system exists no more than one antibound state, or the essential spectrum of the system in the second doublet state operator $\tilde{H}^{d}_{2}$ is the union of two segments and the discrete spectrum of the operator $\tilde{H}^{d}_{2}$ is empty, or the essential spectrum of the system in the second doublet state operator $\tilde{H}^{d}_{2}$ is consists of a single segment, and discrete spectrum is empty, *i.e.*, in the system the antibound state is absent.

The spectrum of the energy operator of system of four electrons in a crystal described by the Hubbard Hamiltonian in the triplet state was studied in [10]. In the four-electron systems are exists quintet state, and three type triplet states, and two type singlet states. The triplet state corresponds to the basic functions

${}^{1}t^{1}_{m,n,p,r} = a^{+}_{m,\uparrow}a^{+}_{n,\uparrow}a^{+}_{p,\uparrow}a^{+}_{r,\downarrow}\varphi_0$, ${}^{2}t^{1}_{m,n,p,r} = a^{+}_{m,\uparrow}a^{+}_{n,\uparrow}a^{+}_{p,\downarrow}a^{+}_{r,\uparrow}\varphi_0$,
${}^{3}t^{1}_{m,n,p,r} = a^{+}_{m,\uparrow}a^{+}_{n,\downarrow}a^{+}_{p,\uparrow}a^{+}_{r,\uparrow}\varphi_0$.

If $\nu = 1$ and $U > 0$, then the essential spectrum of the system first triplet state operator ${}^{1}\tilde{H}^{1}_{t}$ is exactly the union of two segments and the discrete spectrum of operator ${}^{1}\tilde{H}^{1}_{t}$ is empty. In the two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the system first triplet-state operator ${}^{1}\tilde{H}^{1}_{t}$ is the union of two segments and the discrete spectrum of operator ${}^{1}\tilde{H}^{1}_{t}$ is empty, or the essential spectrum of the system first triplet-state operator ${}^{1}\tilde{H}^{1}_{t}$ is single segment and the discrete spectrum of operator ${}^{1}\tilde{H}^{1}_{t}$ is empty. If $\nu = 1$ and $U > 0$, then the essential spectrum of the system second triplet state operator ${}^{2}\tilde{H}^{1}_{t}$ is exactly the union of three segments and the discrete spectrum of operator ${}^{2}\tilde{H}^{1}_{t}$ is consists no more than one point. In the two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the system second triplet-state operator ${}^{2}\tilde{H}^{1}_{t}$ is the union of three segments and the discrete spectrum of the operator ${}^{2}\tilde{H}^{1}_{t}$ is consists no more than one point, or the essential spectrum of the system second triplet-state operator ${}^{2}\tilde{H}_{t}^{1}$ is the union of two segments and the discrete spectrum of the system second triplet state operator ${}^{2}\tilde{H}_{t}^{1}$ is empty, or the essential spectrum of the system second triplet-state operator ${}^{2}\tilde{H}_{t}^{1}$ is consists of single segment and the discrete spectrum of the operator ${}^{2}\tilde{H}_{t}^{1}$ is empty.

If $\nu=1$ and $U>0$, the essential spectrum of the system third triplet-state operator ${}^{3}\tilde{H}_{t}^{1}$ is exactly the union of three segments and the discrete spectrum of the operator ${}^{3}\tilde{H}_{t}^{1}$ is consists no more than one point. In two-dimensional case, we have analogous results. In the three-dimensional case, the essential spectrum of the system third triplet-state operator ${}^{3}\tilde{H}_{t}^{1}$ is the union of three segments, and the discrete spectrum of the operator ${}^{3}\tilde{H}_{t}^{1}$ is consists no more than one point or the essential spectrum of the system third triplet-state operator ${}^{3}\tilde{H}_{t}^{1}$ is the union of two segments, and the discrete spectrum of the operator ${}^{3}\tilde{H}_{t}^{1}$ is empty, or the essential spectrum of the system third triplet-state operator ${}^{3}\tilde{H}_{t}^{1}$ is consists of single segment, and the discrete spectrum of the operator ${}^{3}\tilde{H}_{t}^{1}$ is empty. We see that there are three triplet states, and they have different origins.

The spectrum of the energy operator of four-electron systems in the Hubbard model in the quintet, and singlet states were studied in [11]. The quintet state corresponds to the free motion of four electrons over the lattice with the basic functions $q_{m,n,p,r}^{2}=a_{m,\uparrow}^{+}a_{n,\uparrow}^{+}a_{p,\uparrow}^{+}a_{r,\uparrow}^{+}\varphi_{0}$. In the work [11] proved, that the spectrum of the system in a quintet state is purely continuous and coincides with the segment $[4A-8B\nu,4A+8B\nu]$, and the four-electron bound states or the four-electron antibound states is absent. The singlet state corresponds to the basic functions ${}^{1}s_{p,q,r,t}^{0}=a_{p,\uparrow}^{+}a_{q,\uparrow}^{+}a_{r,\downarrow}^{+}a_{t,\downarrow}^{+}\varphi_{0}$, ${}^{2}s_{p,q,r,t}^{0}=a_{p,\uparrow}^{+}a_{q,\downarrow}^{+}a_{r,\uparrow}^{+}a_{t,\downarrow}^{+}\varphi_{0}$, and these two singlet states have different origins.

If $\nu=1$ and $U>0$, then the essential spectrum of the system of first singlet-state operator ${}^{1}\tilde{H}_{4}^{s}$ is exactly the union of three segments and the discrete spectrum of the operator ${}^{1}\tilde{H}_{4}^{s}$ is consists only one point. In the two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the system first singlet-state operator ${}^{1}\tilde{H}_{4}^{s}$ is the union of three segments and the discrete spectrum of the operator ${}^{1}\tilde{H}_{4}^{s}$ is consists only one point, or the essential spectrum of the system of first singlet-state operator ${}^{1}\tilde{H}_{4}^{s}$ is the union of two segment and the discrete spectrum of the operator ${}^{1}\tilde{H}_{4}^{s}$ is empty, or the essential spectrum of the system of first singlet-state operator ${}^{1}\tilde{H}_{4}^{s}$ is consists of single segment and the discrete spectrum of operator ${}^{1}\tilde{H}_{4}^{s}$ is empty. If $\nu=1$ and $U>0$, then the essential spectrum of the system of second singlet-state operator ${}^{2}\tilde{H}_{4}^{s}$ is exactly the union of three segments and the discrete spectrum of operator ${}^{2}\tilde{H}_{4}^{s}$ is consists only one point. In two-dimensional case, we have the analogous results. In the three-dimensional case, the essential spectrum of the system second singlet-state operator ${}^{2}\tilde{H}_{4}^{s}$ is the union of three segments and the discrete spectrum of the operator ${}^{2}\tilde{H}_{4}^{s}$ is consists only one point, or the essential spectrum of the system of second singlet-state operator ${}^{2}\tilde{H}_{4}^{s}$ is the union of two segment and the discrete spectrum of the operator

${}^2\tilde{H}_4^s$ is empty, or the essential spectrum of the system of second singlet-state operator ${}^2\tilde{H}_4^s$ is consists of single segment and the discrete spectrum of operator ${}^2\tilde{H}_4^s$ is empty.

The structure of essential spectra and discrete spectrum of the energy operator of five-electron systems in the Hubbard model in the fifth doublet state were studied in [12].

In the five-electron systems exists the sextet state, and five type doublet states, and four type quartet states.

The structure of essential spectrum and discrete spectra of the energy operator of five-electron systems in the Hubbard model in the doublet state were investigated in [13] and [14].

The structure of essential spectra and discrete spectrum of the energy operator of five electron systems in the Hubbard model in a sextet and quartet states were studied in [15]. The spectrum of the energy operator of two-electron systems in the impurity Hubbard model in the triplet and singlet state was studied in the work [16].

The spectrum of the energy operator of three-electron systems in the impurity Hubbard model in the second doublet state was studied [17]. The structure of essential spectra and discrete spectrum of three-electron systems in the impurity Hubbard model in the Quartet state were studied in [18]. The structure of essential spectra and discrete spectrum of four-electron systems in the impurity Hubbard model in the first triplet state were studied in [19].

Consequently, in previous works, the spectrum of the energy operator of two, three, four and five electronic systems in the Hubbard model was studied, and the spectrum of the energy operator of two and three electronic systems and four-electron systems for first triplet state in the Impurity Hubbard model was investigated. Naturally, similar problems should be considered for the energy operator of six electronic systems in the Hubbard model.

2. Hamiltonian of the System

We consider the energy operator of six-electron systems in the Hubbard model and describe the structure of the essential spectra and discrete spectrum of the system for octet state, and first quintet and first singlet states in the lattice. The Hamiltonian of the chosen model has the form (2).

In the six electron systems has a octet state, and quintet states, and triplet states, and singlet states. The energy of the system depends on its total spin S. Along with the Hamiltonian, the N_e electron system is characterized by the total spin S, $S = S_{\max}, S_{\max} - 1, \cdots, S_{\min}$, $S_{\max} = \frac{N_e}{2}$, $S_{\min} = 0, \frac{1}{2}$.

Hamiltonian (2) commutes with all components of the total spin operator $S = \left(S^+, S^-, S^z\right)$, and the structure of eigenfunctions and eigenvalues of the system therefore depends on S. The Hamiltonian H acts in the antisymmetric Fock space $\mathcal{H}_{as}$.

3. Six-Electron Octet State in the Hubbard Model

Let φ_0 be the vacuum vector in the space $\mathcal{H}_{as}$. The octet state corresponds to the free motion of six electrons over the lattice with the basic functions $o^3_{p,q,r,t,k,n\in Z^\nu} = a^+_{p,\uparrow}a^+_{q,\uparrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{k,\uparrow}a^+_{n,\uparrow}\varphi_0$. The subspace $\mathcal{H}^o_3$, corresponding to the octet state is the set of all vectors of the form $\psi^o_3 = \sum_{p,q,r,t,k,n\in Z^\nu} f(p,q,r,t,k,n)o^3_{p,q,r,t,k,n\in Z^\nu}$, $f \in l^{as}_2$, where l^{as}_2 is the subspace of antisymmetric functions in the space $l_2\left(\left(Z^\nu\right)^6\right)$. We denote by H^o_3 the restriction of operator H to the subspace $\mathcal{H}^o_3$.

Theorem 1. *The subspace* $\mathcal{H}^o_3$ *is invariant under the operator H, and the restriction* H^o_3 *of operator H to the subspace* $\mathcal{H}^o_3$ *is a bounded self-adjoint operator. It generates a bounded self-adjoint operator* $\bar{H}^o_3$ *acting in the space* l^{as}_2 *as*

$$\begin{aligned}\bar{H}^o_3\psi^o_3 = {} & 6Af(p,q,r,t,k,n) + B\sum_\tau\big[f(p+\tau,q,r,t,k,n) + f(p,q+\tau,r,t,k,n) \\ & + f(p,q,r+\tau,t,k,n) + f(p,q,r,t+\tau,k,n) \\ & + f(p,q,r,t,k+\tau,n) + f(p,q,r,t,k,n+\tau)\big].\end{aligned} \tag{3}$$

The operator H^o_3 acts on a vector $\psi^o_3 \in \mathcal{H}^o_3$ as

$$H^o_3\psi^o_3 = \sum_{p,q,r,t,k,n\in Z^\nu}\left(\bar{H}^o_3 f\right)(p,q,r,t,k,n)o^3_{p,q,r,t,k,n\in Z^\nu}. \tag{4}$$

Proof. We act with the Hamiltonian H on vectors $\psi^o_3 \in \mathcal{H}^o_3$ using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, $\left\{a_{m,\gamma}, a^+_{n,\beta}\right\} = \delta_{m,n}\delta_{\gamma,\beta}$, $\left\{a_{m,\gamma}, a_{n,\beta}\right\} = \left\{a^+_{m,\gamma}, a^+_{n,\beta}\right\} = \theta$, and also take into account that $a_{m,\gamma}\varphi_0 = \theta$, where θ is the zero element of $\mathcal{H}^o_3$. This yields the statement of the theorem. □

Lemma 1. *The spectra of the operators* H^o_3 *and* $\bar{H}^o_3$ *coincide.*

Proof. Because the operators H^o_3 and $\bar{H}^o_3$ are bounded self-adjoint operators, it follows that if $\lambda \in \sigma\left(H^o_3\right)$, then the Weyl criterion (see [20], chapter VII, paragraph 3, pp. 262-263) implies that there is a sequence $\{\psi_i\}^\infty_{i=1}$ such that $\|\psi_i\| = 1$ and $\lim_{i\to\infty}\left\|\left(H^o_3 - \lambda\right)\psi_i\right\| = 0$. We set $\psi_i = \sum_{p,q,r,t,k,n} f_i(p,q,r,t,k,n)a^+_{p,\uparrow}a^+_{q,\uparrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{k,\uparrow}a^+_{n,\uparrow}\varphi_0$. Then

$$\begin{aligned}\left\|\left(H^o_3-\lambda\right)\psi_i\right\|^2 &= \left(\left(H^o_3-\lambda\right)\psi_i, \left(H^o_3-\lambda\right)\psi_i\right) \\ &= \sum_{p,q,r,t,k,n}\left\|\left(\bar{H}^o_3-\lambda\right)f_i(p,q,r,t,k,n)\right\|^2 \\ &\quad \times\left(a^+_{p,\uparrow}a^+_{q,\uparrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{k,\uparrow}a^+_{n,\uparrow}\varphi_0, a^+_{p,\uparrow}a^+_{q,\uparrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{k,\uparrow}a^+_{n,\uparrow}\varphi_0\right)\end{aligned}$$

$$\begin{aligned}&= \sum_{p,q,r,t,k,n}\left\|\left(\bar{H}^3_o-\lambda\right)F_i(p,q,r,t,k,n)\right\|^2 \\ &\quad \times\left(a_{n,\uparrow}a_{k,\uparrow}a_{t,\uparrow}a_{r,\uparrow}a_{q,\uparrow}a_{p,\uparrow}a^+_{p,\uparrow}a^+_{q,\uparrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{k,\uparrow}a^+_{n,\uparrow}\varphi_0, \varphi_0\right) \\ &= \sum_{p,q,r,t,k,n}\left\|\left(\bar{H}^o_3-\lambda\right)F_i(p,q,r,t,k,n)\right\|^2 \times (\varphi_0,\varphi_0) \\ &= \sum_{p,q,r,t,k,n}\left\|\left(\bar{H}^o_3-\lambda\right)F_i(p,q,r,t,k,n)\right\|^2 \to 0,\end{aligned}$$

as $i \to \infty$, where

$F_i=\sum_{p,q,r,t,k,n} f_i(p,q,r,t,k,n)$. It follows that $\lambda\in\sigma(\bar{H}_3^o)$. Consequently, $\sigma(H_3^o)\subset\sigma(\bar{H}_3^o)$.

Conversely, let $\bar{\lambda}\in\sigma(\bar{H}_3^o)$. Then, by the Weyl criterion, there is a sequence $\{F_i\}_{i=1}^{\infty}$ such that $\|F_i\|=1$ and $\lim_{i\to\infty}\|(\bar{H}_3^o-\bar{\lambda})\psi_i\|=0$. Setting $F_i=\sum_{p,q,r,t,k,n} f_i(p,q,r,t,k,n)$, $\|F_i\|=\left(\sum_{p,q,r,t,k,n}|f_i(p,q,r,t,k,n)|^2\right)^{\frac{1}{2}}$, we conclude that $\|\psi_i\|=\|F_i\|=1$ and $\|(\bar{H}_3^o-\bar{\lambda})F_i\|=\|(\bar{H}_3^o-\bar{\lambda})\psi_i\|\to 0$ as $i\to\infty$. This means that $\bar{\lambda}\in\sigma(H_3^o)$ and hence $\sigma(\bar{H}_3^o)\subset\sigma(H_3^o)$. These two relations imply $\sigma(H_3^o)=\sigma(\bar{H}_3^o)$. □

We call the operator H_3^o the six-electron octet state operator in the Hubbard model.

Let $\mathcal{F}: l_2((Z^\nu)^6)\to L_2((T^\nu)^6)\equiv\tilde{\mathcal{H}}_3^o$ be the Fourier transform, where T^ν is the ν-dimensional torus endowed with the normalized Lebesgue measure $d\lambda$, *i.e.* $\lambda(T^\nu)=1$.

We set $\tilde{H}_3^o=\mathcal{F}\bar{H}_3^o\mathcal{F}^{-1}$. In the quasimomentum representation, the operator $\bar{H}_3^o$ acts in the Hilbert space $L_2^{as}((T^\nu)^6)$, where L_2^{as} is the subspace of antisymmetric functions in $L_2((T^\nu)^6)$.

Theorem 2. *The Fourier transform of operator* $\bar{H}_3^o$ *is an operator* $\tilde{H}_3^o=\mathcal{F}\bar{H}_3^o\mathcal{F}^{-1}$ *acting in the space* $L_2^{as}((T^\nu)^6)$ *be the formula*

$$\tilde{H}_3^o\psi_3^o=h(\lambda,\mu,\gamma,\theta,\eta,\chi)f(\lambda,\mu,\gamma,\theta,\eta,\chi),$$

where

$$h(\lambda,\mu,\gamma,\theta,\eta,\chi)=6A+2B\sum_{i=1}^{\nu}\left[\cos(\lambda_i)+\cos(\mu_i)+\cos(\gamma_i)+\cos(\theta_i)+\cos(\eta_i)+\cos(\chi_i)\right],$$

and L_2^{as} is the subspace of antisymmetric functions in $L_2((T^\nu)^6)$.

Proof. The proof is by direct calculation in which we use the Fourier transformation in formula (3). □

The spectrum of operator $\tilde{H}_3^o$ is a purely continuous and coincide with the interval $[6A-12B\nu, 6A+12B\nu]$.

4. Structure of the Essential Spectrum and Discrete Spectrum of Operator First Quintet State ${}^1\tilde{H}_2^q$

The first quintet state corresponding to the free motion of six electrons over lattice and their interactions with the basic functions

$$ {}^1q^2_{p,r,t,s,k,n\in Z^\nu}=a^+_{p,\downarrow}a^+_{r,\uparrow}a^+_{t,\uparrow}a^+_{s,\uparrow}a^+_{k,\uparrow}a^+_{n,\uparrow}\varphi_0. \quad (5)$$

The subspace ${}^1\mathcal{H}_2^q$, corresponding to the first quintet state is the set of all vectors of the form

$$ {}^1\psi_2^q = \sum_{p,r,s,t,k,n\in T^\nu} f(p,r,s,t,k,n)\, {}^1q^2_{p,r,t,s,k,n\in Z^\nu}, \quad f\in l_2^{as}, \tag{6} $$

where l_2^{as} is the subspace of antisymmetric functions in the space $l_2\left(\left(Z^\nu\right)^6\right)$. We denote by ${}^1H_2^q$ the restriction of operator H to the subspace ${}^1\mathcal{H}_2^q$.

Theorem 3. *The subspace* ${}^1\mathcal{H}_2^q$ *is invariant under the operator H, and the restriction* ${}^1H_2^q$ *of operator H to the subspace* ${}^1\mathcal{H}_2^q$ *is a bounded self-adjoint operator. It generates a bounded self-adjoint operator* ${}^1\bar{H}_2^q$ *acting in the space* l_2^{as} *as*

$$ \begin{aligned} {}^1\bar{H}_2^q\, {}^1\psi_2^q = {} & 6Af(p,r,s,t,k,n) + B\sum_\tau \big[f(p+\tau,r,s,t,k,n) + f(p,r+\tau,s,t,k,n) \\ & + f(p,r,s+\tau,t,k,n) + f(p,r,s,t+\tau,k,n) \\ & + f(p,r,s,t,k+\tau,n) + f(p,r,s,t,k,n+\tau)\big] \\ & + U\left[\delta_{p,q}+\delta_{p,r}+\delta_{p,t}+\delta_{p,k}+\delta_{p,n}\right] f(p,r,s,t,k,n). \end{aligned} \tag{7} $$

The operator ${}^1H_2^q$ acts on a vector ${}^1\psi_2^q \in {}^1\mathcal{H}_2^q$ as

$$ {}^1H_2^q\, {}^1\psi_2^q = \sum_{p,r,s,t,k,n\in Z^\nu} \left({}^1\bar{H}_2^q f\right)(p,r,s,t,k,n)\, {}^1q^2_{p,r,s,t,k,n\in Z^\nu}. \tag{8} $$

Lemma 2. *The spectra of the operators* ${}^1H_2^q$ *and* ${}^1\bar{H}_2^q$ *coincide.*

The proof of the lemma 2 similarly to proof the lemma 1.

We set ${}^1\tilde{H}_2^q = \mathcal{F}\, {}^1\bar{H}_2^q \mathcal{F}^{-1}$. In the quasimomentum representation, the operator ${}^1\tilde{H}_2^q$ acts in the Hilbert space $L_2^{as}\left(\left(T^\nu\right)^6\right)$, where L_2^{as} is the subspace of antisymmetric functions in $L_2\left(\left(T^\nu\right)^6\right)$.

Theorem 4. *The Fourier transform of operator* ${}^1\bar{H}_2^q$ *is an operator* ${}^1\tilde{H}_2^q = \mathcal{F}\, {}^1\bar{H}_2^q \mathcal{F}^{-1}$, *acting in the space* $L_2^{as}\left(\left(T^\nu\right)^6\right)$ *be the formula*

$$ \begin{aligned} {}^1\tilde{H}_2^q\, {}^1\psi_2^q = {} & h(\lambda,\mu,\gamma,\theta,\eta,\chi) f(\lambda,\mu,\gamma,\theta,\eta,\chi) + U\int_{T^\nu}\big[f(s,\lambda+\mu-s,\gamma,\theta,\eta,\chi) \\ & + f(s,\mu,\lambda+\gamma-s,\theta,\eta,\chi) + f(s,\mu,\gamma,\lambda+\theta-s,\eta,\chi) \\ & + f(s,\mu,\gamma,\theta,\lambda+\eta-s,\chi) + f(s,\mu,\gamma,\theta,\eta,\lambda+\chi-s)\big]\mathrm{d}s, \end{aligned} \tag{9} $$

where

$$ h(\lambda,\mu,\gamma,\theta,\eta,\chi) = 6A + 2B\sum_{i=1}^{\nu}\left[\cos(\lambda_i)+\cos(\mu_i)+\cos(\gamma_i)+\cos(\theta_i)+\cos(\eta_i)+\cos(\chi_i)\right], $$

and L_2^{as} is the subspace of antisymmetric functions in $L_2\left(\left(T^\nu\right)^6\right)$.

Proof. The proof is by direct calculation in which we use the Fourier transformation in formula (7). □

Using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces [21], and taking into account that the function $f(\lambda,\mu,\gamma,\theta,\eta,\chi)$ is an antisymmetric function, we can verify that the operator ${}^1\tilde{H}_2^q$ can be represented in the form

$$^{1}\tilde{H}_2^q = \tilde{H}_{2\Lambda_1}^1 \otimes I \otimes I + I \otimes \tilde{H}_{2\Lambda_2}^2 \otimes I + I \otimes I \otimes \tilde{H}_{2\Lambda_3}^3, \qquad (10)$$

where

$$\left(\tilde{H}_{2\Lambda_1}^1 f_{\Lambda_1}\right)(\lambda) = \left\{2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos\left(\frac{\Lambda_1^i}{2} - \lambda_i\right)\right\} f_{\Lambda_1}(\lambda) + U\int_{T^\nu} f_{\Lambda_1}(s)\,\mathrm{d}s, \quad (11)$$

here $\Lambda_1 = \lambda + \mu$,

$$\left(\tilde{H}_{2\Lambda_2}^2 f_{\Lambda_2}\right)(\gamma) = \left\{2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}\cos\left(\frac{\Lambda_2^i}{2} - \gamma_i\right)\right\} f_{\Lambda_2}(\gamma) - 2U\int_{T^\nu} f_{\Lambda_2}(s)\,\mathrm{d}s, \quad (12)$$

here $\Lambda_2 = \gamma + \theta$,

$$\left(\tilde{H}_{2\Lambda_3}^3 f_{\Lambda_3}\right)(\eta) = \left\{2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2}\cos\left(\frac{\Lambda_3^i}{2} - \eta_i\right)\right\} f_{\Lambda_3}(\eta) + 2U\int_{T^\nu} f_{\Lambda_3}(s)\,\mathrm{d}s, \quad (13)$$

here $\Lambda_3 = \eta + \chi$, and I is the unit operator in space $\tilde{\mathcal{H}}_2$.

Consequently, we must investigate the spectrum of the operators $\tilde{H}_{2\Lambda_1}^1$, $\tilde{H}_{2\Lambda_2}^2$, and $\tilde{H}_{2\Lambda_3}^3$.

Let the total quasimomentum of the two-electron system $\lambda + \mu = \Lambda_1$ be fixed. We let $L_2(\Gamma_{\Lambda_1})$ denote the space of functions that are square integrable on the manifold $\Gamma_{\Lambda_1} = \{(\lambda,\mu) : \lambda + \mu = \Lambda_1\}$. It is known [22] that the operator $\tilde{H}_2^1$ and the space $\tilde{\mathcal{H}}_2^1 \equiv L_2\left(\left(T^\nu\right)^2\right)$ can be decomposed into a direct integral $\tilde{H}_2^1 = \oplus\int_{T^\nu} \tilde{H}_{2\Lambda_1}^1 \mathrm{d}\Lambda_1$, $\tilde{\mathcal{H}}_2^1 = \oplus\int_{T^\nu} \tilde{\mathcal{H}}_{2\Lambda_1}^1 \mathrm{d}\Lambda_1$ of operators $\tilde{H}_{2\Lambda_1}^1$ and spaces $\tilde{\mathcal{H}}_{2\Lambda_1}^1 = L_2(\Gamma_{\Lambda_1})$, such that the spaces $\tilde{\mathcal{H}}_{2\Lambda_1}^1$ are invariant under the operators $\tilde{H}_{2\Lambda_1}^1$ and each operator $\tilde{H}_{2\Lambda_1}^1$ acts in $\tilde{\mathcal{H}}_{2\Lambda_1}^1$ according to the formula

$$\left(\tilde{H}_{2\Lambda_1}^1 f_{\Lambda_1}\right)(\lambda) = \left\{2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos\left(\frac{\Lambda_1^i}{2} - \lambda_i\right)\right\} f_{\Lambda_1}(\lambda) + U\int_{T^\nu} f_{\Lambda_1}(s)\,\mathrm{d}s,$$

where $f_{\Lambda_1}(x) = f(x, \Lambda_1 - x)$. Therefore, the function $f(\lambda,\mu,\gamma,\theta,\eta,\chi)$ is an antisymmetric function of parameters $\lambda,\mu,\gamma,\theta,\eta,\chi$, and all parameters $\lambda,\mu,\gamma,\theta,\eta,\chi$ a changed in the ν-dimensional torus T^ν, because all integral's $\int_{T^\nu} f_{\Lambda_i}(s)\,\mathrm{d}s, i = \overline{1,12}$ are equal.

First, we investigate the spectrum of the operator $\tilde{H}_{2\Lambda_1}^1$.

It is known that the continuous spectrum of $\tilde{H}_{2\Lambda_1}^1$ is independent of the parameter U and consists of the intervals

$$\sigma_{cont}\left(\tilde{H}_{2\Lambda_1}^1\right) = G_{\Lambda_1}^\nu = \left[m_{\Lambda_1}^\nu, M_{\Lambda_1}^\nu\right] = \left[2A - 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}, 2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\right].$$

Definition 1. *The eigenfunction* $\varphi_{\Lambda_1} \in L_2(T^\nu \times T^\nu)$ *of the operator* $\tilde{H}_{2\Lambda_1}^1$ *corresponding to an eigenvalue* $z_{\Lambda_1} \notin G_{\Lambda_1}^\nu$ *is called a bound state* (*BS*) (*anti-bound state* (*ABS*)) *of* $\tilde{H}_2^1$ *with the quasi momentum* Λ_1, *and the quantity* z_{Λ_1} *is called the energy of this state.*

We consider the operator K_{Λ_1} acting the space $\tilde{\mathcal{H}}_{2\Lambda_1}^1$ according to the formula

$$\left(K_{\Lambda_1}(z) f_{\Lambda_1}\right)(x) = \int_{T^\nu} \frac{U}{2A + 4B\sum_{i=1}^{\nu} \cos\frac{\Lambda_1^i}{2} \cos\left(\frac{\Lambda_1^i}{2} - t_i\right) - z} f_{\Lambda_1}(t) \mathrm{d}t. \quad (14)$$

It is a completely continuous operator in $\tilde{H}_{2\Lambda_1}^1$ for $z \notin G_{\Lambda_1}^\nu$.

We set

$$D_{\Lambda_1}^\nu(z) = 1 + U\int_{T^\nu} \frac{\mathrm{d}s_1 \mathrm{d}s_2 \cdots \mathrm{d}s_\nu}{2A + 4B\sum_{i=1}^{\nu} \cos\frac{\Lambda_1^i}{2} \cos\left(\frac{\Lambda_1^i}{2} - s_i\right) - z}. \quad (15)$$

Lemma 3. *A number* $z_0 \notin G_{\Lambda_1}^\nu$ *is an eigenvalue of the operator* $\tilde{H}_{2\Lambda_1}^1$ *if and only if it is a zero of the function* $D_{\Lambda_1}^\nu(z)$, *i.e.,* $D_{\Lambda_1}^\nu(z_0) = 0$.

Proof. Let the number $z = z_0 \notin G_{\Lambda_1}^\nu$ be an eigenvalue of the operator $\tilde{H}_{2\Lambda_1}^1$, and $\varphi_{\Lambda_1}(x)$ be the corresponding eigenfunction, *i.e.,*

$$\left\{2A + 4B\sum_{i=1}^{\nu} \cos\frac{\Lambda_1^i}{2} \cos\left(\frac{\Lambda_1^i}{2} - \lambda_i\right)\right\} \varphi_{\Lambda_1}(\lambda) + U\int_{T^\nu} \varphi_{\Lambda_1}(s) \mathrm{d}s = z_0 \varphi_{\Lambda_1}(\lambda). \quad (16)$$

Let $\psi_{\Lambda_1}(x) = \left[2A + 4B\sum_{i=1}^{\nu} \cos\frac{\Lambda_1^i}{2} \cos\left(\frac{\Lambda_1^i}{2} - \lambda_i\right) - z\right] \varphi_{\Lambda_1}(x)$. Then

$$\psi_{\Lambda_1}(x) + U\int_{T^\nu} \frac{1}{2A + 4B\sum_{i=1}^{\nu} \cos\frac{\Lambda_1^i}{2} \cos\left(\frac{\Lambda_1^i}{2} - \lambda_i\right) - z} \psi_{\Lambda_1}(s) \mathrm{d}s = 0, \quad (17)$$

i.e., the number $\mu = -1$ is an eigenvalue of the operator $K_{\Lambda_1}(z)$. It then follows that $D_{\Lambda_1}^\nu(z_0) = 0$.

Now let $z = z_0$ be a zero of the function $D_{\Lambda_1}^\nu(z)$, *i.e.,* $D_{\Lambda_1}^\nu(z_0) = 0$. It follows from the Fredholm theorem than the homogeneous equation

$$\psi_{\Lambda_1}(x) + U\int_{T^\nu} \frac{1}{2A + 4B\sum_{i=1}^{\nu} \cos\frac{\Lambda_1^i}{2} \cos\left(\frac{\Lambda_1^i}{2} - \lambda_i\right) - z} \psi_{\Lambda_1}(s) \mathrm{d}s = 0$$

has a nontrivial solution. This means that the number $z = z_0$ is an eigenvalue of the operator $\tilde{H}_{2\Lambda_1}^1$. □

We consider the one-dimensional case.

Theorem 5.

a) Let $\nu = 1$ and $U < 0$, then for all values of parameters of the Hamiltonian, the operator $\tilde{H}_{2\Lambda_1}^1$ has a unique eigenvalue $z_1 = 2A - \sqrt{U^2 + 16B^2 \cos^2\frac{\Lambda_1}{2}}$, that is below the continuous spectrum of the operator $\tilde{H}_{2\Lambda_1}^1$, *i.e.,* $\tilde{z}_1 < m_{\Lambda_1}^1$.

b) Let $\nu = 1$ and $U > 0$, then for all values of parameters of the Hamiltonian, the operator $\tilde{H}_{2\Lambda_1}^1$ has a unique eigenvalue $\tilde{z}_1 = 2A + \sqrt{U^2 + 16B^2 \cos^2\frac{\Lambda_1}{2}}$, that is above the continuous spectrum of the operator $\tilde{H}_{2\Lambda_1}^1$, *i.e.,* $\tilde{z}_1 > M_{\Lambda_1}^1$.

Proof. If $U < 0$, then in the one-dimensional case, the function $D_{\Lambda_1}^1(z)$ decreases monotonically outside the continuous spectrum domain of the operator

$\tilde{H}^1_{2\Lambda_1}$, *i.e.*, in the intervals $\left(-\infty, m^1_{\Lambda_1}\right)$ and $\left(M^1_{\Lambda_1}, +\infty\right)$. For $z < m^1_{\Lambda_1}$ the function $D^1_{\Lambda_1}(z)$ decreases from 1 to $-\infty$, $D^1_{\Lambda_1}(z) \to 1$ as $z \to -\infty$, $D^1_{\Lambda_1}(z) \to -\infty$ as $z \to m^1_{\Lambda_1} - 0$. Therefore, below the value $m^1_{\Lambda_1}$, the function $D^1_{\Lambda_1}(z)$ has a single zero at the point $z = z_1 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}} < m^1_{\Lambda_1}$. For $z > M^1_{\Lambda_1}$, and $U < 0$, the function $D^1_{\Lambda_1}(z)$ decreases from $+\infty$ to 1, $D^1_{\Lambda_1}(z) \to +\infty$ as $z \to M^1_{\Lambda_1} + 0$, $D^1_{\Lambda_1}(z) \to 1$ as $z \to +\infty$. Therefore, above the value $M^1_{\Lambda_1}$, the function $D^1_{\Lambda_1}(z)$ cannot vanish. If $U > 0$, and $z < m^1_{\Lambda_1}$, the function $D^1_{\Lambda_1}(z)$ increases from 1 to $+\infty$, $D^1_{\Lambda_1}(z) \to 1$ as $z \to -\infty$, $D^1_{\Lambda_1}(z) \to +\infty$ as $z \to m^1_{\Lambda_1} - 0$. Therefore, belove the value $m^1_{\Lambda_1}$, the function $D^1_{\Lambda_1}(z)$ cannot vanish. For $z > M^1_{\Lambda_1}$, and $U > 0$, the function $D^1_{\Lambda_1}(z)$ increases from $-\infty$ to 1, $D^1_{\Lambda_1}(z) \to 1$ as $z \to +\infty$, $D^1_{\Lambda_1}(z) \to -\infty$ as $z \to M^1_{\Lambda_1} + 0$. Therefore, above the value $M^1_{\Lambda_1}$, the function $D^1_{\Lambda_1}(z)$ vanishes on a single point $z = \tilde{z}_1 = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$. □

In two-dimensional case, we have analogously results. We now consider the three-dimensional case. Let $\nu = 3$, and $U < 0$. We denote

$$m_{\Lambda_1} = \int_{T^3} \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3}{\sum_{i=1}^3 \cos \frac{\Lambda_1^i}{2} \left(1 + \cos\left(\frac{\Lambda_1^i}{2} - s_i\right)\right)}, \text{ and}$$

$$M_{\Lambda_1} = \int_{T^3} \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3}{\sum_{i=1}^3 \cos \frac{\Lambda_1^i}{2} \left(1 - \cos\left(\frac{\Lambda_1^i}{2} - s_i\right)\right)}.$$

Theorem 6. *Let* $\nu = 3$.

a) If $U < 0$, and $U < -\frac{4B}{m_{\Lambda_1}}$, then the operator $\tilde{H}^1_{2\Lambda_1}$ has a unique eigenvalue z_1, that is below the continuous spectrum of $\tilde{H}^1_{2\Lambda_1}$, *i.e.*, $z_1 < m^3_{\Lambda_1}$.

b) If $U < 0$, and $-\frac{4B}{m_{\Lambda_1}} \le U < 0$, then the operator $\tilde{H}^1_{2\Lambda_1}$ has no eigenvalue, that is below the continuous spectrum of $\tilde{H}^1_{2\Lambda_1}$.

c) If $U > 0$, and $U > \frac{4B}{M_{\Lambda_1}}$, then the operator $\tilde{H}^1_{2\Lambda_1}$ has a unique eigenvalue $\tilde{z}_1$, that is above the continuous spectrum of $\tilde{H}^1_{2\Lambda_1}$, *i.e.*, $\tilde{z}_1 > M^3_{\Lambda_1}$.

d). If $U > 0$, and $0 < U \le \frac{4B}{M_{\Lambda_1}}$, then the operator $\tilde{H}^1_{2\Lambda_1}$ has no eigenvalue, that is above the continuous spectrum of $\tilde{H}^1_{2\Lambda_1}$.

Now, we consider the operator $\tilde{H}^2_{2\Lambda_2}$, and investigated the spectra this operator. We set

$$D_{\Lambda_2}^{\nu}(z)=1-2U\int_{T^{\nu}}\frac{\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_{\nu}}{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}\cos\left(\frac{\Lambda_2^i}{2}-s_i\right)-z}. \quad (18)$$

It is known the continuous spectrum of the operator $\tilde{H}_{2\Lambda_2}^2$, is consists of the segment $\sigma_{cont}\left(\tilde{H}_{2\Lambda_2}^2\right)=\left[2A-4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2},2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}\right]$.

Theorem 7. *a) Let* $\nu=1$ *and* $U<0$, *then for all values of parameters of the Hamiltonian, the operator* $\tilde{H}_{2\Lambda_2}^2$ *has a unique eigenvalue* $z_2=2A+2\sqrt{U^2+4B^2\cos^2\frac{\Lambda_2}{2}}$, *that is above the continuous spectrum of* $\tilde{H}_{2\Lambda_2}^2$, *i.e.,* $z_2>M_{\Lambda_2}^1$.

b) Let $\nu=1$ and $U>0$, then for all values of parameters of the Hamiltonian, the operator $\tilde{H}_{2\Lambda_2}^2$ has a unique eigenvalue $\tilde{z}_2=2A-2\sqrt{U^2+4B^2\cos^2\frac{\Lambda_2}{2}}$, that is below the continuous spectrum of $\tilde{H}_{2\Lambda_2}^2$, *i.e.,* $\tilde{z}_2<m_{\Lambda_2}^1$.

The proof of Theorem 7 are similarly the proof of Theorem 5.

We denote $m_{\Lambda_2}=\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{\sum_{i=1}^{3}\cos\frac{\Lambda_2^i}{2}\left(1+\cos\left(\frac{\Lambda_2^i}{2}-s_i\right)\right)}$, and

$$M_{\Lambda_2}=\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{\sum_{i=1}^{3}\cos\frac{\Lambda_2^i}{2}\left(1-\cos\left(\frac{\Lambda_2^i}{2}-s_i\right)\right)}.$$

Theorem 8. *Let* $\nu=3$.

a) If $U<0$, and $U<-\frac{2B}{M_{\Lambda_2}}$, then the operator $\tilde{H}_{2\Lambda_2}^2$ has a unique eigenvalue z_2, that is above the continuous spectrum of $\tilde{H}_{2\Lambda_2}^2$, *i.e.,* $z_2>M_{\Lambda_2}^3$.

b) If $U<0$, and $-\frac{2B}{M_{\Lambda_2}}\le U<0$, then the operator $\tilde{H}_{2\Lambda_2}^2$ has no eigenvalue, that is above the continuous spectrum of $\tilde{H}_{2\Lambda_2}^2$.

c) If $U>0$, and $U>\frac{2B}{m_{\Lambda_2}}$, then the operator $\tilde{H}_{2\Lambda_2}^2$ has a unique eigenvalue $\tilde{z}_2$, that is below the continuous spectrum of $\tilde{H}_{2\Lambda_2}^2$, *i.e.,* $\tilde{z}_2<m_{\Lambda_2}^3$.

d) If $U>0$, and $0<U\le\frac{2B}{m_{\Lambda_2}}$, then the operator $\tilde{H}_{2\Lambda_2}^2$ has no eigenvalue, that is below the continuous spectrum of $\tilde{H}_{2\Lambda_2}^2$.

Now, we consider the operator $\tilde{H}_{2\Lambda_3}^3$, and investigated the spectra this operator. We set

$$D_{\Lambda_3}^{\nu}(z)=1+2U\int_{T^{\nu}}\frac{\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_{\nu}}{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2}\cos\left(\frac{\Lambda_3^i}{2}-s_i\right)-z}. \quad (19)$$

It is known the continuous spectrum of the operator $\tilde{H}^3_{2\Lambda_3}$, is consists of the segment $\sigma_{cont}\left(\tilde{H}^3_{2\Lambda_3}\right)=\left[2A-4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2},2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2}\right]$.

Theorem 9. a) *Let* $\nu=1$ *and* $U<0$, *then for all values of parameters of the Hamiltonian, the operator* $\tilde{H}^3_{2\Lambda_3}$ *has a unique eigenvalue*

$z_3=2A-2\sqrt{U^2+4B^2\cos^2\frac{\Lambda_3}{2}}$, *that is below the continuous spectrum of* $\tilde{H}^3_{2\Lambda_3}$, *i.e.,* $z_3<m^1_{\Lambda_3}$.

b) Let $\nu=1$ and $U>0$, then for all values of parameters of the Hamiltonian, the operator $\tilde{H}^3_{2\Lambda_3}$ has a unique eigenvalue

$\tilde{z}_3=2A+2\sqrt{U^2+4B^2\cos^2\frac{\Lambda_3}{2}}$, that is above the continuous spectrum of $\tilde{H}^3_{2\Lambda_3}$, *i.e.,* $\tilde{z}_3>M^1_{\Lambda_3}$.

In the two-dimensional case, we have analogously results.

We consider the three-dimensional case.

We denote $m_{\Lambda_3}=\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{\sum_{i=1}^{3}\cos\frac{\Lambda_3^i}{2}\left(1+\cos\left(\frac{\Lambda_3^i}{2}-s_i\right)\right)}$, and

$M_{\Lambda_3}=\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{\sum_{i=1}^{3}\cos\frac{\Lambda_3^i}{2}\left(1-\cos\left(\frac{\Lambda_3^i}{2}-s_i\right)\right)}$.

Theorem 10. *Let* $\nu=3$.

a) If $U<0$, and $U<-\frac{2B}{m_{\Lambda_3}}$, then the operator $\tilde{H}^3_{2\Lambda_3}$ has a unique eigenvalue z_3, that is below the continuous spectrum of $\tilde{H}^3_{2\Lambda_3}$, *i.e.,* $z_3<m^3_{\Lambda_3}$.

b) If $U<0$, and $-\frac{2B}{m_{\Lambda_3}}\le U<0$, then the operator $\tilde{H}^3_{2\Lambda_3}$ has no eigenvalue, that is below the continuous spectrum of $\tilde{H}^3_{2\Lambda_3}$.

c) If $U>0$, and $U>\frac{2B}{M_{\Lambda_3}}$, then the operator $\tilde{H}^3_{2\Lambda_3}$ has a unique eigenvalue $\tilde{z}_3$, that is above the continuous spectrum of $\tilde{H}^3_{2\Lambda_3}$, *i.e.,* $\tilde{z}_3>M^3_{\Lambda_3}$.

d) If $U>0$, and $0<U\le\frac{2B}{M_{\Lambda_3}}$, then the operator $\tilde{H}^3_{2\Lambda_3}$ has no eigenvalue, that is above the continuous spectrum of $\tilde{H}^3_{2\Lambda_3}$.

Now, using the obtained results and representation (10), we describe the structure of essential spectrum and discrete spectrum of the energy operator of six electron systems in the Hubbard model in the first quintet state.

Theorem 11. *Let* $\nu=1$. *Then*

A) If $\nu=1$ and $U<0$, then the essential spectrum of the operator ${}^1\tilde{H}^q_2$ is consists of the union of seven segments:

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=[a+c+e,b+d+f]\cup[a+c+z_3,b+d+z_3]$$
$$\cup[a+e+z_2,b+f+z_2]\cup[a+z_2+z_3,b+z_2+z_3]$$
$$\cup[c+e+z_1,d+f+z_1]\cup[c+z_1+z_3,d+z_1+z_3]$$
$$\cup[e+z_1+z_2,f+z_1+z_2],$$

and the discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of no more one eigenvalue: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\{z_1+z_2+z_3\}$, or $\sigma_{disc}\left({}^1\tilde{H}_0^s\right)=\varnothing$. Here and hereafter $a=2A-4B\cos\frac{\Lambda_1}{2}$, $b=2A+4B\cos\frac{\Lambda_1}{2}$, $c=2A-4B\cos\frac{\Lambda_2}{2}$, $d=2A+4B\cos\frac{\Lambda_2}{2}$, $e=2A-4B\cos\frac{\Lambda_3}{2}$, $f=2A+4B\cos\frac{\Lambda_3}{2}$, and $z_1=2A-\sqrt{U^2+16B^2\cos^2\frac{\Lambda_1}{2}}$, $z_2=2A+2\sqrt{U^2+4B^2\cos^2\frac{\Lambda_2}{2}}$, $z_3=2A-2\sqrt{U^2+4B^2\cos^2\frac{\Lambda_3}{2}}$.

B) If $\nu=1$, and $U>0$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of seven segments:

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=[a+c+e,b+d+f]\cup[a+c+\tilde{z}_3,b+d+\tilde{z}_3]$$
$$\cup[a+e+\tilde{z}_2,b+f+\tilde{z}_2]\cup[a+\tilde{z}_2+\tilde{z}_3,b+\tilde{z}_2+\tilde{z}_3]$$
$$\cup[c+e+\tilde{z}_1,d+f+\tilde{z}_1]\cup[c+\tilde{z}_1+\tilde{z}_3,d+\tilde{z}_1+\tilde{z}_3]$$
$$\cup[e+\tilde{z}_1+\tilde{z}_2,f+\tilde{z}_1+\tilde{z}_2],$$

and the discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of no more one eigenvalue: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3\}$, or $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$. Here $\tilde{z}_1=2A+\sqrt{U^2+16B^2\cos^2\frac{\Lambda_1}{2}}$,

$$\tilde{z}_2=2A-2\sqrt{U^2+4B^2\cos^2\frac{\Lambda_2}{2}},\quad \tilde{z}_3=2A+2\sqrt{U^2+4B^2\cos^2\frac{\Lambda_3}{2}}.$$

Proof. A) It follows from representation (10), and from Theorems 5 and 7 and 9, that in one-dimensional case, if $U<0$, then the continuous spectrum of the operator $\tilde{H}_{2\Lambda_1}^1$ is consists from $\sigma_{cont}\left(\tilde{H}_{2\Lambda_1}^1\right)=\left[2A-4B\cos\frac{\Lambda_1}{2},2A+4B\cos\frac{\Lambda_1}{2}\right]$ and discrete spectrum of the operator $\tilde{H}_{2\Lambda_1}^1$ is consists of unique eigenvalue z_1, the continuous spectrum of the operator $\tilde{H}_{2\Lambda_2}^2$ is consists from $\sigma_{cont}\left(\tilde{H}_{2\Lambda_2}^2\right)=\left[2A-4B\cos\frac{\Lambda_2}{2},2A+4B\cos\frac{\Lambda_2}{2}\right]$, and discrete spectrum of the operator $\tilde{H}_{2\Lambda_2}^2$ is consists of unique eigenvalue z_2, and the continuous spectrum of the operator $\tilde{H}_{2\Lambda_3}^3$ is consists from

$\sigma_{cont}\left(\tilde{H}^3_{2\Lambda_3}\right)=\left[2A-4B\cos\frac{\Lambda_3}{2},2A+4B\cos\frac{\Lambda_3}{2}\right]$, and discrete spectrum of the operator $\tilde{H}^3_{2\Lambda_3}$ is consists of unique eigenvalue z_3. Therefore, the essential spectrum of the operator ${}^1\tilde{H}^q_2$ is consists from of the union of seven segments:

$$\begin{aligned}\sigma_{ess}\left({}^1\tilde{H}^q_2\right)=&\left[a+c+e,b+d+f\right]\cup\left[a+c+z_3,b+d+z_3\right]\\&\cup\left[a+e+z_2,b+f+z_2\right]\cup\left[a+z_2+z_3,b+z_2+z_3\right]\\&\cup\left[c+e+z_1,d+f+z_1\right]\cup\left[c+z_1+z_3,d+z_1+z_3\right]\\&\cup\left[e+z_1+z_2,f+z_1+z_2\right],\end{aligned}$$

and discrete spectrum of the operator ${}^1\tilde{H}^q_2$ is consists is the no more then one eigenvalues: $\sigma_{disc}\left({}^1\tilde{H}^q_2\right)=\left\{z_1+z_2+z_3\right\}$, or $\sigma_{disc}\left({}^1\tilde{H}^q_2\right)=\varnothing$. These is given to the proof of statement A) from Theorem 11.

The statements B) from Theorem 11. are proved similarly. □

In the two-dimensional case, we have the analogously results.

Now, we consider the three-dimensional case.

Theorem 12. *Let* $\nu=3$. *Then*

A) If $\nu=3$ and $U<0$, and $U<-\frac{4B}{m_{\Lambda_1}}$, and $M_{\Lambda_2}>\frac{1}{2}m_{\Lambda_1}$, and $M_{\Lambda_2}<m_{\Lambda_3}$, or $U<-\frac{2B}{M_{\Lambda_2}}$, and $M_{\Lambda_2}<\frac{1}{2}m_{\Lambda_1}$, and $m_{\Lambda_1}<2m_{\Lambda_3}$, or $U<-\frac{2B}{m_{\Lambda_3}}$, and $M_{\Lambda_2}>m_{\Lambda_3}$, and $M_{\Lambda_2}<\frac{1}{2}m_{\Lambda_1}$, then the essential spectrum of the operator ${}^1\tilde{H}^q_2$ is consists of the union of seven segments:

$$\begin{aligned}\sigma_{ess}\left({}^1\tilde{H}^q_2\right)=&\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3,b_1+d_1+z_3\right]\\&\cup\left[a_1+e_1+z_2,b_1+f_1+z_2\right]\cup\left[a_1+z_2+z_3,b_1+z_2+z_3\right]\\&\cup\left[c_1+e_1+z_1,d_1+f_1+z_1\right]\cup\left[c_1+z_1+z_3,d_1+z_1+z_3\right]\\&\cup\left[e_1+z_1+z_2,f_1+z_1+z_2\right],\end{aligned}$$

and the discrete spectrum of the operator ${}^1\tilde{H}^q_2$ is consists of no more one eigenvalue: $\sigma_{disc}\left({}^1\tilde{H}^q_2\right)=\left\{z_1+z_2+z_3\right\}$, or $\sigma_{disc}\left({}^1\tilde{H}^q_2\right)=\varnothing$. Here and hereafter $a_1=2A-4B\sum_{i=1}^{3}\cos\frac{\Lambda_1^i}{2}$, $b_1=2A+4B\sum_{i=1}^{3}\cos\frac{\Lambda_1^i}{2}$, $c_1=2A-4B\sum_{i=1}^{3}\cos\frac{\Lambda_2^i}{2}$, $d_1=2A+4B\sum_{i=1}^{3}\cos\frac{\Lambda_2^i}{2}$, $e_1=2A-4B\sum_{i=1}^{3}\cos\frac{\Lambda_3^i}{2}$, $f_1=2A+4B\sum_{i=1}^{3}\cos\frac{\Lambda_3^i}{2}$, and z_1, z_2, and z_3 are the eigenvalues, correspondingly, the operator's $\tilde{H}^1_{2\Lambda_1}$, $\tilde{H}^2_{2\Lambda_2}$ and $\tilde{H}^3_{2\Lambda_3}$.

B) If $\nu=3$, and $U<0$, $-\frac{4B}{m_{\Lambda_1}}\le U<-\frac{2B}{M_{\Lambda_2}}$, and $M_{\Lambda_2}<m_{\Lambda_3}$, or $-\frac{2B}{M_{\Lambda_2}}\le U<-\frac{4B}{m_{\Lambda_1}}$, and $m_{\Lambda_1}<2m_{\Lambda_3}$, or $-\frac{2B}{m_{\Lambda_3}}\le U<-\frac{2B}{M_{\Lambda_2}}$, and $m_{\Lambda_1}>2M_{\Lambda_2}$, then the essential spectrum of the operator ${}^1\tilde{H}^q_2$ is consists of the union of four segments:

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3,b_1+d_1+z_3\right]\cup\left[a_1+e_1+z_2,b_1+f_1+z_2\right]\cup\left[a_1+z_2+z_3,b_1+z_2+z_3\right],$$ or

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3,b_1+d_1+z_3\right]\cup\left[c_1+e_1+z_1,d_1+f_1+z_1\right]\cup\left[c_1+z_1+z_3,d_1+z_1+z_3\right],$$ or

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+e_1+z_2,b_1+f_1+z_2\right]\cup\left[c_1+e_1+z_1,d_1+f_1+z_1\right]\cup\left[e_1+z_1+z_2,f_1+z_1+z_2\right],$$ and the discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$.

C) If $\nu=3$, and $U<0$, and $-\frac{2B}{M_{\Lambda_2}}\le U<-\frac{2B}{m_{\Lambda_3}}$, and $m_{\Lambda_1}<2M_{\Lambda_2}$, and $m_{\Lambda_3}>M_{\Lambda_2}$, or $-\frac{4B}{m_{\Lambda_1}}\le U<-\frac{2B}{m_{\Lambda_3}}$, and $m_{\Lambda_1}<2m_{\Lambda_3}$, and $m_{\Lambda_1}>2M_{\Lambda_2}$, and or $-\frac{2B}{M_{\Lambda_2}}\le U<-\frac{4B}{m_{\Lambda_1}}$, and $M_{\Lambda_2}>m_{\Lambda_3}$, and $m_{\Lambda_1}>2M_{\Lambda_2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of two segments:

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3,b_1+d_1+z_3\right],$$

or

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+e_1+z_2,b_1+f_1+z_2\right],$$

or

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[c_1+e_1+z_1,d_1+f_1+z_1\right],$$

and the discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$.

D) If $\nu=3$, and $U<0$, and $-\frac{2B}{m_{\Lambda_3}}\le U<0$, and $m_{\Lambda_1}>2M_{\Lambda_2}$, and $m_{\Lambda_1}<2m_{\Lambda_3}$, or $U<0$, and $-\frac{4B}{m_{\Lambda_1}}\le U<0$, $M_{\Lambda_2}>m_{\Lambda_3}$, and $m_{\Lambda_1}>2M_{\Lambda_2}$, or $U<0$, and $-\frac{2B}{M_{\Lambda_2}}\le U<0$, $m_{\Lambda_1}<M_{\Lambda_2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of single segments:

$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]$, and the discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$.

Theorem 13. *Let* $\nu=3$, *then*

A) If $\nu=3$ and $U>0$, and $U>\frac{2B}{M_{\Lambda_3}}$, and $M_{\Lambda_1}>2m_{\Lambda_2}$, and $M_{\Lambda_3}<m_{\Lambda_2}$, or $M_{\Lambda_1}<2m_{\Lambda_2}$, and $M_{\Lambda_1}>2M_{\Lambda_3}$, or $U>0$, and $U>\frac{2B}{m_{\Lambda_2}}$, and $M_{\Lambda_1}>2M_{\Lambda_3}$, and $M_{\Lambda_3}>m_{\Lambda_2}$, or $M_{\Lambda_1}<2M_{\Lambda_3}$, and $M_{\Lambda_1}>2m_{\Lambda_2}$, or $U>0$, $U>\frac{4B}{M_{\Lambda_1}}$, and $m_{\Lambda_2}<M_{\Lambda_3}$, and $M_{\Lambda_1}<2m_{\Lambda_2}$, or $M_{\Lambda_3}<m_{\Lambda_2}$, and

$M_{\Lambda_1} < 2M_{\Lambda_3}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of seven segments:

$$\begin{aligned}\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = &[a_1+c_1+e_1, b_1+d_1+f_1]\cup[a_1+c_1+\tilde{z}_3, b_1+d_1+\tilde{z}_3]\\ &\cup[a_1+e_1+\tilde{z}_2, b_1+f_1+\tilde{z}_2]\cup[a_1+\tilde{z}_2+\tilde{z}_3, b_1+\tilde{z}_2+\tilde{z}_3]\\ &\cup[c_1+e_1+\tilde{z}_1, d_1+f_1+\tilde{z}_1]\cup[c_1+\tilde{z}_1+\tilde{z}_3, d_1+\tilde{z}_1+\tilde{z}_3]\\ &\cup[e_1+\tilde{z}_1+\tilde{z}_2, f_1+\tilde{z}_1+\tilde{z}_2],\end{aligned}$$

and the discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of no more one eigenvalue: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3\}$, or $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$. Here and hereafter $a_1 = 2A-4B\sum_{i=1}^3\cos\frac{\Lambda_1^i}{2}$, $b_1 = 2A+4B\sum_{i=1}^3\cos\frac{\Lambda_1^i}{2}$, $c_1 = 2A-4B\sum_{i=1}^3\cos\frac{\Lambda_2^i}{2}$, $d_1 = 2A+4B\sum_{i=1}^3\cos\frac{\Lambda_2^i}{2}$, $e_1 = 2A-4B\sum_{i=1}^3\cos\frac{\Lambda_3^i}{2}$, $f_1 = 2A+4B\sum_{i=1}^3\cos\frac{\Lambda_3^i}{2}$, and $\tilde{z}_1$, $\tilde{z}_2$, and $\tilde{z}_3$ are the eigenvalues, correspondingly, the operator's $\tilde{H}_{2\Lambda_1}^1$, $\tilde{H}_{2\Lambda_2}^2$ and $\tilde{H}_{2\Lambda_3}^3$.

B). If $\nu=3$, and $U>0$, $\frac{2B}{m_{\Lambda_2}}<U\le\frac{2B}{M_{\Lambda_3}}$, and $M_{\Lambda_3}<m_{\Lambda_2}$, and $M_{\Lambda_1}>2m_{\Lambda_2}$, or $U>0$, $\frac{4B}{M_{\Lambda_1}}<U\le\frac{2B}{M_{\Lambda_3}}$, and $M_{\Lambda_1}>2M_{\Lambda_3}$, and $M_{\Lambda_1}<2m_{\Lambda_2}$ or $U>0$, and $\frac{2B}{M_{\Lambda_3}}<U\le\frac{2B}{m_{\Lambda_2}}$, and $M_{\Lambda_1}>2M_{\Lambda_3}$, and $m_{\Lambda_2}<M_{\Lambda_3}$, or $U>0$, and $\frac{4B}{M_{\Lambda_1}}<U\le\frac{2B}{m_{\Lambda_2}}$, and $M_{\Lambda_1}>2m_{\Lambda_2}$, and $M_{\Lambda_1}<2M_{\Lambda_3}$, or $U>0$, and $\frac{2B}{m_{\Lambda_2}}<U\le\frac{4B}{M_{\Lambda_1}}$, and $m_{\Lambda_2}<M_{\Lambda_3}$, and $M_{\Lambda_1}<2m_{\Lambda_2}$, or $U>0$, $\frac{2B}{M_{\Lambda_3}}<U\le\frac{4B}{M_{\Lambda_1}}$, and $M_{\Lambda_1}<2M_{\Lambda_3}$, and $M_{\Lambda_3}<m_{\Lambda_2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of four segments:

$$\begin{aligned}\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = &[a_1+c_1+e_1, b_1+d_1+f_1]\cup[a_1+e_1+\tilde{z}_2, b_1+f_1+\tilde{z}_2]\\ &\cup[c_1+e_1+\tilde{z}_1, d_1+f_1+\tilde{z}_1]\cup[e_1+\tilde{z}_1+\tilde{z}_2, f_1+\tilde{z}_1+\tilde{z}_2],\end{aligned}$$

or

$$\begin{aligned}\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = &[a_1+c_1+e_1, b_1+d_1+f_1]\cup[a_1+c_1+\tilde{z}_3, b_1+d_1+\tilde{z}_3]\\ &\cup[a_1+e_1+\tilde{z}_2, b_1+f_1+\tilde{z}_2]\cup[a_1+\tilde{z}_2+\tilde{z}_3, b_1+\tilde{z}_2+\tilde{z}_3],\end{aligned}$$

or

$$\begin{aligned}\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = &[a_1+c_1+e_1, b_1+d_1+f_1]\cup[a_1+c_1+\tilde{z}_3, b_1+d_1+\tilde{z}_3]\\ &\cup[c_1+e_1+\tilde{z}_1, d_1+f_1+\tilde{z}_1]\cup[c_1+\tilde{z}_1+\tilde{z}_3, d_1+\tilde{z}_1+\tilde{z}_3],\end{aligned}$$

and the discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$.

C). If $\nu=3$, and $U>0$, and $\frac{4B}{M_{\Lambda_1}}<U\le\frac{2B}{m_{\Lambda_2}}$, $M_{\Lambda_1}>2m_{\Lambda_2}$, and

$M_{\Lambda_3} < m_{\Lambda_2}$ or $U > 0$, and $\frac{2B}{m_{\Lambda_2}} < U \le \frac{4B}{M_{\Lambda_1}}$, and $M_{\Lambda_1} < 2m_{\Lambda_2}$, and $M_{\Lambda_1} > 2M_{\Lambda_3}$, or $U > 0$, and $\frac{4B}{M_{\Lambda_1}} < U \le \frac{2B}{M_{\Lambda_3}}$, and $M_{\Lambda_1} > 2M_{\Lambda_3}$, and $m_{\Lambda_2} < M_{\Lambda_3}$, or $U > 0$ and $\frac{2B}{M_{\Lambda_3}} < U \le \frac{4B}{M_{\Lambda_1}}$, and $M_{\Lambda_1} < 2M_{\Lambda_2}$ and $M_{\Lambda_1} > 2m_{\Lambda_2}$, or $U > 0$ and $\frac{2B}{M_{\Lambda_3}} < U \le \frac{2B}{m_{\Lambda_2}}$, and $m_{\Lambda_2} < M_{\Lambda_3}$ and $M_{\Lambda_1} < m_{\Lambda_2}$ or $U > 0$ and $\frac{2B}{m_{\Lambda_2}} < U \le \frac{2B}{M_{\Lambda_3}}$, and $M_{\Lambda_3} < m_{\Lambda_2}$ and $M_{\Lambda_1} < 2M_{\Lambda_3}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of two segments:

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = \left[a_1 + c_1 + e_1, b_1 + d_1 + f_1\right] \cup \left[c_1 + e_1 + \tilde{z}_1, d_1 + f_1 + \tilde{z}_1\right],$$

or

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = \left[a_1 + c_1 + e_1, b_1 + d_1 + f_1\right] \cup \left[a_1 + c_1 + \tilde{z}_3, b_1 + d_1 + \tilde{z}_3\right],$$

or

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = \left[a_1 + c_1 + e_1, b_1 + d_1 + f_1\right] \cup \left[a_1 + e_1 + \tilde{z}_2, b_1 + f_1 + \tilde{z}_2\right],$$

and the discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right) = \varnothing$.

D) If $\nu = 3$, and $U > 0$, and $0 < U \le \frac{4B}{M_{\Lambda_1}}$, and $M_{\Lambda_1} > 2m_{\Lambda_2}$, and $M_{\Lambda_3} < m_{\Lambda_2}$, or $M_{\Lambda_3} > m_{\Lambda_2}$, or $U > 0$, and $0 < U \le \frac{2B}{m_{\Lambda_2}}$, and $M_{\Lambda_1} < 2m_{\Lambda_2}$, and $M_{\Lambda_1} > 2M_{\Lambda_3}$, or $M_{\Lambda_3} < m_{\Lambda_2}$, and $M_{\Lambda_1} < 2M_{\Lambda_3}$, or $U > 0$ and $0 < U \le \frac{2B}{M_{\Lambda_3}}$, and $M_{\Lambda_1} < 2M_{\Lambda_3}$, and $M_{\Lambda_1} > 2m_{\Lambda_2}$, or $m_{\Lambda_2} < M_{\Lambda_3}$ and $M_{\Lambda_1} < 2m_{\Lambda_2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of single segment: $\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = \left[a_1 + c_1 + e_1, b_1 + d_1 + f_1\right]$, and the discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right) = \varnothing$.

We now consider the three-dimensional case, while as $\Lambda_1 = \left(\Lambda_1^0, \Lambda_1^0, \Lambda_1^0\right)$. Then the continuous spectrum of the operator $\tilde{H}_{2\Lambda_1}^1$ is consists of the segment: $\sigma_{cont}\left(\tilde{H}_{2\Lambda_1}^1\right) = \left[2A - 12B\cos\frac{\Lambda_1^0}{2}, 2A + 12B\cos\frac{\Lambda_1^0}{2}\right]$. We consider the Watson integral [23] $W = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{3\mathrm{d}x\mathrm{d}y\mathrm{d}z}{3 - \cos x - \cos y - \cos z} \approx 1,516$.

Theorem 14. *Let* $\nu = 3$ *and* $\Lambda_1 = \left(\Lambda_1^0, \Lambda_1^0, \Lambda_1^0\right)$. *Then*

A) If $U < 0$, and $U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, then the operator $\tilde{H}_{2\Lambda_1}^1$ has a unique eigenvalue z_1 the below of the continuous spectrum of the operator $\tilde{H}^1_{2\Lambda_1}$. If $U<0$, and $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \le U < 0$. then the operator $\tilde{H}^1_{2\Lambda_1}$ has no eigenvalue of the below the continuous spectrum of the operator $\tilde{H}^1_{2\Lambda_1}$.

B) If $U>0$, and $U > \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, then the operator $\tilde{H}^1_{2\Lambda_1}$ has a unique eigenvalue $\tilde{z}_1$ the above of the continuous spectrum of the operator $\tilde{H}^1_{2\Lambda_1}$. If $U>0$, and $0 < U \le \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, then the operator $\tilde{H}^1_{2\Lambda_1}$ has no eigenvalue of the above the continuous spectrum of the operator $\tilde{H}^1_{2\Lambda_1}$.

We now consider the three-dimensional case, while as $\Lambda_2 = \left(\Lambda_2^0, \Lambda_2^0, \Lambda_2^0\right)$. Then the continuous spectrum of the operator $\tilde{H}^2_{2\Lambda_2}$ is consists of the segment:

$$\sigma_{cont}\left(\tilde{H}^2_{2\Lambda_2}\right) = \left[2A - 12B\cos\frac{\Lambda_2^0}{2}, 2A + 12B\cos\frac{\Lambda_2^0}{2}\right].$$

Theorem 15. *Let* $\nu = 3$ *and* $\Lambda_2 = \left(\Lambda_2^0, \Lambda_2^0, \Lambda_2^0\right)$. *Then*

A) If $U<0$, and $U < -\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, then the operator $\tilde{H}^2_{2\Lambda_2}$ has a unique eigenvalue z_2 the above of the continuous spectrum of the operator $\tilde{H}^2_{2\Lambda_2}$. If $U<0$, and $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < 0$, then the operator $\tilde{H}^2_{2\Lambda_2}$ has no eigenvalue of the below the continuous spectrum of the operator $\tilde{H}^2_{2\Lambda_2}$.

B) If $U>0$, and $U > \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, then the operator $\tilde{H}^2_{2\Lambda_2}$ has a unique eigenvalue $\tilde{z}_2$ the below of the continuous spectrum of the operator $\tilde{H}^2_{2\Lambda_2}$. If $U>0$, and $0 < U \le \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, then the operator $\tilde{H}^2_{2\Lambda_2}$ has no eigenvalue of the below the continuous spectrum of the operator $\tilde{H}^2_{2\Lambda_2}$.

We now consider the three-dimensional case, while as $\Lambda_3 = \left(\Lambda_3^0, \Lambda_3^0, \Lambda_3^0\right)$. Then the continuous spectrum of the operator $\tilde{H}^3_{2\Lambda_3}$ is consists of the segment:

$$\sigma_{cont}\left(\tilde{H}^3_{2\Lambda_3}\right) = \left[2A - 12B\cos\frac{\Lambda_3^0}{2}, 2A + 12B\cos\frac{\Lambda_3^0}{2}\right].$$

Theorem 16. *Let* $\nu = 3$ *and* $\Lambda_3 = \left(\Lambda_3^0, \Lambda_3^0, \Lambda_3^0\right)$. *Then*

A) If $U<0$, and $U < -\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, then the operator $\tilde{H}^3_{2\Lambda_3}$ has a unique eigenvalue z_3 the below of the continuous spectrum of the operator $\tilde{H}^3_{2\Lambda_3}$. If $U<0$, and $-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}\le U<0$, then the operator $\tilde{H}^3_{2\Lambda_3}$ has no eigenvalue of the below the continuous spectrum of the operator $\tilde{H}^3_{2\Lambda_3}$.

B) If $U>0$, and $U>\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, then the operator $\tilde{H}^3_{2\Lambda_3}$ has a unique eigenvalue $\tilde{z}_3$ the above of the continuous spectrum of the operator $\tilde{H}^3_{2\Lambda_3}$. If $U>0$, and $0<U\le\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, then the operator $\tilde{H}^3_{2\Lambda_3}$ has no eigenvalue of the above the continuous spectrum of the operator $\tilde{H}^3_{2\Lambda_3}$.

Theorem 17. *Let* $\nu=3$ *and* $\Lambda_1=(\Lambda_1^0,\Lambda_1^0,\Lambda_1^0)$, $\Lambda_2=(\Lambda_2^0,\Lambda_2^0,\Lambda_2^0)$, *and* $\Lambda_3=(\Lambda_3^0,\Lambda_3^0,\Lambda_3^0)$. *Then*

A) If $U<0$, and $U<-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, and $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, or $\cos\frac{\Lambda_2^0}{2}>\cos\frac{\Lambda_3^0}{2}$, or $U<0$, $U<-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_2^0}{2}>\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $U<0$, $U<-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of seven intervals:

$$\begin{aligned}\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=&[a_1+c_1+e_1,b_1+d_1+f_1]\cup[a_1+c_1+z_3,b_1+d_1+z_3]\\ &\cup[a_1+e_1+z_2,b_1+f_1+z_2]\cup[a_1+z_2+z_3,b_1+z_2+z_3]\\ &\cup[c_1+e_1+z_1,d_1+f_1+z_1]\cup[c_1+z_1+z_3,d_1+z_1+z_3]\\ &\cup[e_1+z_1+z_2,f_1+z_1+z_2],\end{aligned}$$

and discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of no more then one eigenvalue: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\{z_1+z_2+z_3\}$, or $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$. Here and hereafter $a_1=2A-12B\cos\frac{\Lambda_1^0}{2}$, $b_1=2A+12B\cos\frac{\Lambda_1^0}{2}$, $c_1=2A-12B\cos\frac{\Lambda_2^0}{2}$, $d_1=2A+12B\cos\frac{\Lambda_2^0}{2}$, $e_1=2A-12B\cos\frac{\Lambda_3^0}{2}$, $f_1=2A+12B\cos\frac{\Lambda_3^0}{2}$, and z_1, z_2, and z_3 are the eigenvalues of the operators $\tilde{H}^1_{2\Lambda_1},\tilde{H}^2_{2\Lambda_2},\tilde{H}^3_{2\Lambda_3}$, correspondingly.

B) If $U<0$, $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}\le U<-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, and

$\cos\frac{\Lambda_2^0}{2} > \cos\frac{\Lambda_3^0}{2}$, or $U<0$, $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2} < \cos\frac{\Lambda_3^0}{2}$, or $U<0$, $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2} > \cos\frac{\Lambda_3^0}{2}$, or $U<0$, $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, and $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $U<0$, $-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W} \le U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $U<0$, $-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_2^0}{2} < \cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of four intervals: $\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+c_1+z_3, b_1+d_1+z_3] \cup [a_1+e_1+z_2, b_1+f_1+z_2] \cup [a_1+z_2+z_3, b_1+z_2+z_3]$, or $\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+c_1+z_3, b_1+d_1+z_3] \cup [c_1+e_1+z_1, d_1+f_1+z_1] \cup [c_1+z_1+z_3, d_1+z_1+z_3]$, or $\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+e_1+z_2, b_1+f_1+z_2] \cup [c_1+e_1+z_1, d_1+f_1+z_1] \cup [e_1+z_1+z_2, f_1+z_1+z_2]$, and discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right) = \varnothing$.

C) If $U<0$, $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, and $\cos\frac{\Lambda_2^0}{2} > \cos\frac{\Lambda_3^0}{2}$, $U<0$, $-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, and $\cos\frac{\Lambda_2^0}{2} < \cos\frac{\Lambda_3^0}{2}$, or $U<0$, $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} \le U < -\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $U<0$, $-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W} \le U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2} > \cos\frac{\Lambda_3^0}{2}$, or $U<0$, $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} \le U < -\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, and $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, and

$\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, or $U<0$, $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}\le U<-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, and $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of two intervals:

$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=[a_1+c_1+e_1,b_1+d_1+f_1]\cup[a_1+c_1+z_3,b_1+d_1+z_3]$, or

$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=[a_1+c_1+e_1,b_1+d_1+f_1]\cup[a_1+e_1+z_2,b_1+f_1+z_2]$, or

$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=[a_1+c_1+e_1,b_1+d_1+f_1]\cup[c_1+e_1+z_1,d_1+f_1+z_1]$, and discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$.

D) If $U<0$, $-\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}\le U<0$, $\cos\frac{\Lambda_2^0}{2}>\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $U<0$, $-\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}\le U<0$, and $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $U<0$, $-\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}\le U<0$, $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2}>\cos\frac{\Lambda_3^0}{2}$, or $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of unique interval: $\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=[a_1+c_1+e_1,b_1+d_1+f_1]$, and discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$.

Theorem 18. *Let* $\nu=3$ *and* $\Lambda_1=\left(\Lambda_1^0,\Lambda_1^0,\Lambda_1^0\right)$, $\Lambda_2=\left(\Lambda_2^0,\Lambda_2^0,\Lambda_2^0\right)$, *and* $\Lambda_3=\left(\Lambda_3^0,\Lambda_3^0,\Lambda_3^0\right)$. *Then*

A) If $U>0$, $U>\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $U>0$, $U>\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_2^0}{2}>\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $U>0$, and $U>\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, or $\cos\frac{\Lambda_2^0}{2}>\cos\frac{\Lambda_3^0}{2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of seven intervals:

$$\begin{aligned}\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=&[a_1+c_1+e_1,b_1+d_1+f_1]\cup[a_1+c_1+\tilde{z}_3,b_1+d_1+\tilde{z}_3]\\&\cup[a_1+e_1+\tilde{z}_2,b_1+f_1+\tilde{z}_2]\cup[a_1+\tilde{z}_2+\tilde{z}_3,b_1+\tilde{z}_2+\tilde{z}_3]\\&\cup[c_1+e_1+\tilde{z}_1,d_1+f_1+\tilde{z}_1]\cup[c_1+\tilde{z}_1+\tilde{z}_3,d_1+\tilde{z}_1+\tilde{z}_3]\\&\cup[e_1+\tilde{z}_1+\tilde{z}_2,f_1+\tilde{z}_1+\tilde{z}_2],\end{aligned}$$

and discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of no more then one eigenvalue: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3\}$, or $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$. Here and hereafter $a_1=2A-12B\cos\frac{\Lambda_1^0}{2}$, $b_1=2A+12B\cos\frac{\Lambda_1^0}{2}$, $c_1=2A-12B\cos\frac{\Lambda_2^0}{2}$, $d_1=2A+12B\cos\frac{\Lambda_2^0}{2}$, $e_1=2A-12B\cos\frac{\Lambda_3^0}{2}$, $f_1=2A+12B\cos\frac{\Lambda_3^0}{2}$, and $\tilde{z}_1$, $\tilde{z}_2$, and $\tilde{z}_3$ are the eigenvalues of the operators $\tilde{H}_{2\Lambda_1}^1,\tilde{H}_{2\Lambda_2}^2,\tilde{H}_{2\Lambda_3}^3$, correspondingly.

B) If $U>0$, and $\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}<U\le\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$ and $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $U>0$, $\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}<U\le\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$ and $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $U>0$, $\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}<U\le\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_3^0}{2}<\cos\frac{\Lambda_2^0}{2}$ and $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $U>0$, $\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}<U\le\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, and $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, or $U>0$, $\frac{6B\cos\frac{\Lambda_3^0}{2}}{W}<U\le\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, or $U>0$, $\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}<U\le\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}>\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, and $\cos\frac{\Lambda_3^0}{2}<\cos\frac{\Lambda_2^0}{2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of four intervals:

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=[a_1+c_1+e_1,b_1+d_1+f_1]\cup[a_1+e_1+\tilde{z}_2,b_1+f_1+\tilde{z}_2]\cup[c_1+e_1+\tilde{z}_1,d_1+f_1+\tilde{z}_1]\cup[e_1+\tilde{z}_1+\tilde{z}_2,f_1+\tilde{z}_1+\tilde{z}_2],$$ or

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=[a_1+c_1+e_1,b_1+d_1+f_1]\cup[a_1+c_1+\tilde{z}_3,b_1+d_1+\tilde{z}_3]\cup[c_1+e_1+\tilde{z}_1,d_1+f_1+\tilde{z}_1]\cup[c_1+\tilde{z}_1+\tilde{z}_3,d_1+\tilde{z}_1+\tilde{z}_3],$$ or

$$\sigma_{ess}\left({}^1\tilde{H}_2^q\right)=[a_1+c_1+e_1,b_1+d_1+f_1]\cup[a_1+c_1+\tilde{z}_3,b_1+d_1+\tilde{z}_3]\cup[a_1+e_1+\tilde{z}_2,b_1+d_1+\tilde{z}_2]\cup[a_1+\tilde{z}_2+\tilde{z}_3,b_1+\tilde{z}_2+\tilde{z}_3],$$ and discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right)=\varnothing$.

C) If $U>0$, and $\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}<U\le\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}<\frac{1}{2}\cos\frac{\Lambda_2^0}{2}$ and $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$, or $U>0$, $\frac{6B\cos\frac{\Lambda_2^0}{2}}{W}<U\le\frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_2^0}{2}<\cos\frac{\Lambda_3^0}{2}$

and $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $U > 0$, $\frac{12B\cos\frac{\Lambda_1^0}{2}}{W} < U \le \frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$ and $\cos\frac{\Lambda_3^0}{2} < \cos\frac{\Lambda_2^0}{2}$, or $U > 0$, $\frac{6B\cos\frac{\Lambda_3^0}{2}}{W} < U \le \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2} > \cos\frac{\Lambda_3^0}{2}$, or $U > 0$, $\frac{6B\cos\frac{\Lambda_2^0}{2}}{W} < U \le \frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2} < \cos\frac{\Lambda_3^0}{2}$, or $U > 0$, $\frac{6B\cos\frac{\Lambda_3^0}{2}}{W} < U \le \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_3^0}{2} < \cos\frac{\Lambda_2^0}{2}$, and $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the union of two intervals:
$\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [c_1 + e_1 + \tilde{z}_1, d_1 + f_1 + \tilde{z}_1]$, or
$\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + \tilde{z}_2, b_1 + f_1 + \tilde{z}_2]$, or
$\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + \tilde{z}_3, b_1 + d_1 + \tilde{z}_3]$, and discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right) = \varnothing$.

D) If $U > 0$, and $0 < U \le \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$ and $\cos\frac{\Lambda_2^0}{2} < \cos\frac{\Lambda_3^0}{2}$, or $U > 0$, $0 < U \le \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$ and $\cos\frac{\Lambda_2^0}{2} < \cos\frac{\Lambda_3^0}{2}$, or $U > 0$, $0 < U \le \frac{12B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$ and $\cos\frac{\Lambda_3^0}{2} < \cos\frac{\Lambda_2^0}{2}$, or $U > 0$, $0 < U \le \frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_2^0}{2} > \cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_1^0}{2} < \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, or $U > 0$, $0 < U \le \frac{6B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_3^0}{2}$, and $\cos\frac{\Lambda_2^0}{2} < \cos\frac{\Lambda_3^0}{2}$, or $U > 0$, $0 < U \le \frac{6B\cos\frac{\Lambda_3^0}{2}}{W}$, $\cos\frac{\Lambda_3^0}{2} < \cos\frac{\Lambda_2^0}{2}$, and $\cos\frac{\Lambda_1^0}{2} > \frac{1}{2}\cos\frac{\Lambda_2^0}{2}$, then the essential spectrum of the operator ${}^1\tilde{H}_2^q$ is consists of the unique interval: $\sigma_{ess}\left({}^1\tilde{H}_2^q\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$, and discrete spectrum of the operator ${}^1\tilde{H}_2^q$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_2^q\right) = \varnothing$.

5. Six-Electron First Singlet State in the Hubbard Model

Let φ_0 be the vacuum vector in the space $\mathcal{H}_{as}$. The first singlet state corresponds to the free motion of six electrons over the lattice and their interactions with the basic functions ${}^1s^0_{p,q,r,t,k,n\in Z^\nu}=a^+_{p,\uparrow}a^+_{q,\uparrow}a^+_{r,\uparrow}a^+_{t,\downarrow}a^+_{k,\downarrow}a^+_{n,\downarrow}\varphi_0$. The subspace ${}^1\mathcal{H}^0_s$, corresponding to the first singlet state is the set of all vectors of the form ${}^1\psi^0_s=\sum_{p,q,r,t,k,n\in Z^\nu} f(p,q,r,t,k,n)\,{}^1s^0_{p,q,r,t,k,n\in Z^\nu}$, $f\in l_2^{as}$ where l_2^{as} is the subspace of antisymmetric functions in the space $l_2\left(\left(Z^\nu\right)^6\right)$.

Theorem 19. *The subspace* ${}^1\mathcal{H}^0_s$ *is invariant under the operator H, and the restriction* ${}^1H^0_s$ *of operator H to the subspace* ${}^1\mathcal{H}^0_s$ *is a bounded self-adjoint operator. It generates a bounded self-adjoint operator* ${}^1\bar{H}^0_s$ *acting in the space* l_2^{as} *as*

$$\begin{aligned}{}^1\bar{H}^0_s\,{}^1\psi^0_s &= 6Af(p,q,r,t,k,n)+B\sum_\tau\big[f(p+\tau,q,r,t,k,n)+f(p,q+\tau,r,t,k,n)\\&+f(p,q,r+\tau,t,k,n)+f(p,q,r,t+\tau,k,n)+f(p,q,r,t,k+\tau,n)\\&+f(p,q,r,t,k,n+\tau)\big]+U\big[\delta_{p,t}+\delta_{p,k}+\delta_{p,n}+\delta_{q,t}+\delta_{q,k}+\delta_{q,n}\\&+\delta_{r,t}+\delta_{r,k}+\delta_{r,n}\big]f(p,q,r,t,k,n).\end{aligned}\tag{20}$$

The operator ${}^1H^0_s$ acts on a vector ${}^1\psi^0_s\in{}^1\mathcal{H}^0_s$ as

$${}^1H^0_s\,{}^1\psi^0_s=\sum_{p,q,r,t,k,n\in Z^\nu}\left({}^1\bar{H}^0_s f\right)(p,q,r,t,k,n)\,{}^1s^0_{p,q,r,t,k,n\in Z^\nu}.$$

Lemma 4. *The spectra of the operators* ${}^1H^0_s$ *and* ${}^1\bar{H}^0_s$ *coincide.*

We call the operator ${}^1H^0_s$ the six-electron first singlet state operator.

Let $\mathcal{F}:l_2\left(\left(T^\nu\right)^6\right)\to L_2\left(\left(T^\nu\right)^6\right)\equiv{}^1\tilde{\mathcal{H}}^0_s$ be the Fourier transform, where T^ν is the ν-dimensional torus endowed with the normalized Lebesgue measure $d\lambda$, *i.e.* $\lambda\left(T^\nu\right)=1$.

We set ${}^1\tilde{H}^0_s=\mathcal{F}\,{}^1\bar{H}^0_s\mathcal{F}^{-1}$. In the quasimomentum representation, the operator ${}^1\bar{H}^0_s$ acts in the Hilbert space $L_2^{as}\left(\left(T^\nu\right)^6\right)$, where L_2^{as} is the subspace of antisymmetric functions in $L_2\left(\left(T^\nu\right)^6\right)$.

Theorem 20. *The Fourier transform of operator* ${}^1\bar{H}^0_s$ *is an bounded self-adjoint operator* ${}^1\tilde{H}^0_s=\mathcal{F}\,{}^1\bar{H}^0_s\mathcal{F}^{-1}$ *acting in the space* ${}^1\tilde{\mathcal{H}}^0_s$ *be the formula*

$$\begin{aligned}{}^1\tilde{H}^0_s\,{}^1\psi^0_s &= h(\lambda,\mu,\gamma,\theta,\eta,\xi)f(\lambda,\mu,\gamma,\theta,\eta,\xi)+U\int_{T^\nu}\big[f(s,\mu,\gamma,\lambda+\theta-s,\eta,\xi)\\&+f(s,\mu,\gamma,\theta,\lambda+\eta-s,\xi)+f(s,\mu,\gamma,\theta,\eta,\lambda+\xi-s)\\&+f(\lambda,s,\gamma,\mu+\theta-s,\eta,\xi)+f(\lambda,s,\gamma,\theta,\mu+\eta-s,\xi)\\&+f(\lambda,s,\gamma,\theta,\eta,\mu+\xi-s)+f(\lambda,\mu,s,\gamma+\theta-s,\eta,\xi)\\&+f(\lambda,\mu,s,\theta,\gamma+\eta-s,\xi)+f(\lambda,\mu,s,\theta,\eta,\gamma+\xi-s)\big]\mathrm{d}s,\end{aligned}\tag{21}$$

where

$h(\lambda,\mu,\gamma,\theta,\eta,\xi)=6A+2B\sum_{i=1}^{\nu}\left[\cos\lambda_i+\cos\mu_i+\cos\gamma_i+\cos\theta_i+\cos\eta_i+\cos\xi_i\right]$.

The proof Theorem 20, is straightforward of (20) using the Fourier transformation.

Taking into account that the function $f(\lambda,\mu,\gamma,\theta,\eta,\xi)$ is antisymmetric, and using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces, we can verify that the operator ${}^1\tilde{H}^0_s$ can be represented in the form

$$ {}^{1}\tilde{H}_{s}^{0}\,{}^{1}\psi_{s}^{0}=\tilde{H}_{2}^{1}(\lambda,\mu)\otimes I\otimes I+I\otimes\tilde{H}_{2}^{2}(\gamma,\theta)\otimes I+I\otimes I\otimes\tilde{H}_{2}^{3}(\eta,\xi) \tag{22}$$

where

$$\left(\tilde{H}_{2}^{1}f\right)(\lambda,\mu)=\left\{2A+2B\sum_{i=1}^{\nu}\left(\cos\lambda_{i}+\cos\mu_{i}\right)\right\}f(\lambda,\mu)+U\int_{T^{\nu}}f(s,\lambda+\mu-s)\mathrm{d}s$$
$$+U\int_{T^{\nu}}f(s,\lambda+\xi-s)\mathrm{d}s+U\int_{T^{\nu}}f(s,\gamma+\xi-s)\mathrm{d}s,$$

$$\left(\tilde{H}_{2}^{2}f\right)(\gamma,\theta)=\left\{2A+2B\sum_{i=1}^{\nu}\left(\cos\gamma_{i}+\cos\theta_{i}\right)\right\}f(\gamma,\theta)-U\int_{T^{\nu}}f(s,\lambda+\theta-s)\mathrm{d}s$$
$$-U\int_{T^{\nu}}f(s,\mu+\theta-s)\mathrm{d}s-U\int_{T^{\nu}}f(s,\mu+\xi-s)\mathrm{d}s,$$

$$\left(\tilde{H}_{2}^{3}f\right)(\eta,\xi)=\left\{2A+2B\sum_{i=1}^{\nu}\left[\cos\eta_{i}+\cos\xi_{i}\right]\right\}f(\eta,\xi)+U\int_{T^{\nu}}f(s,\mu+\eta-s)\mathrm{d}s$$
$$+U\int_{T^{\nu}}f(s,\gamma+\theta-s)\mathrm{d}s-U\int_{T^{\nu}}f(s,\gamma+\eta-s)\mathrm{d}s,$$

and I is the unit operator in the two-electron space $\tilde{\mathcal{H}}_2$.

Consequently, we must investigate the spectrum of the operators $\tilde{H}_2^1$, $\tilde{H}_2^2$, and $\tilde{H}_2^3$.

Let the total quasimomentum of the two-electron system $\lambda+\mu=\Lambda_1$ be fixed. We let $L_2\left(\Gamma_{\Lambda_1}\right)$ denote the space of functions that are square integrable on the manifold $\Gamma_{\Lambda_1}=\left\{(\lambda,\mu):\lambda+\mu=\Lambda_1\right\}$. It is known that the operator $\tilde{H}_2^1$ and the space $\tilde{\mathcal{H}}_2^1\equiv L_2\left(\left(T^{\nu}\right)^2\right)$ can be decomposed into a direct integral $\tilde{H}_2^1=\oplus\int_{T^{\nu}}\tilde{H}_{2\Lambda_1}^1\mathrm{d}\Lambda_1$, $\tilde{\mathcal{H}}_2^1=\oplus\int_{T^{\nu}}\tilde{\mathcal{H}}_{2\Lambda_1}^1\mathrm{d}\Lambda_1$ of operators $\tilde{H}_{2\Lambda_1}^1$ and spaces $\tilde{\mathcal{H}}_{2\Lambda_1}^1=L_2\left(\Gamma_{\Lambda_1}\right)$, such that the spaces $\tilde{\mathcal{H}}_{2\Lambda_1}^1$ are invariant under the operators $\tilde{H}_{2\Lambda_1}^1$ and each operator $\tilde{H}_{2\Lambda_1}^1$ acts in $\tilde{\mathcal{H}}_{2\Lambda_1}^1$ according to the formula

$$\left(\tilde{H}_{2\Lambda_1}^1 f_{\Lambda_1}\right)(\lambda)=\left\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos\left(\frac{\Lambda_1^i}{2}-\lambda_i\right)\right\}f_{\Lambda_1}(\lambda)+3U\int_{T^{\nu}}f_{\Lambda_1}(s)\mathrm{d}s,$$

where $f_{\Lambda_1}(x)=f\left(x,\Lambda_1-x\right)$.

It is known that the continuous spectrum of operator $\tilde{H}_{2\Lambda_1}^1$ is independent of the parameter U and consists of the intervals

$$\sigma_{cont}\left(\tilde{H}_{2\Lambda_1}^1\right)=G_{\Lambda_1}^{\nu}=\left[m_{\Lambda_1}^{\nu},M_{\Lambda_1}^{\nu}\right]=\left[2A-4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2},2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\right].$$

We set $D_{\Lambda_1}^{\nu}(z)=1+3U\int_{T^{\nu}}\dfrac{\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_{\nu}}{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_1^i}{2}\cos\left(\frac{\Lambda_1^i}{2}-s_i\right)-z}$.

Lemma 5. *A number* $z_0\notin G_{\Lambda_1}^{\nu}$ *is an eigenvalue of the operator* $\tilde{H}_{2\Lambda_1}^1$ *if and only if it is a zero of the function* $D_{\Lambda_1}^{\nu}(z)$, *i.e.,* $D_{\Lambda_1}^{\nu}(z_0)=0$.

The proof the Lemma 5 is analogously to the proof of Lemma 3 in the these work.

We consider the one-dimensional case.

Theorem 21. a) *If* $\nu = 1$ *and* $U < 0$, *then for all values of parameters of the Hamiltonian, the operator* $\tilde{H}^1_{2\Lambda_1}$ *has a unique eigenvalue*

$z_1 = 2A - \sqrt{9U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$, *that is below the continuous spectrum of* $\tilde{H}^1_{2\Lambda_1}$, *i.e.,* $z_1 < m^1_{\Lambda_1}$.

b) If $\nu = 1$ *and* $U > 0$, *then for all values of parameters of the Hamiltonian, the operator* $\tilde{H}^1_{2\Lambda_1}$ *has a unique eigenvalue* $\tilde{z}_1 = 2A + \sqrt{9U^2 + 16B^2\cos^2\frac{\Lambda_1}{2}}$, *that is above the continuous spectrum of* $\tilde{H}^1_{2\Lambda_1}$, *i.e.,* $\tilde{z}_1 > M^1_{\Lambda_1}$.

The proof of Theorem 21 are similarly the proof of Theorem 5.

In the two-dimensional case, we have analogous results.

We consider the Watson integral. Because the measure ν is normalized,

$$J = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{3-\cos x-\cos y-\cos z} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{3+\cos x+\cos y+\cos z} = \frac{W}{3}.$$

Let $\nu = 3$, and $\Lambda_1 = \left(\Lambda_1^0, \Lambda_1^0, \Lambda_1^0\right)$.

It is known that the continuous spectrum of operator $\tilde{H}^1_{2\Lambda_1}$ is independent of U and coincides with the segment

$$\sigma_{cont}\left(\tilde{H}^1_{2\Lambda_1}\right) = G^3_{\Lambda_1} = \left[2A - 12B\cos\frac{\Lambda_1^0}{2}, 2A + 12B\cos\frac{\Lambda_1^0}{2}\right].$$

Theorem 22. *a) Let* $\nu = 3$ *and* $U < 0$, *and the total quasimomentum* Λ_1 *of the system have the form* $\Lambda_1 = \left(\Lambda_1^0, \Lambda_1^0, \Lambda_1^0\right)$. *Then the operator* $\tilde{H}^1_{2\Lambda_1}$ *has a unique eigenvalue* z_1', *if* $U < -\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *that is below the continuous spectrum of operator* $\tilde{H}^1_{2\Lambda_1}$. *Otherwise, the operator* $\tilde{H}^1_{2\Lambda_1}$ *has no eigenvalue, that is below the continuous spectrum of operator* $\tilde{H}^1_{2\Lambda_1}$.

b) Let $\nu = 3$ *and* $U > 0$, *and the total quasimomentum* Λ_1 *of the system have the form* $\Lambda_1 = \left(\Lambda_1^0, \Lambda_1^0, \Lambda_1^0\right)$. *Then the operator* $\tilde{H}^1_{2\Lambda_1}$ *has a unique eigenvalue* z_1'', *if* $U > \frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *that is above the continuous spectrum of operator* $\tilde{H}^1_{2\Lambda_1}$. *Otherwise, the operator* $\tilde{H}^1_{2\Lambda_1}$ *has no eigenvalue, that is above the continuous spectrum of operator* $\tilde{H}^1_{2\Lambda_1}$.

We let $\Lambda_2 = \gamma + \theta$. We now investigated the spectrum of the operator $\tilde{H}^2_{2\Lambda_2}$, *i.e.*, the operator

$$\left(\tilde{H}^2_{2\Lambda_2} f_{\Lambda_2}\right)(\gamma) = \left\{2A + 4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}\cos\left(\frac{\Lambda_2^i}{2} - \gamma_i\right)\right\} f_{\Lambda_2}(\gamma) - 3U\int_{T^\nu} f_{\Lambda_2}(s)\mathrm{d}s.$$

It is known that the continuous spectrum of the operator $\tilde{H}^2_{2\Lambda_2}$ is independent of U and coincides with the segment

$$\sigma_{cont}\left(\tilde{H}^2_{2\Lambda_2}\right)=G^{\nu}_{\Lambda_2}=\left[m^{\nu}_{\Lambda_2},M^{\nu}_{\Lambda_2}\right]=\left[2A-4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2},2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}\right].$$

We set $D^{\nu}_{\Lambda_2}(z)=1-3U\int_{T^{\nu}}\frac{ds_1ds_2\cdots ds_{\nu}}{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_2^i}{2}\cos\left(\frac{\Lambda_2^i}{2}-s_i\right)-z}$.

Theorem 23. a) *If* $\nu=1$ *and* $U<0$, *then for all values of parameters of the Hamiltonian, the operator* $\tilde{H}^2_{2\Lambda_2}$ *has a unique eigenvalue*

$z_2=2A+\sqrt{9U^2+16B^2\cos^2\frac{\Lambda_2}{2}}$, *that is above the continuous spectrum of* $\tilde{H}^2_{2\Lambda_2}$, *i.e.*, $z_2>M^1_{\Lambda_2}$.

b) If $\nu=1$ *and* $U>0$, *then for all values of parameters of the Hamiltonian, the operator* $\tilde{H}^2_{2\Lambda_2}$ *has a unique eigenvalue* $\tilde{z}_2=2A-\sqrt{9U^2+16B^2\cos^2\frac{\Lambda_2}{2}}$, *that is below the continuous spectrum of* $\tilde{H}^2_{2\Lambda_2}$, *i.e.*, $\tilde{z}_2<m^1_{\Lambda_2}$.

Now, we consider three-dimensional case. Let $\nu=3$ and $\Lambda_2=\left(\Lambda_2^0,\Lambda_2^0,\Lambda_2^0\right)$.

It is known that the continuous spectrum of $\tilde{H}^2_{2\Lambda_2}$ is independent of U and coincides with the segment

$$\sigma_{cont}\left(\tilde{H}^2_{2\Lambda_2}\right)=G^3_{\Lambda_2}=\left[2A-12B\cos\frac{\Lambda_2^0}{2},2A+12B\cos\frac{\Lambda_2^0}{2}\right].$$

Theorem 24. *a). Let* $\nu=3$ *and* $U<0$, *and the total quasimomentum* Λ_2 *of the system have the form* $\Lambda_2=\left(\Lambda_2^0,\Lambda_2^0,\Lambda_2^0\right)$. *Then the operator* $\tilde{H}^2_{2\Lambda_2}$ *has a unique eigenvalue* z_2', *if* $U<-\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *that is above the continuous spectrum of operator* $\tilde{H}^2_{2\Lambda_2}$. *Otherwise, the operator* $\tilde{H}^2_{2\Lambda_2}$ *has no eigenvalue, that is above the continuous spectrum of operator* $\tilde{H}^2_{2\Lambda_2}$.

b). Let $\nu=3$ *and* $U>0$, *and the total quasimomentum* Λ_2 *of the system have the form* $\Lambda_2=\left(\Lambda_2^0,\Lambda_2^0,\Lambda_2^0\right)$. *Then the operator* $\tilde{H}^2_{2\Lambda_2}$ *has a unique eigenvalue* z_2'', *if* $U>\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *that is below the continuous spectrum of operator* $\tilde{H}^2_{2\Lambda_2}$. *Otherwise, the operator* $\tilde{H}^2_{2\Lambda_2}$ *has no eigenvalue, that is above the continuous spectrum of operator* $\tilde{H}^2_{2\Lambda_2}$.

Let $\Lambda_3=\eta+\xi$. We now investigated the spectra of operator $\tilde{H}^3_{2\Lambda_3}$, *i.e.*, the operator

$$\left(\tilde{H}^3_{2\Lambda_3}f_{\Lambda_3}\right)(\eta)=\left\{2A+4B\sum_{i=1}^{\nu}\cos\frac{\Lambda_3^i}{2}\cos\left(\frac{\Lambda_3^i}{2}-\eta_i\right)\right\}f_{\Lambda_3}(\eta)+U\int_{T^{\nu}}f_{\Lambda_3}(s)ds.$$

It is known that the continuous spectrum of the operator $\tilde{H}^3_{2\Lambda_3}$ is independent of U and coincides with the segment

$$\sigma_{cont}\left(\tilde{H}^3_{2\Lambda_3}\right)=G^{\nu}_{\Lambda_3}=\left[m^{\nu}_{\Lambda_3},M^{\nu}_{\Lambda_3}\right]=\left[2A-4B\sum\nolimits_{i=1}^{\nu}\cos\frac{\Lambda^i_3}{2},2A+4B\sum\nolimits_{i=1}^{\nu}\cos\frac{\Lambda^i_3}{2}\right].$$

Denote $D^{\nu}_{\Lambda_3}(z)=1+U\int_{T^{\nu}}\frac{\mathrm{d}s_1\mathrm{d}s_2\cdots\mathrm{d}s_{\nu}}{2A+4B\sum\nolimits_{i=1}^{\nu}\cos\frac{\Lambda^i_3}{2}\cos\left(\frac{\Lambda^i_3}{2}-s_i\right)-z}$.

Theorem 25. a). *If* $\nu=1$ *and* $U<0$, *and for all values of parameters of the Hamiltonian, the operator* $\tilde{H}^3_{2\Lambda_3}$ *has a unique eigenvalue* $z_3=2A-\sqrt{U^2+16B^2\cos^2\frac{\Lambda_3}{2}}$, *that is below the continuous spectrum of operator* $\tilde{H}^3_{2\Lambda_3}$, *i.e.,* $z_3<m^3_{\Lambda_3}$.

b) If $\nu=1$ *and* $U>0$, *and for all values of parameters of the Hamiltonian, the operator* $\tilde{H}^3_{2\Lambda_3}$ *has a unique eigenvalue* $\tilde{z}_3=2A+\sqrt{U^2+16B^2\cos^2\frac{\Lambda_3}{2}}$, *that is above the continuous spectrum of operator* $\tilde{H}^3_{2\Lambda_3}$, *i.e.,* $\tilde{z}_3>M^3_{\Lambda_3}$.

Let $\nu=3$ and $\Lambda_3=\left(\Lambda^0_3,\Lambda^0_3,\Lambda^0_3\right)$.

It is known that the continuous spectrum of $\tilde{H}^3_{2\Lambda_3}$ is independent of U and coincides with the segment

$$\sigma_{cont}\left(\tilde{H}^3_{2\Lambda_3}\right)=G^3_{\Lambda_3}=\left[2A-12B\cos\frac{\Lambda^0_3}{2},2A+12B\cos\frac{\Lambda^0_3}{2}\right].$$

Theorem 26. *a). Let* $\nu=3$ *and* $U<0$, *and the total quasimomentum* Λ_3 *of the system have the form* $\Lambda_3=\left(\Lambda^0_3,\Lambda^0_3,\Lambda^0_3\right)$. *Then the operator* $\tilde{H}^3_{2\Lambda_3}$ *has a unique eigenvalue* z'_3, *if* $U<-\frac{12B\cos\frac{\Lambda^0_3}{2}}{W}$, *that is below the continuous spectrum of operator* $\tilde{H}^3_{2\Lambda_3}$. *Otherwise, the operator* $\tilde{H}^3_{2\Lambda_3}$ *has no eigenvalue, that is below the continuous spectrum of operator* $\tilde{H}^3_{2\Lambda_3}$.

b) Let $\nu=3$ *and* $U>0$, *and the total quasimomentum* Λ_3 *of the system have the form* $\Lambda_3=\left(\Lambda^0_3,\Lambda^0_3,\Lambda^0_3\right)$. *Then the operator* $\tilde{H}^2_{2\Lambda_2}$ *has a unique eigenvalue* z''_3, *if* $U>\frac{12B\cos\frac{\Lambda^0_3}{2}}{W}$, *that is above the continuous spectrum of operator* $\tilde{H}^3_{2\Lambda_3}$. *Otherwise, the operator* $\tilde{H}^3_{2\Lambda_3}$ *has no eigenvalue, that is above the continuous spectrum of operator* $\tilde{H}^3_{2\Lambda_3}$.

We now using the obtaining results and the representation (22), we can describe the structure of essential spectrum and discrete spectrum of the operator of first six-electron singlet state operator ${}^1\tilde{H}^0_s$:

Theorem 27. *If* $\nu=1$ *and* $U<0$, *then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}^0_s$ *is exactly the union of seven segments*:

$$\sigma_{ess}\left({}^1\tilde{H}_s^0\right)=\left[a+c+e,b+d+f\right]\cup\left[a+c+z_3,b+d+z_3\right]\cup\left[a+e+z_2,b+f+z_2\right]\cup\left[a+z_2+z_3,b+z_2+z_3\right]\cup\left[c+e+z_1,d+f+z_1\right]\cup\left[c+z_1+z_3,d+z_1+z_3\right]\cup\left[e+z_1+z_2,f+z_1+z_3\right].$$

The discrete spectrum of operator ${}^1\tilde{H}_s^0$ consists of no more than one point: $\sigma_{disc}\left({}^1\tilde{H}_s^0\right)=\left\{z_1+z_2+z_3\right\}$, or $\sigma_{disc}\left({}^1\tilde{H}_s^0\right)=\varnothing$.

Here and hereafter $a=2A-4B\cos\frac{\Lambda_1}{2}$, $b=2A+4B\cos\frac{\Lambda_1}{2}$, $c=2A-4B\cos\frac{\Lambda_2}{2}$, $d=2A+4B\cos\frac{\Lambda_2}{2}$, $e=2A-4B\cos\frac{\Lambda_3}{2}$, $f=2A+4B\cos\frac{\Lambda_3}{2}$, $z_1=2A-\sqrt{9U^2+16B^2\cos^2\frac{\Lambda_1}{2}}$, $z_2=2A+\sqrt{9U^2+16B^2\cos^2\frac{\Lambda_2}{2}}$, $z_3=2A-\sqrt{U^2+16B^2\cos^2\frac{\Lambda_3}{2}}$.

The proof of Theorem 27 are similarly the proof of Theorem 11.

Theorem 28. *If* $\nu=1$ *and* $U>0$, *then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}_s^0$ *is exactly the union of seven segments*:

$$\sigma_{ess}\left({}^1\tilde{H}_s^0\right)=\left[a+c+e,b+d+f\right]\cup\left[a+c+\tilde{z}_3,b+d+\tilde{z}_3\right]\cup\left[a+e+\tilde{z}_2,b+f+\tilde{z}_2\right]\cup\left[a+\tilde{z}_2+\tilde{z}_3,b+\tilde{z}_2+\tilde{z}_3\right]\cup\left[c+e+\tilde{z}_1,d+f+\tilde{z}_1\right]\cup\left[c+\tilde{z}_1+\tilde{z}_3,d+\tilde{z}_1+\tilde{z}_3\right]\cup\left[e+\tilde{z}_1+\tilde{z}_2,f+\tilde{z}_1+\tilde{z}_2\right].$$

The discrete spectrum of operator ${}^1\tilde{H}_s^0$ is consists of no more than one point: $\sigma_{disc}\left({}^1\tilde{H}_s^0\right)=\left\{\tilde{z}_1+\tilde{z}_2+\tilde{z}_3\right\}$, or $\sigma_{disc}\left({}^1\tilde{H}_s^0\right)=\varnothing$.

Here $\tilde{z}_1=2A+\sqrt{9U^2+16B^2\cos^2\frac{\Lambda_1}{2}}$, $\tilde{z}_2=2A-\sqrt{9U^2+16B^2\cos^2\frac{\Lambda_2}{2}}$,

$$\tilde{z}_3=2A+\sqrt{U^2+16B^2\cos^2\frac{\Lambda_3}{2}}.$$

Let $\nu=3$, and $\Lambda_1=\left(\Lambda_1^0,\Lambda_1^0,\Lambda_1^0\right)$, and $\Lambda_2=\left(\Lambda_2^0,\Lambda_2^0,\Lambda_2^0\right)$, and $\Lambda_3=\left(\Lambda_3^0,\Lambda_3^0,\Lambda_3^0\right)$.

Now, using the obtained results and representation (22), we describe the structure of the essential spectrum and the discrete spectrum of the system first six-electron singlet state operator ${}^1\tilde{H}_s^0$:

Let $\nu=3$ and $\Lambda_1=\left(\Lambda_1^0,\Lambda_1^0,\Lambda_1^0\right)$, and $\Lambda_2=\left(\Lambda_2^0,\Lambda_2^0,\Lambda_2^0\right)$, and $\Lambda_3=\left(\Lambda_3^0,\Lambda_3^0,\Lambda_3^0\right)$.

Theorem 29. *The following statements hold*:

a) Let $U<0$, *and* $U<-\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2}>\cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2}>3\cos\frac{\Lambda_3^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2}>3\cos\frac{\Lambda_3^0}{2}$, *or* $U<0$, *and* $U<-\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$,

$\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *or* $U < 0$, *and* $U < -\frac{12B\cos\frac{\Lambda_2^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$. *Then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}_s^0$ *is consists of the union of seven segments*:

$$\sigma_{ess}\left({}^1\tilde{H}_s^0\right) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+c_1+z_3', b_1+d_1+z_3']$$
$$\cup [a_1+e_1+z_2', b_1+f_1+z_2'] \cup [a_1+z_2'+z_3', b_1+z_2'+z_3']$$
$$\cup [c_1+e_1+z_1', d_1+f_1+z_1'] \cup [c_1+z_1'+z_3', d_1+z_1'+z_3']$$
$$\cup [e_1+z_1'+z_2', f_1+z_1'+z_2'].$$

The discrete spectrum of the operator ${}^1\tilde{H}_s^0$ is consists of no more one point: $\sigma_{disc}\left({}^1\tilde{H}_s^0\right) = \{z_1'+z_2'+z_3'\}$, or $\sigma_{disc}\left({}^1\tilde{H}_s^0\right) = \varnothing$.

Here and hereafter $a_1 = 2A - 12B\cos\frac{\Lambda_1^0}{2}$, $b_1 = 2A + 12B\cos\frac{\Lambda_1^0}{2}$, $c_1 = 2A - 12B\cos\frac{\Lambda_2^0}{2}$, $d_1 = 2A + 12B\cos\frac{\Lambda_2^0}{2}$, $e_1 = 2A - 12B\cos\frac{\Lambda_3^0}{2}$, $f_1 = 2A + 12B\cos\frac{\Lambda_3^0}{2}$, and z_1' is an eigenvalue of the operator $\tilde{H}_{2\Lambda_1}^1$, and z_2' is an eigenvalue of the operator $\tilde{H}_{2\Lambda_2}^2$, and z_3' is an eigenvalue of the operator $\tilde{H}_{2\Lambda_3}^3$.

b) *Let* $U < 0$, *and* $-\frac{4B\cos\frac{\Lambda_1^0}{2}}{W} \le U < -\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, $\cos\frac{\Lambda_2^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *or* $U < 0$, *and* $-\frac{4B\cos\frac{\Lambda_1^0}{2}}{W} \le U < -\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *or* $U < 0$, *and* $-\frac{4B\cos\frac{\Lambda_2^0}{2}}{W} \le U < -\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *or* $U < 0$, *and* $-\frac{4B\cos\frac{\Lambda_2^0}{2}}{W} \le U < -\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_2^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *or* $U < 0$, *and* $-\frac{12B\cos\frac{\Lambda_3^0}{2}}{W} \le U < -\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *or* $U < 0$, *and* $-\frac{12B\cos\frac{\Lambda_3^0}{2}}{W} \le U < -\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *and* $\cos\frac{\Lambda_2^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *and*

$\cos\frac{\Lambda_1^0}{2}<\cos\frac{\Lambda_2^0}{2}$, *then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}_s^0$ *is consists of the union of four segments*:

$$\sigma_{ess}\left({}^1\tilde{H}_s^0\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+e_1+z_2',b_1+f_1+z_2'\right]\cup\left[c_1+e_1+z_1',d_1+f_1+z_1'\right]\cup\left[e_1+z_1'+z_2',f_1+z_1'+z_2'\right],$$

or

$$\sigma_{ess}\left({}^1\tilde{H}_s^0\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3',b_1+d_1+z_3'\right]\cup\left[a_1+e_1+z_2',b_1+f_1+z_2'\right]\cup\left[a_1+z_2'+z_3',b_1+z_2'+z_3'\right],$$

or

$$\sigma_{ess}\left({}^1\tilde{H}_s^0\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3',b_1+d_1+z_3'\right]\cup\left[c_1+e_1+z_1',d_1+f_1+z_1'\right]\cup\left[c_1+z_1'+z_3',d_1+z_1'+z_3'\right],$$

and the discrete spectrum of the operator ${}^1\tilde{H}_s^0$ is empty: $\sigma_{disc}\left({}^1\tilde{H}_s^0\right)=\varnothing$.

c) Let $U<0$, *and* $-\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}\le U<-\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}>\cos\frac{\Lambda_2^0}{2}$, $\cos\frac{\Lambda_2^0}{2}>3\cos\frac{\Lambda_3^0}{2}$, *or* $U<0$, *and* $-\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}\le U<-\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}>\cos\frac{\Lambda_2^0}{2}$ *and* $\cos\frac{\Lambda_1^0}{2}>3\cos\frac{\Lambda_3^0}{2}$ *or* $U<0$, *and* $-\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}\le U<-\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}>\cos\frac{\Lambda_2^0}{2}$ *and* $\cos\frac{\Lambda_1^0}{2}>3\cos\frac{\Lambda_3^0}{2}$ *or* $U<0$, *and* $-\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}\le U<-\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}<\cos\frac{\Lambda_2^0}{2}$ *and* $\cos\frac{\Lambda_1^0}{2}<3\cos\frac{\Lambda_3^0}{2}$ *or* $U<0$, *and* $-\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}\le U<-\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}>\cos\frac{\Lambda_2^0}{2}$ *and* $\cos\frac{\Lambda_1^0}{2}<3\cos\frac{\Lambda_3^0}{2}$ *or* $U<0$, *and* $-\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}\le U<-\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}<\cos\frac{\Lambda_2^0}{2}$ *and* $\cos\frac{\Lambda_2^0}{2}>3\cos\frac{\Lambda_3^0}{2}$, *then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}_s^0$ *is consists of the union of two segments*:

$\sigma_{ess}\left({}^1\tilde{H}_s^0\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3',b_1+d_1+z_3'\right]$, *or*

$\sigma_{ess}\left({}^1\tilde{H}_s^0\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+e_1+z_2',b_1+f_1+z_2'\right]$, *or*

$\sigma_{ess}\left({}^1\tilde{H}_s^0\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[c_1+e_1+z_1',d_1+f_1+z_1'\right]$. *The discrete spectrum of the operator* ${}^1\tilde{H}_s^0$ *is empty set*: $\sigma_{disc}\left({}^1\tilde{H}_s^0\right)=\varnothing$.

d) Let $U<0$, $-\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}\le U<0$ *and* $\cos\frac{\Lambda_1^0}{2}>\cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2}>3\cos\frac{\Lambda_3^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2}<\cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2}>3\cos\frac{\Lambda_3^0}{2}$, *or* $U<0$, *and* $-\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}\le U<0$ *and* $\cos\frac{\Lambda_1^0}{2}>\cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2}<3\cos\frac{\Lambda_3^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2}<3\cos\frac{\Lambda_3^0}{2}$, *or* $U<0$, *and* $-\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}\le U<0$ *and* $\cos\frac{\Lambda_1^0}{2}<\cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2}<3\cos\frac{\Lambda_3^0}{2}$, *or* $\cos\frac{\Lambda_2^0}{2}<3\cos\frac{\Lambda_3^0}{2}$, *then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}_s^0$ *is consists of single segments:* $\sigma_{ess}\left({}^1\tilde{H}_s^0\right)=\left[a_1+c_1+e_1,b_1+d_1+f_1\right]$, *and the discrete spectrum of the operator* ${}^1\tilde{H}_{3/2}^q$ *is empty set:* $\sigma_{disc}\left({}^1\tilde{H}_s^0\right)=\varnothing$.

Theorem 30. *The following statements hold:*

a) Let $U>0$, *and* $U>\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}<\cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2}>3\cos\frac{\Lambda_3^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2}<3\cos\frac{\Lambda_3^0}{2}$, *or* $U>0$, *and* $U>\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}>\cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2}>3\cos\frac{\Lambda_3^0}{2}$, *or* $\cos\frac{\Lambda_2^0}{2}<3\cos\frac{\Lambda_3^0}{2}$, *or* $U>0$, *and* $U>\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}<3\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2}>\cos\frac{\Lambda_2^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2}<\cos\frac{\Lambda_2^0}{2}$, *then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}_s^0$ *is consists of the union of seven segments:*

$$\begin{aligned}\sigma_{ess}\left({}^1\tilde{H}_{3/2}^q\right)=&\left[a_1+c_1+e_1,b_1+d_1+f_1\right]\cup\left[a_1+c_1+z_3'',b_1+d_1+z_3''\right]\\&\cup\left[a_1+e_1+z_2'',b_1+f_1+z_2''\right]\cup\left[a_1+z_2''+z_3'',b_1+z_2''+z_3''\right]\\&\cup\left[c_1+e_1+z_1'',d_1+f_1+z_1''\right]\cup\left[c_1+z_1''+z_3'',d_1+z_1''+z_3''\right]\\&\cup\left[e_1+z_1''+z_2'',f_1+z_1''+z_2''\right].\end{aligned}$$

The discrete spectrum of the operator ${}^1\tilde{H}_s^0$ is consists of no more one point: $\sigma_{disc}\left({}^1\tilde{H}_s^0\right)=\left\{z_1''+z_2''+z_3''\right\}$, or $\sigma_{disc}\left({}^1\tilde{H}_s^0\right)=\varnothing$.

Here and hereafter, z_1'' is an eigenvalue of the operator $\tilde{H}_{2\Lambda_1}^1$, and z_2'' is an eigenvalue of the operator $\tilde{H}_{2\Lambda_2}^2$, and z_3'' is an eigenvalue of the operator $\tilde{H}_{2\Lambda_3}^3$.

b) Let $U>0$, *and* $\frac{4B\cos\frac{\Lambda_1^0}{2}}{W}<U\le\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2}<\cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2}>3\cos\frac{\Lambda_3^0}{2}$, *or* $U>0$, *and* $\frac{12B\cos\frac{\Lambda_3^0}{2}}{W}<U\le\frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *and*

$\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, *and* $\frac{4B\cos\frac{\Lambda_2^0}{2}}{W} < U \le \frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, *and* $\frac{12B\cos\frac{\Lambda_3^0}{2}}{W} < U \le \frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, *and* $\frac{4B\cos\frac{\Lambda_1^0}{2}}{W} < U \le \frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, *and* $\frac{4B\cos\frac{\Lambda_2^0}{2}}{W} < U \le \frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}_s^0$ *is consists of the union of four segments*:

$$\sigma_{ess}\left({}^1\tilde{H}_s^0\right) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+c_1+z_3'', b_1+d_1+z_3''] \cup [c_1+e_1+z_1'', d_1+f_1+z_1''] \cup [c_1+z_1''+z_3'', d_1+z_1''+z_3''],$$

or

$$\sigma_{ess}\left({}^1\tilde{H}_s^0\right) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+c_1+z_3'', b_1+d_1+z_3''] \cup [a_1+e_1+z_2'', b_1+f_1+z_2''] \cup [a_1+z_2''+z_3'', b_1+z_2''+z_3''],$$

or

$$\sigma_{ess}\left({}^1\tilde{H}_s^0\right) = [a_1+c_1+e_1, b_1+d_1+f_1] \cup [a_1+e_1+z_2'', b_1+f_1+z_2''] \cup [c_1+e_1+z_1'', d_1+f_1+z_1''] \cup [e_1+z_1''+z_2'', f_1+z_1''+z_2''].$$

The discrete spectrum of the operator ${}^1\tilde{H}_s^0$ is empty set: $\sigma_{disc}\left({}^1\tilde{H}_s^0\right) = \varnothing$.

c) Let $U > 0$, *and* $\frac{12B\cos\frac{\Lambda_3^0}{2}}{W} < U \le \frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, *and* $\frac{4B\cos\frac{\Lambda_1^0}{2}}{W} < U \le \frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, *and* $\frac{12B\cos\frac{\Lambda_3^0}{2}}{W} < U \le \frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, *and* $\frac{4B\cos\frac{\Lambda_2^0}{2}}{W} < U \le \frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, *and* $\frac{4B\cos\frac{\Lambda_2^0}{2}}{W} < U \le \frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and*

$\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *or* $U > 0$, *and* $\frac{4B\cos\frac{\Lambda_1^0}{2}}{W} < U \le \frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *and* $\cos\frac{\Lambda_2^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}_s^0$ *is consists of the union of two segments:*

$\sigma_{ess}\left({}^1\tilde{H}_s^0\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + c_1 + z_3'', b_1 + d_1 + z_3'']$, *or*

$\sigma_{ess}\left({}^1\tilde{H}_s^0\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [c_1 + e_1 + z_1'', d_1 + f_1 + z_1'']$, *or*

$\sigma_{ess}\left({}^1\tilde{H}_s^0\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1] \cup [a_1 + e_1 + z_2'', b_1 + f_1 + z_2'']$. *The discrete spectrum of the operator* ${}^1\tilde{H}_s^0$ *is empty set:* $\sigma_{disc}\left({}^1\tilde{H}_s^0\right) = \varnothing$.

d) Let $U > 0$, $0 < U \le \frac{12B\cos\frac{\Lambda_3^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} > 3\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *or* $U > 0$, *and* $0 < U \le \frac{4B\cos\frac{\Lambda_1^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *or* $U > 0$, *and* $0 < U \le \frac{4B\cos\frac{\Lambda_2^0}{2}}{W}$, *and* $\cos\frac{\Lambda_1^0}{2} < 3\cos\frac{\Lambda_3^0}{2}$, *and* $\cos\frac{\Lambda_1^0}{2} > \cos\frac{\Lambda_2^0}{2}$, *or* $\cos\frac{\Lambda_1^0}{2} < \cos\frac{\Lambda_2^0}{2}$, *then the essential spectrum of the system first six-electron singlet state operator* ${}^1\tilde{H}_s^0$ *is consists of single segments:*

$\sigma_{ess}\left({}^1\tilde{H}_s^0\right) = [a_1 + c_1 + e_1, b_1 + d_1 + f_1]$, *and the discrete spectrum of the operator* ${}^1\tilde{H}_s^0$ *is empty set:* $\sigma_{disc}\left({}^1\tilde{H}_s^0\right) = \varnothing$.

6. Conclusion

In this paper, we consider six-electron systems in the octet, and first quintet and first singlet states. In the six-electron systems, the total spin S takes the values $S = 3, 2, 1$ and 0. The states with total spin value $S = 3$ so-called the octet state, and with total spin value $S = 2$ so-called the quintet states, and with total spin value $S = 0$, so-called the singlet state. We proved in the octet state the spectrum of the operator $\tilde{H}_3^o$ is purely continuous and coincides with the segment $[6A - 12B\nu, 6A + 12B\nu]$. We proved in the first singlet state in the one-dimensional case the essential spectrum of the system consists of the union of seven segments and discrete spectrum of the system in this case consists of no more than one eigenvalue, *i.e.*, or discrete spectrum of the system consists of unique eigenvalue, or discrete spectrum of the system is empty set. In the three-dimensional case, or the essential spectra of the system consist of the union of seven segments, and discrete spectrum of the system consists of no more than one eigenvalue, or the essential spectrum of the system consists of the union of four segments, and discrete spectra of the system is empty set, or the essential spectrum of the system consists the union of two segments, and discrete spectra of the system is empty set, or the essential spectrum of the system consists of single segment, and discrete spectra of the system are empty set. We have analogous results for the first quintet state. Note that the spectrum of the Hamiltonian in the first quintet state and in the first singlet state differs from each other.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Hubbard, J. (1963) Electron Correlations in Narrow Energy Bands. *Proceedings of the Royal Society of London*, *Series A*: *Mathematical and Physical Sciences*, **276**, 238-257. https://doi.org/10.1098/rspa.1963.0204

[2] Gutzwiller, M.C. (1963) Effect of Correlation on the Ferromagnetism of Transition Metals. *Physical Review Letters*, **10**, 159-162. https://doi.org/10.1103/PhysRevLett.10.159

[3] Kanamori, J. (1963) Electron Correlation and Ferromagnetism of Transition Metals. *Progress of Theoretical Physics*, **30**, 275-289. https://doi.org/10.1143/PTP.30.275

[4] Anderson, P.W. (1961) Localized Magnetic States in Metals. *Physical Review Journals Archive*, **124**, 41-53. https://doi.org/10.1103/PhysRev.124.41

[5] Shubin, S.P. and Wonsovsky, S.V. (1934) On the Electron Theory of Metals. *Proceedings of the Royal Society of London*, *Series A*: *Containing Papers of a Mathematical and Physical Character*, **145**, 159-172. https://doi.org/10.1098/rspa.1934.0089

[6] Karpenko, B.V., Dyakin, V.V. and Budrina, G.L. (1986) Two Electrons in the Hubbard Model. *Physics of Metals and Metallography*, **61**, 702-706.

[7] Mattis, D.C. (1986) The Few-Body Problems on a Lattice. *Reviews of Modern Physics*, **58**, 370-379. https://doi.org/10.1103/RevModPhys.58.361

[8] Izyumov and Yu. N. (1988) Skryabin, Statistical Mechanics of Magnetically Ordered Systems. Springer, New York.

[9] Tashpulatov, S.M. (2014) Spectral Properties of Three-Electron Systems in the Hubbard Model. *Theoretical and Mathematical Physics*, **179**, 712-728. https://doi.org/10.1007/s11232-014-0173-y

[10] Tashpulatov, S.M. (2016) Spectra of the Energy Operator of Four-Electron Systems in the Triplete State in the Hubbard Model. *Journal of Physics*: *Conference Series*, **697**, Article ID: 012025. https://doi.org/10.1088/1742-6596/697/1/012025

[11] Tashpulatov, S.M. (2017) The Structure of Essential Spectra and Discrete Spectrum of Four-Electron Systems in the Hubbard Model in a Singlet State. *Lobachevskii Journal of Mathematics*, **38**, 530-541. https://doi.org/10.1134/S1995080217030246

[12] Tashpulatov, S.M. (2018) The Structure of Essential Spectra and Discrete Spectrum of the Energy Operator of Five-Electron Systems in the Hubbard Model. Fifth Doublet State. *Bulletin of the Institute of Mathematics*, No. 5, 43-52.

[13] Tashpulatov, S.M. (2021) Structure of Essential Spectrum and Discrete Spectra of the Energy Operator of Five-Electron Systems in the Hubbard Model—Doublet State. In: Kusraev, A.G. and Totieva, Z.D., Eds., *Operator Theory and Differential Equations. Trends in Mathematics*, Birkhäuser, Cham, 275-301. https://doi.org/10.1007/978-3-030-49763-7_19

[14] Tashpulatov, S.M. (2020) Structure of Essential Spectrum and Discrete Spectra of the Energy Operator of Five-Electron Systems in the Hubbard Model. Third and Fourth Doublet States. *Journal of Applied Mathematics and Physics*, **8**, 2886-2918. https://doi.org/10.4236/jamp.2020.812214

[15] Tashpulatov, S.M. (2021) Structure of Essential Spectra and Discrete Spectrum of the Energy Operator of Five Electron Systems in the Hubbard Model. Sextet and Quartet States. *American Review of Mathematics and Statistics*, **9**, 12-40. https://doi.org/10.15640/arms.v9n1a2

[16] Tashpulatov, S.M. (2022) Spectra of the Energy Operator of Two-Electron System in the Impurity Hubbard Model. *Journal of Applied Mathematics and Physics*, **10**, 2743-2779. https://doi.org/10.4236/jamp.2022.109184

[17] Tashpulatov, S.M. (2019) Spectra of the Energy Operator of Three-Electron Systems in the Impurity Hubbard Model. The Second Doublet State. *Contemporary Problems in Mathematics and Physics*, **65**, 109-123. (In Russian) https://doi.org/10.22363/2413-3639-2019-65-1-109-123

[18] Tashpulatov, S.M. (2021) The Structure of Essential Spectra and Discrete Spectrum of Three-Electron Systems in the Impurity Hubbard Model. Quartet State. *Journal of Applied Mathematics and Physic*, **9**, 1391-1421. https://doi.org/10.4236/jamp.2021.96094

[19] Tashpulatov, S.M. and Parmanova, R.T. (2021) Spectra of the Energy Operator of Four-Electron Systems in the Impurity Hubbard Model. Triplet State. *Journal of Applied Mathematics and Physics*, **9**, 2776-2795. https://doi.org/10.4236/jamp.2021.911179

[20] Reed, M. and Simon, B. (1972) Functional Analysis. In: *Methods of Modern Mathematical Physics*, Vol. 1, Academic Press, Cambridge. https://doi.org/10.1016/B978-0-12-585001-8.50007-6

[21] Reed, M. and Simon, B. (1982) Operator Analysis. In: *Methods of Modern Mathematical Physics*, Vol. 4, Academic Press, Cambridge.

[22] Naimark, M.A. (1986) Normed Rings. Nauka, Moscow.

[23] Valkov, V.V., Ovchinnikov, S.G. and Petrakovskii, O.P. (1986) The Excitation Spectra of Two-Magnon Systems in Easy-Axis Quasidimensional Ferromagnets. *Physics of the Solid State*, **30**, 3044-3047.