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Four-Electron Systems in the Impurity Hubbard Model. Second Triplet State. Spectra of the System in the ν-Dimensional Lattice Z^ν

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Abstract

We consider an energy operator of four-electron system in the Impurity Hubbard model with a coupling between nearest-neighbors. The spectrum of the systems in the second triplet state in a ν-dimensional lattice is investigated. For investigation the structure of essential spectra and discrete spectrum of the energy operator of four-electron systems in an impurity Hubbard model, for which the momentum representation is convenient. In addition, we used the tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces and described the structure of essential spectrum and discrete spectrum of the energy operator of four-electron systems in an impurity Hubbard model for the second triplet state of the system. The investigations show that the essential spectrum of the system consists of the union of no more than sixteen segments, and the discrete spectrum of the system consists of no more than eleven eigenvalues.

Keywords

Spectra of Four-Electron System, Bound State, Anti-Bound State, Impurity Hubbard Model, Quintet State, Singlet State, Triplet State

1. Introduction

The Hubbard model first appeared in 1963 in the works [1] [2] [3]. The model proposed in [1] [2] [3] was called the Hubbard model after John Hubbard, who made a fundamental contribution to studying the statistical mechanics of that system, although the local form of Coulomb interaction was first introduced for an impurity model in a metal by Anderson [4]. We also recall that the Hubbard model is a particular case of the Shubin-Wonsowsky polaron model [5], which had appeared 30 years before [1] [2] [3]. In the Shubin-Wonsowsky model, along with the on-site Coulomb interaction, the interaction of electrons on neighboring sites is also taken into account.

The Hubbard model well describes the behavior of particles in a periodic potential at sufficiently low temperatures such that all particles are in the lower Bloch band and long-range interactions can be neglected. If the interaction between particles on different sites is taken into account, then the model is often called the extended Hubbard model. It was proposed for describing electrons in solids, and it remains especially interesting since then for studying high-temperature superconductivity. Later, the extended Hubbard model also found applications in describing the behavior of ultracold atoms in optical lattices. In considering electrons in solids, the Hubbard model can be considered a sophisticated version of the model of strongly bound electrons, involving only the electron hopping term in the Hamiltonian. In the case of strong interactions, these two models can give essentially different results. The Hubbard model exactly predicts the existence of so-called Mott insulators, where conductance is absent due to strong repulsion between particles. The Hubbard model is based on the approximation of strongly coupled electrons. In the strongcoupling approximation, electrons initially occupy orbital's in atoms (lattice sites) and then hop over to other atoms, thus conducting the current. Mathematically, this is represented by the so-called hopping integral. This process can be considered the physical phenomenon underlying the occurrence of electron bands in crystal materials. But the interaction between electrons is not considered in more general band theories. In addition to the hopping integral, which explains the conductance of the material, the Hubbard model contains the so-called on-site repulsion, corresponding to the Coulomb repulsion between electrons. This leads to a competition between the hopping integral, which depends on the mutual position of lattice sites, and the on-site repulsion, which is independent of the atom positions. As a result, the Hubbard model explains the metal-insulator transition in oxides of some transition metals. When such a material is heated, the distance between nearest-neighbor sites increases, the hopping integral decreases, and on-site repulsion becomes dominant.

The Hubbard model is currently one of the most extensively studied multielectron models of metals [6]-[12] and [13], chapter III, PP. 75-191. In the review [7], the results obtained on the Hubbard model are summarized. According to the Hubbard model, the more progress that is made in obtaining theoretical solutions, the clearer it becomes that this simple model can exhibit a startling array of phases and regimes, many of which have clear parallels with observed behaviors of a wide variety of complex materials.

For instance, there is compelling evidence that ferromagnetism, various forms of antiferromagnetism, unconventional superconductivity, charge-density waves, electronic liquid crystalline phases, and topologically ordered phases (e.g., "spin liquids"), among other phases, occur in specific realizations of the Hubbard model.

It is our purpose here to summarize, to the extent possible in a brief article, what is established concerning the quantum phases of the Hubbard model. The role of the Hubbard model, which it played in the study of high-temperature superconductivity in cuprates, is discussed.

It is shown that the positive eigenvalues in the Hubbard model (corresponding to repulsive effectual interactions) weaken, and the negative ones grow. The various eigenfunctions correspond to, but are not completely determined by, an irreducible representation of a group of crystal points in the Hubbard model.

Obtaining exact results for the spectrum and wave functions of the crystal described by the Hubbard model and impurity Hubbard model is of great interest.

The spectrum and wave functions of the system of two electrons in a crystal described by the Hubbard Hamiltonian were studied in [14]. It is known that two-electron systems can be in two states, triplet and singlet [14]. In the work [14] is considered the Hamiltonian of the form

$$H = A\sum_{m,\gamma} a^{+}_{m,\gamma} a_{m,\gamma} + B\sum_{m,\tau,\gamma} a^{+}_{m,\gamma} a_{m+\tau,\gamma} + U\sum_{m} a^{+}_{m,\uparrow} a_{m,\uparrow} a^{+}_{m,\downarrow} a_{m,\downarrow}. \tag{1}$$

Here A is the electron energy at a lattice site, B is the transfer integral between neighboring sites, $\tau = \pm e_j, j = 1,2,\cdots,\nu$, where e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, U is the parameter of the on-site Coulomb interaction of two electrons, γ is the spin index, $\gamma = \uparrow$ or $\gamma = \downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2}$, and $a^{+}_{m,\gamma}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^{\nu}$. It was proved in [14] that the spectrum of the system Hamiltonian H^{t} in the triplet state is purely continuous and coincides with a segment $[m, M] = [2A - 4B\nu, 2A + 4B\nu]$, where ν is the lattice dimensionality, and the operator H^{s} of the system in the singlet state, in addition to the continuous spectrum $[m, M]$, has a unique antibound state for some values of the quasimomentum. For the antibound state, correlated motion of the electrons is realized under which the contribution of binary states is large. Because the system is closed, the energy must remain constant and large. This prevents the electrons from being separated by long distances. Next, an essential point is that bound states (sometimes called scattering-type states) do not form below the continuous spectrum. This can be easily understood because the interaction is repulsive. We note that a converse situation is realized for $U < 0$: below the continuous spectrum, there is a bound state (antibound states are absent) because the electrons are then attracted to one another.

For the first band, the spectrum is independent of the parameter U of the on-site Coulomb interaction of two electrons and corresponds to the energy of two noninteracting electrons, being exactly equal to the triplet band. The second band is determined by Coulomb interaction to a much greater degree: both the amplitudes and the energy of two electrons depend on U, and the band itself disappears as $U \to 0$ and increases without bound as $U \to \infty$. The second band largely corresponds to a one-particle state, namely, the motion of the doublet, *i.e.*, two-electron bound states.

The spectrum and wave functions of the system of three electrons in a crystal described by the Hubbard Hamiltonian were studied in [15]. In the three-electron systems are exists quartet state, and two type doublet states.

The spectrum of the energy operator of system of four electrons in a crystal described by the Hubbard Hamiltonian in the triplet state was studied in [16]. In the four-electron systems are exists quintet state, and three type triplet states, and two type singlet states. The spectrum of the energy operator of four-electron systems in the Hubbard model in the quintet, and singlet states were studied in [17].

In the work [18] is considered dominant correlation effects in two-electron atoms.

The use of films in various areas of physics and technology arouses great interest in studying a localized impurity state (LIS) of magnet. Therefore, it is important to study the spectral properties of electron systems in the impurity Hubbard model.

The spectrum and wave functions of the system of two electrons in a crystal described by the impurity Hubbard Hamiltonian were studied in [19] and [20].

The spectrum of the energy operator of three-electron systems in the impurity Hubbard model in the second doublet state was studied [21]. The structure of essential spectra and discrete spectrum of three-electron systems in the impurity Hubbard model in the Quartet state were studied in [22]. The structure of essential spectra and discrete spectrum of four-electron systems in the impurity Hubbard model in the Quartet state and in the first triplet state were studied in [23] and [24] in the one-dimensional lattice.

In this paper we give a full description of the structure of the essential spectra and discrete spectrum of four-electron systems in the impurity Hubbard model for second triplet state. In contrast to the works [23] and [24], not only the one-dimensional case is considered here, but the cases when $\nu = 1,2,3$ and the spectrum of the system for the second triplet state is described for all values of the parameters of the Hamiltonian. First, using the standard anticommutation relations between the operators of electron creation and annihilation at the lattice sites, we get a coordinate representation of the Hamiltonian action, and then moving on to the Fourier transformation we get a quasi-pulse representation of the Hamiltonian action. Using the concept of tensor products of Hilbert spaces, and tensor products of operators in Hilbert spaces, we bring the problems of studying the spectrum of the energy operator of four electron systems in the Impurity Hubbard model to the study of the spectrum of the energy operator of one electron system in the Impurity Hubbard model. Then, using the results obtained from the study of the spectrum of the energy operator of one-electron systems in the impurity Hubbard model, we describe the spectrum of four electron systems in the Impurity Hubbard model for the second triplet state. The results obtained show how the results of this work differ from the results of the works [23] and [24]. The main result of this paper is Theorems 8 and 9, which describe the spectrum of considered model for second triplet state. The results of sections 2 and 3 and 4 (Theorem 7) there are preliminary facts for the proof of Theorems 8 and 9.

2. Preliminaries

We consider the energy operator of four-electron systems in the Impurity Hubbard model and describe the structure of the essential spectra and discrete spectrum of the system for second triplet state in the lattice. The Hamiltonian of the chosen model has the form

$$\begin{aligned} H = {} & A\sum_{m,\gamma} a_{m,\gamma}^{+} a_{m,\gamma} + B\sum_{m,\tau,\gamma} a_{m,\gamma}^{+} a_{m+\tau,\gamma} + U\sum_{m} a_{m,\uparrow}^{+} a_{m,\uparrow} a_{m,\downarrow}^{+} a_{m,\downarrow} \\ & + \left(A_0 - A\right)\sum_{\gamma} a_{0,\gamma}^{+} a_{0,\gamma} + \left(B_0 - B\right)\sum_{\tau,\gamma}\left(a_{0,\gamma}^{+} a_{\tau,\gamma} + a_{\tau,\gamma}^{+} a_{0,\gamma}\right) \\ & + \left(U_0 - U\right) a_{0,\uparrow}^{+} a_{0,\uparrow} a_{0,\downarrow}^{+} a_{0,\downarrow}. \end{aligned} \tag{2}$$

Here A (A_0) is the electron energy at a regular (impurity) lattice site; $B > 0$ ($B_0 > 0$) is the transfer integral between electrons (between electron and impurity) in a neighboring sites, $\tau = \pm e_j, j = 1, 2, \cdots, \nu$, where e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, U (U_0) is the parameter of the on-site Coulomb interaction of two electrons, correspondingly in the regular (impurity) lattice site; γ is the spin index, $\gamma = \uparrow$ or $\gamma = \downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2}$, and $a_{m,\gamma}^{+}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^{\nu}$. The second triplet state corresponds four-electron bound states (or antibound states) to the basis functions: ${}^{2}t_{p,q,r,k\in Z^{\nu}}^{1} = a_{p,\uparrow}^{+} a_{q,\uparrow}^{+} a_{r,\downarrow}^{+} a_{k,\uparrow}^{+} \varphi_0$. The subspace ${}^{2}\tilde{\mathcal{H}}_t^1$, corresponding to the second triplet state is the set of all vectors of the form ${}^{2}\psi_t^1 = \sum_{p,q,r,k\in Z^{\nu}} f\left(p,q,r,k\right) {}^{2}t_{p,q,r,k\in Z^{\nu}}^{1}$, $f \in l_2^{as}$, where l_2^{as} is the subspace of antisymmetric functions in $l_2\left(\left(Z^{\nu}\right)^4\right)$. In this case, the Hamiltonian H acts in the antisymmetric Fock space ${}^{2}\tilde{\mathcal{H}}_t^1$. Let φ_0 be the vacuum vector in the antisymmetrical Fock space ${}^{2}\tilde{\mathcal{H}}_t^1$. Let ${}^{2}\tilde{H}_t^1$ be the restriction H to the subspace ${}^{2}\tilde{\mathcal{H}}_t^1$. The second triplet state corresponds the free motions of four-electrons in the lattice and their interactions. Let $\varepsilon_1 = A_0 - A$, $\varepsilon_2 = B_0 - B$, $\varepsilon_3 = U_0 - U$.

The energy of the system depends on its total spin S. Along with the Hamiltonian, the N_e electron system is characterized by the total spin S,

$$S = S_{\max}, S_{\max} - 1, \cdots, S_{\min},\quad S_{\max} = \frac{N_e}{2},\quad S_{\min} = 0, \frac{1}{2}.$$

Hamiltonian (2) commutes with all components of the total spin operator $S = \left(S^{+}, S^{-}, S^{z}\right)$, and the structure of eigenfunctions and eigenvalues of the system therefore depends on S. The Hamiltonian H acts in the antisymmetric Fo'ck space $\mathcal{H}_{as}$.

Below we give the constructions of the Fo'ck space $\mathcal{F}(\mathcal{H})$.

Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{H}^n$ the *n*- fold tensor product $\mathcal{H}^n = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$. We set $\mathcal{H}^0 = C$ and $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n$. The $\mathcal{F}(\mathcal{H})$ is called the Fo'ck space over $\mathcal{H}$; it will be separably, if $\mathcal{H}$ is. For example, if $\mathcal{H} = L_2(R)$, then an element $\psi \in \mathcal{F}(\mathcal{H})$ is a sequence of functions $\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \psi_3(x_1, x_2, x_3), \cdots\}$, so that $|\psi_0|^2 + \sum_{n=1}^{\infty} \int_{R^n} |\psi_n(x_1, x_2, \cdots, x_n)|^2 \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n < \infty$. Actually, it is not $\mathcal{F}(\mathcal{H})$, itself, but two of its subspaces which are used most frequently in quantum field theory. These two subspaces are constructed as follows: Let $\mathcal{P}_n$ be the permutation group on *n* elements, and let $\{\psi_n\}$ be a basis for space $\mathcal{H}$. For each $\sigma \in \mathcal{P}_n$, we define an operator (which we also denote by σ) on basis elements $\mathcal{H}^n$, by $\sigma(\varphi_{k_1} \otimes \varphi_{k_2} \otimes \cdots \otimes \varphi_{k_n}) = \varphi_{k_{\sigma(1)}} \otimes \varphi_{k_{\sigma(2)}} \otimes \cdots \otimes \varphi_{k_{\sigma(n)}}$. The operator σ extends by linearity to a bounded operator (of norm one) on space $\mathcal{H}^n$, so we can define $S_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \sigma$. That the operator S_n is the operator of orthogonal projection: $S_n^2 = S_n$, and $S_n^* = S_n$. The range of S_n is called *n*- fold symmetric tensor product of $\mathcal{H}$. In the case, where $\mathcal{H} = L_2(R)$ and $\mathcal{H}^n = L_2(R) \otimes L_2(R) \otimes \cdots \otimes L_2(R) = L_2(R^n)$, $S_n \mathcal{H}^n$ is just the subspace of $L_2(R^n)$, of all functions, left invariant under any permutation of the variables. We now define $\mathcal{F}_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_n \mathcal{H}^n$. The space $\mathcal{F}_s(\mathcal{H})$ is called the symmetrical Fo'ck space over $\mathcal{H}$, or Boson Fo'ck space over $\mathcal{H}$.

Let $\varepsilon(.)$ is function from $\mathcal{P}_n$ to $\{1, -1\}$, which is one on even permutations and minus one on odd permutations. Define $A_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \varepsilon(\sigma) \sigma$; then A_n is an orthogonal projector on $\mathcal{H}^n$. $A_n \mathcal{H}^n$ is called the *n*- fold antisymmetrical tensor product of $\mathcal{H}$. In the case where $\mathcal{H} = L_2(R)$, $A_n \mathcal{H}^n$ is just the subspace of $L_2(R^n)$, consisting of those functions odd under interchange of two coordinates. The subspace $\mathcal{F}_a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} A_n \mathcal{H}^n$ is called the antisymmetrical Fo'ck space over $\mathcal{H}$, or the Fermion Fo'ck space over $\mathcal{H}$.

Let φ_0 be the vacuum vector in the antisymmetrical Fock space ${}^2\tilde{\mathcal{H}}_t^1$. Let ${}^2\tilde{H}_t^1$ be the restriction H to the subspace ${}^2\tilde{\mathcal{H}}_t^1$.

Theorem 1. *The subspace* ${}^2\mathcal{H}_t^1$ *is invariant under the operator H, and the restriction* ${}^2H_t^1$ *of operator H to the subspace* ${}^2\mathcal{H}_t^1$ *is a bounded self-adjoint operator. It generates a bounded self-adjoint operator* ${}^2\bar{H}_t^1$ *acting in the space* l_2^{as} *as*

$$
\begin{aligned}
{}^2\bar{H}_t^1\, {}^2\psi_t^1 = {} & 4Af(p,q,r,k) + B\sum_{\tau}\left[f(p+\tau,q,r,k) + f(p,q+\tau,r,k)\right. \\
& \left. + f(p,q,r+\tau,k) + f(p,q,r,k+\tau)\right] + U\left[\delta_{p,r} + \delta_{q,r} + \delta_{r,k}\right] f(p,q,r,k)
\end{aligned}
$$

$$\begin{aligned}&+\left(A_0-A\right)\left(\delta_{p,0}+\delta_{q,0}+\delta_{r,0}+\delta_{k,0}\right)f\left(p,q,r,k\right)+\left(B_0-B\right)\sum_{\tau}\Big[\delta_{p,0}f\left(\tau,q,r,k\right)\\&+\delta_{q,0}f\left(p,\tau,r,k\right)+\delta_{r,0}f\left(p,q,\tau,k\right)+\delta_{k,0}f\left(p,q,r,\tau\right)+\delta_{p,\tau}f\left(0,q,r,k\right)\\&+\delta_{q,\tau}f\left(p,0,r,k\right)+\delta_{r,\tau}f\left(p,q,0,k\right)+\delta_{k,\tau}f\left(p,q,r,0\right)\Big]\\&+\left(U_0-U\right)\left[\delta_{p,r}\delta_{p,0}+\delta_{q,r}\delta_{q,0}+\delta_{r,k}\delta_{r,0}\right]f\left(p,q,r,k\right).\end{aligned} \tag{3}$$

The operator ${}^2H_t^1$ acts on a vector ${}^2\psi_t^1\in{}^2\mathcal{H}_t^1$ as

$$ {}^2H_t^1\,{}^2\psi_t^1=\sum_{p,q,r,k\in Z^\nu}\left({}^2\bar{H}_t^1f\right)\left(p,q,r,k\right)^2t^1_{p,q,r,k\in Z^\nu}. \tag{4}$$

Proof. We act with the Hamiltonian H on vectors ${}^2\psi_t^1\in{}^2\mathcal{H}_t^1$ using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, $\left\{a_{m,\gamma},a_{n,\beta}^+\right\}=\delta_{m,n}\delta_{\gamma,\beta}$, $\left\{a_{m,\gamma},a_{n,\beta}\right\}=\left\{a_{m,\gamma}^+,a_{n,\beta}^+\right\}=\theta$, and also take into account that $a_{m,\gamma}\varphi_0=\theta$, where θ is the zero element of ${}^2\mathcal{H}_t^1$. This yields the statement of the theorem.

Lemma 1. *The spectra of the operators* ${}^2H_t^1$ *and* ${}^2\bar{H}_t^1$ *coincide.*

Proof. Because the operators ${}^2H_t^1$ and ${}^2\bar{H}_t^1$ are bounded self-adjoint operators, it follows that if $\lambda\in\sigma\left({}^2H_t^1\right)$, then the Weyl criterion (see [25], chapter VII, paragraph 3, pp. 262-263) implies that there is a sequence $\left\{\psi_i\right\}_{i=1}^\infty$ such that $\left\|\psi_i\right\|=1$ and $\lim_{i\to\infty}\left\|\left({}^2H_t^1-\lambda\right)\psi_i\right\|=0$. We set $\psi_i=\sum_{p,q,r,k}f_i\left(p,q,r,k\right)a_{p,\uparrow}^+a_{q,\uparrow}^+a_{r,\downarrow}^+a_{k,\uparrow}^+\varphi_0$. Then

$$\begin{aligned}\left\|\left({}^2H_t^1-\lambda\right)\psi_i\right\|^2&=\left(\left({}^2H_t^1-\lambda\right)\psi_i,\left({}^2H_t^1-\lambda\right)\psi_i\right)\\&=\sum_{p,q,r,k}\left\|\left({}^2\bar{H}_t^1-\lambda\right)f_i\left(p,q,r,k\right)\right\|^2\left(a_{p,\uparrow}^+a_{q,\uparrow}^+a_{r,\downarrow}^+a_{k,\uparrow}^+\varphi_0,a_{p,\uparrow}^+a_{q,\uparrow}^+a_{r,\downarrow}^+a_{k,\uparrow}^+\varphi_0\right)\\&=\sum_{p,q,r,k}\left\|\left({}^2\bar{H}_t^1-\lambda\right)F_i\left(p,q,r,k\right)\right\|^2\left(a_{k,\uparrow}a_{r,\downarrow}a_{q,\uparrow}a_{p,\uparrow}a_{p,\uparrow}^+a_{q,\uparrow}^+a_{r,\downarrow}^+a_{k,\uparrow}^+\varphi_0,\varphi_0\right)\\&=\sum_{p,q,r,k}\left\|\left({}^2\bar{H}_t^1-\lambda\right)F_i\left(p,q,r,k\right)\right\|^2\left(\varphi_0,\varphi_0\right)\\&=\sum_{p,q,r,k}\left\|\left({}^2\bar{H}_t^1-\lambda\right)F_i\left(p,q,r,k\right)\right\|^2\to0\end{aligned}$$

as $i\to\infty$, where $F_i=\sum_{p,q,r,k}f_i\left(p,q,r,k\right)$. It follows that $\lambda\in\sigma\left({}^2\bar{H}_t^1\right)$. Consequently, $\sigma\left({}^2H_t^1\right)\subset\sigma\left({}^2\bar{H}_t^1\right)$.

Conversely, let $\bar{\lambda}\in\sigma\left({}^2\bar{H}_t^1\right)$. Then, by the Weyl criterion, there is a sequence $\left\{F_i\right\}_{i=1}^\infty$ such that $\left\|F_i\right\|=1$ and $\lim_{i\to\infty}\left\|\left({}^2\bar{H}_t^1-\bar{\lambda}\right)\psi_i\right\|=0$. Setting $F_i=\sum_{p,q,r,k}f_i\left(p,q,r,k\right)$, $\left\|F_i\right\|=\left(\sum_{p,q,r,k}\left|f_i\left(p,q,r,k\right)\right|^2\right)^{\frac{1}{2}}$, we conclude that $\left\|\psi_i\right\|=\left\|F_i\right\|=1$ and $\left\|\left({}^2\bar{H}_t^1-\bar{\lambda}\right)F_i\right\|=\left\|\left({}^1\bar{H}_t^1-\bar{\lambda}\right)\psi_i\right\|\to0$ as $i\to\infty$. This means that $\bar{\lambda}\in\sigma\left({}^2H_t^1\right)$ and hence $\sigma\left({}^2\bar{H}_t^1\right)\subset\sigma\left({}^2H_t^1\right)$. These two relations imply $\sigma\left({}^2H_t^1\right)=\sigma\left({}^2\bar{H}_t^1\right)$. □

We call the operator ${}^2H_t^1$ the four-electron second triplet state operator in the impurity Hubbard model.

Let $\mathcal{F}: l_2\left(\left(Z^\nu\right)^4\right) \to L_2\left(\left(T^\nu\right)^4\right) \equiv {}^2\tilde{\mathcal{H}}_t^1$ be the Fourier transform, where T^ν is the ν-dimensional torus endowed with the normalized Lebesgue measure $d\lambda$, *i.e.* $\lambda\left(T^\nu\right)=1$.

We set ${}^2\tilde{H}_t^1 = \mathcal{F}\,{}^2\bar{H}_t^1\mathcal{F}^{-1}$. In the quasimomentum representation, the operator ${}^2\bar{H}_t^1$ acts in the Hilbert space $L_2^{as}\left(\left(T^\nu\right)^4\right)$, where L_2^{as} is the subspace of antisymmetric functions in $L_2\left(\left(T^\nu\right)^4\right)$.

Theorem 2. *The Fourier transform of operator* ${}^2\bar{H}_t^1$ *is an operator* ${}^2\tilde{H}_t^1 = \mathcal{F}\,{}^2\bar{H}_t^1\mathcal{F}^{-1}$ *acting in the space* ${}^2\tilde{\mathcal{H}}_t^1$ *be the formula*

$$\begin{aligned}
&{}^2\tilde{H}_t^1\,{}^2\psi_t^1 = h(\lambda,\mu,\gamma,\theta)f(\lambda,\mu,\gamma,\theta)+U\int_{T^\nu}\left[f(s,\mu,\lambda+\gamma-s,\theta)\right.\\
&\left.+f(\lambda,s,\mu+\gamma-s,\theta)+f(\lambda,\mu,s,\gamma+\theta-s)\right]\mathrm{d}s+(A_0-A)\left[\int_{T^\nu}f(s,\mu,\gamma,\theta)\mathrm{d}s\right.\\
&\left.+\int_{T^\nu}f(\lambda,l,\gamma,\theta)\mathrm{d}l+\int_{T^\nu}f(\lambda,\mu,\xi,\theta)\mathrm{d}\xi+\int_{T^\nu}f(\lambda,\mu,\gamma,n)\mathrm{d}n\right]+(B_0-B)\\
&\times\left[\sum_{j=1}^{\nu}\int_{T^\nu}2\left[\cos\lambda_j+\cos s_j\right]f(s,\mu,\gamma,\theta)\mathrm{d}s+\sum_{j=1}^{\nu}\int_{T^\nu}2\left[\cos\mu_j+\cos l_j\right]\right.\\
&\times f(\lambda,l,\gamma,\theta)\mathrm{d}l+\sum_{j=1}^{\nu}\int_{T^\nu}2\left[\cos\gamma_j+\cos\xi_j\right]f(\lambda,\mu,\xi,\theta)\mathrm{d}\xi\\
&\left.+\sum_{j=1}^{\nu}\int_{T^\nu}2\left[\cos\theta_j+\cos n_j\right]f(\lambda,\mu,\gamma,n)\mathrm{d}n\right]+(U_0-U)\left[\int_{T^\nu}\int_{T^\nu}f(s,\mu,\xi,\theta)\mathrm{d}s\mathrm{d}\xi\right.\\
&\left.+\int_{T^\nu}\int_{T^\nu}f(\lambda,l,\xi,\theta)\mathrm{d}l\mathrm{d}\xi+\int_{T^\nu}\int_{T^\nu}f(\lambda,\mu,\xi,n)\mathrm{d}\xi\mathrm{d}n\right],
\end{aligned} \tag{5}$$

where $h(\lambda,\mu,\gamma,\theta)=4A+2B\sum_{j=1}^{\nu}\left[\cos\lambda_j+\cos\mu_j+\cos\gamma_j+\cos\theta_j\right]$.

The proof Theorem 2, is straightforward of (3) using the Fourier transformation.

The spectral properties of four-electron systems in the impurity Hubbard model in the second triplet state are closely related to those of its one-electron subsystems in the impurity Hubbard model. Therefore we first study the spectrum and localized impurity states of one-electron impurity systems.

3. One-Electron Impurity Systems

The Hamiltonian of one-electron impurity system has the form:

$$\begin{aligned}
H = {} & A\sum_{m,\gamma}a_{m,\gamma}^{+}a_{m,\gamma}+B\sum_{m,\tau,\gamma}a_{m,\gamma}^{+}a_{m+\tau,\gamma}+(A_0-A)\sum_{\gamma}a_{0,\gamma}^{+}a_{0,\gamma}\\
&+(B_0-B)\sum_{\tau,\gamma}\left(a_{0,\gamma}^{+}a_{\tau,\gamma}+a_{\tau,\gamma}^{+}a_{0,\gamma}\right),
\end{aligned} \tag{6}$$

here A (A_0) is the electron energy at a regular (impurity) lattice site; $B>0$ ($B_0>0$) is the transfer integral between electrons (between electron and impurity)in a neighboring sites, $\tau=\pm e_j, j=1,2,\cdots,\nu$, where e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors; γ is the spin index, $\gamma=\uparrow$ or $\gamma=\downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2}$, and $a_{m,\gamma}^{+}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^{\nu}$.

We let $\mathcal{H}_1$ denote the Hilbert space spanned by the vectors in the form $\psi = \sum_p a^+_{p,\uparrow}\varphi_0$. It is called the space of one-electron states of the operator *H*. The space $\mathcal{H}_1$ is invariant with respect to action of the operator *H*. Denote by H_1 the restriction of *H* to the subspace $\mathcal{H}_1$.

As in the proof of Theorem 1, using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, we get the following

Theorem 3. *The subspace* $\mathcal{H}_1$ *is invariant with respect to the action of the operator H, and the operator* H_1 *is a linear bounded self-adjoint operator, acting in* $\mathcal{H}_1$ *as*

$$H_1\psi = \sum_p \left(\bar{H}_1 f\right)(p) a^+_{p,\uparrow}\varphi_0,\ \psi \in \mathcal{H}_1, \tag{7}$$

where $\bar{H}_1$ is a linear bounded self-adjoint operator acting in the space l_2 as

$$\left(\bar{H}_1 f\right)(p) = Af(p) + B\sum_\tau f(p+\tau) + \varepsilon_1\delta_{p,0}f(p) + \varepsilon_2\sum_\tau\left(\delta_{p,\tau}f(0) + \delta_{p,0}f(\tau)\right). \tag{8}$$

Lemma 2. *The spectra of the operators* $\bar{H}_1$ *and* H_1 *coincide.*

The proof of Lemma 2 is the same as the proof of the Lemma 1.

As in section 2 denote by $\mathcal{F}: l_2\left(Z^{\nu}\right) \to L_2\left(T^{\nu}\right) \equiv \tilde{\mathcal{H}}_1$ the Fourier transform. Setting $\tilde{H}_1 = \mathcal{F}\bar{H}_1\mathcal{F}^{-1}$ we get that the operator $\bar{H}_1$ acts in the Hilbert space $L_2\left(T^{\nu}\right)$.

Using the equality (8) and properties of the Fourier transform we have the following

Theorem 4. *The operator* $\tilde{H}_1$ *acting in the space* $\tilde{\mathcal{H}}_1$ *as*

$$\begin{aligned}\left(\tilde{H}_1 f\right)(\mu) &= \left[A + 2B\sum_{i=1}^{\nu}\cos\mu_i\right]f(\mu) + \varepsilon_1\int_{T^{\nu}} f(s)\,\mathrm{d}s \\ &+ 2\varepsilon_2\int_{T^{\nu}}\sum_{i=1}^{\nu}\left[\cos\mu_i + \cos s_i\right]f(s)\,\mathrm{d}s,\ \mu = (\mu_1,\cdots,\mu_n),\ s = (s_1,\cdots,s_n) \in T^{\nu}.\end{aligned} \tag{9}$$

Let *A* be an operator acting in Banach space *E* over *C*. The number λ is called regular for the operator *A* if the operator $R(\lambda) = (A - \lambda I)^{-1}$, called the resolvent of the operator *A*, is defined throughout *E* and is continuous. The set of regular values of operator *A* is called the resolvent set of this operator, and the complement of the resolvent set is the spectrum of this operator $\sigma(A)$. The spectrum of a bounded operator is compact in *C* or is empty. The spectrum of a linear bounded operator is not empty. A discrete spectrum $\sigma_{pp}(A)$ is a set of such λ, in which the operator $A - I\lambda$ is not injective.

The number λ is called the eigenvalue of the operator *A*, if there exists such a nonzero vector *x* that the equality $A(x) = \lambda x$ is valid. Any nonzero vector $x \neq 0$, satisfying this equation is called the eigenvector of the operator *A*, corresponding to the eigenvalue of λ.

The discrete spectrum is all the eigenvalues of the operator *A*.

A continuous spectrum $\sigma_{cont}(A)$ is a set of values λ, for which the resolvent $(A - \lambda I)^{-1}$ is not defined everywhere in a dense set in *E*, but is not continuous (that is, the operator $A-\lambda I$ is injective, but not surjective, and its image is dense everywhere).

The set of all isolated, finite-fold eigenvalues of operator A is called the discrete spectrum of this operator and is denoted by $\sigma_{disc}(A)$.

The entire spectrum of A without the discrete spectrum of this operator is called the essential spectrum of this operator A, and is denoted by $\sigma_{ess}(A)$.

It is clear that the continuous spectrum of operator $\tilde{H}_1$ is independent of the numbers ε_1 and ε_2, and is equal to segment $[m_\nu, M_\nu]=[A-2B\nu, A+2B\nu]$, where $m_\nu=\min_{x\in T^\nu} h(x)$, $M_\nu=\max_{x\in T^\nu} h(x)$ (here $h(x)=A+2B\sum_{i=1}^{\nu}\cos x_i$).

To find the eigenvalues and eigenfunctions of operator $\tilde{H}_1$ we rewrite (9) in following form:

$$\left\{A+2B\sum_{i=1}^{\nu}\cos\mu_i-z\right\}f(\mu)+\varepsilon_1\int_{T^\nu}f(s)\mathrm{d}s+2\varepsilon_2\int_{T^\nu}\sum_{i=1}^{\nu}[\cos\mu_i+\cos s_i]f(s)\mathrm{d}s=0, \tag{10}$$

where $z\in R$.

Suppose first that $\nu=1$ and denote $a=\int_T f(s)\mathrm{d}s$, $b=\int_T f(s)\cos s\mathrm{d}s$, $h(\mu)=A+2B\cos$. From (10) it follows that

$$f(\mu)=-\frac{(\varepsilon_1+2\varepsilon_2\cos\mu)a+2\varepsilon_2 b}{h(\mu)-z}. \tag{11}$$

Now substitute (10) in expressing of *a* and *b* we get the following system of two linear homogeneous algebraic equations:

$$\left(1+\int_T\frac{\varepsilon_1+2\varepsilon_2\cos s}{h(s)-z}\mathrm{d}s\right)a+2\varepsilon_2\int_T\frac{\mathrm{d}s}{h(s)-z}b=0;$$

$$\int_T\frac{\cos s(\varepsilon_1+2\varepsilon_2\cos s)}{h(s)-z}\mathrm{d}sa+\left(1+2\varepsilon_2\int_T\frac{\cos s\mathrm{d}s}{h(s)-z}\right)b=0.$$

This system has a nontrivial solution if and only if the determinant $\Delta_1(z)$ of this system is equal to zero, where

$$\Delta_1(z)=\left(1+\int_T\frac{(\varepsilon_1+2\varepsilon_2\cos s)\mathrm{d}s}{h(s)-z}\right)\cdot\left(1+2\varepsilon_2\int_T\frac{\cos s\mathrm{d}s}{h(s)-z}\right)$$
$$-2\varepsilon_2\int_T\frac{\mathrm{d}s}{h(s)-z}\int_T\frac{\cos s(\varepsilon_1+2\varepsilon_2\cos s)}{h(s)-z}\mathrm{d}s.$$

Therefore, it is true the following

Lemma 3. *If a real number* $z\notin[m_1,M_1]$ *then z is an eigenvalue of the operator* $\tilde{H}_1$ *if and only if* $\Delta_1(z)=0$.

The following Theorem describe of the exchange of the spectrum of operator $\tilde{H}_1$ in the case $\nu=1$. We consider every possible cases.

Theorem 5. *Let* $\nu=1$. *Then*

A).1). If $\varepsilon_2=-B$ and $\varepsilon_1<-2B$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z=A+\varepsilon_1$, lying the below of the continuous spectrum of the operator

$\tilde{H}_1$.

2). If $\varepsilon_2 = -B$ and $\varepsilon_1 > 2B$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A + \varepsilon_1$, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

B). 1). If $\varepsilon_1 < 0$ and $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A - \sqrt{4B^2 + \varepsilon_1^2}$, lying the below of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_1 > 0$ and $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A + \sqrt{4B^2 + \varepsilon_1^2}$, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

C). If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$ or $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, then the operator $\tilde{H}_1$ has a two eigenvalues $z_1 = A - \frac{2BE}{\sqrt{E^2-1}}$, and $z_2 = A + \frac{2BE}{\sqrt{E^2-1}}$, where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, lying the below and above of the continuous spectrum of the operator $\tilde{H}_1$.

D). 1). If $\varepsilon_1 = \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A + \frac{2B(E^2+1)}{E^2-1}$, where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_1 = -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A - \frac{2B(E^2+1)}{E^2-1}$, where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, lying the below of the continuous spectrum of the operator $\tilde{H}_1$.

E). 1). If $\varepsilon_2 > 0$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2-1}$, where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $\alpha > 1$, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_2 < -2B$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2-1}$, where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $\alpha > 1$, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

F). 1). If $\varepsilon_2 > 0$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A - \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2-1} < m_1$, where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $\alpha > 1$, lying the below of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_2 < -2B$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A - \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} < m_1$, where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $\alpha > 1$, lying the below of the continuous spectrum of the operator $\tilde{H}_1$.

K). 1). If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, then the operator $\tilde{H}_1$ has a exactly two eigenvalues $z_1 = A + \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} < m_1$, and $z_2 = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} > M_1$, where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $0 < \alpha < 1$, lying correspondingly, the below and above of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, then the operator $\tilde{H}_1$ has a exactly two eigenvalues $z_1 = A + \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} < m_1$, and $z_2 = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} > M_1$, where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $0 < \alpha < 1$, lying correspondingly, the below and above of the continuous spectrum of the operator $\tilde{H}_1$.

M). 1). If $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$, then the operator $\tilde{H}_1$ has a exactly two eigenvalues $z_1 = A - \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} < m_1$, and $z_2 = A - \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} > M_1$, where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $0 < \alpha < 1$, lying, correspondingly the below and above of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$, then the operator $\tilde{H}_1$ has a exactly two eigenvalues $z_1 = A - \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} < m_1$, and $z_2 = A - \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} > M_1$, where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $0<\alpha<1$, lying, correspondingly the below and above of the continuous spectrum of the operator $\tilde{H}_1$.

N). If $-2B<\varepsilon_2<0$, then the operator $\tilde{H}_1$ has no eigenvalues lying the outside of the continuous spectrum of the operator $\tilde{H}_1$.

Here we will not give a proof of Theorem 5. For the proof of this theorem, see the proof of Theorem 8 from [19], pp. 2750-2757.

Now we consider the two-dimensional case. In two-dimensional case, we have, what the equation $\Delta_2(z)=0$, is equivalent to the equation of the form $(\varepsilon_2+B)^2+\{\varepsilon_1 B^2+(\varepsilon_2^2+2B\varepsilon_2)(z-A)\}J(z)=0$, where

$J(z)=\int_{T^2}\frac{\mathrm{d}s_1\mathrm{d}s_2}{A+2B(\cos s_1+\cos s_2)-z}$. In this case, also $J(z)\to+0$, as $z\to-\infty$, and $J(z)\to+\infty$, as $z\to m_2-0$, and $J(z)\to-\infty$, as $z\to M_2+0$, and $J(z)\to-0$, as $z\to+\infty$. In one- and two-dimensional case the behavior of function $J(z)$ be similarly. Therefore, we have the analogously results, what is find the one-dimensional case.

We consider the three-dimensional case. We denote by W Watson integral [26]

$$W=\frac{1}{\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{3\mathrm{d}x\mathrm{d}y\mathrm{d}z}{3-\cos x-\cos y-\cos z}\simeq 1{,}516.$$

In the three-dimensional case, the integral

$\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{3+\cos s_1+\cos s_2+\cos s_2}=\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{3-\cos s_1-\cos s_2-\cos s_2}$ have the finite value. Expressing these integral via Watson integral W, and taking into account, what the measure is normalized, we have, that $J(z)=\frac{W}{6B}$.

The following Theorem describe of the exchange of the spectrum of operator $\tilde{H}_1$ in the case $\nu=3$.

Theorem 6. *Let* $\nu=3$. *Then*

A). 1). If $\varepsilon_2=-B$ and $\varepsilon_1<-6B$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z=A+\varepsilon_1$, lying the below of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_2=-B$ and $\varepsilon_1>6B$, then the operator $\tilde{H}_1$ has a unique eigenvalue $z=A+\varepsilon_1$, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

3). If $\varepsilon_2=-B$ and $-6B\le\varepsilon_1<-2B$, then the operator $\tilde{H}_1$ has no eigenvalue, lying the below of the continuous spectrum of the operator $\tilde{H}_1$.

4). If $\varepsilon_2=-B$ and $2B<\varepsilon_1\le 6B$ then the operator $\tilde{H}_1$ has no eigenvalue, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

B). 1). If $\varepsilon_2=-2B$ or $\varepsilon_2=0$ and $\varepsilon_1<0$, $\varepsilon_1\le-\frac{6B}{W}$, then the operator $\tilde{H}_1$ has a unique eigenvalue z_1, lying the below of the continuous spectrum of the operator $\tilde{H}_1$. If $\varepsilon_2=0$ and $\varepsilon_1<0$, and $-\frac{6B}{W}\le\varepsilon_1<0$, then the operator $\tilde{H}_1$ has no eigenvalue the outside of the continuous spectrum of operator $\tilde{H}_1$.

2). If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, $\varepsilon_1 \geq \frac{6B}{W}$, then the operator $\tilde{H}_1$ has a unique eigenvalue z_2, lying the above of the continuous spectrum of the operator $\tilde{H}_1$. If $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, and $0 < \varepsilon_1 \leq \frac{6B}{W}$, then the operator $\tilde{H}_1$ has no eigenvalue the outside of the continuous spectrum of operator $\tilde{H}_1$.

C). 1). If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, $E < W$, then the operator $\tilde{H}_1$ has a unique eigenvalue z, where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, lying the below of the continuous spectrum of the operator $\tilde{H}_1$. If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, $E > W$, then the operator $\tilde{H}_1$ has no eigenvalues the outside the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, $E < W$, then the operator $\tilde{H}_1$ has a unique eigenvalue $\tilde{z}$, where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, lying the above of the continuous spectrum of the operator $\tilde{H}_1$. If $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, $E > W$, then the operator $\tilde{H}_1$ has no eigenvalues the outside the continuous spectrum of the operator $\tilde{H}_1$.

D). 1). If $\varepsilon_1 = \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \frac{4}{3}W$, then the operator $\tilde{H}_1$ has a unique eigenvalue z, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_1 = -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \frac{4}{3}W$, then the operator $\tilde{H}_1$ has a unique eigenvalue $\tilde{z}$, lying the below of the continuous spectrum of the operator $\tilde{H}_1$.

E). 1). If $\varepsilon_2 > 0$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$, and the real number $\alpha > 1$, then the operator $\tilde{H}_1$ has a unique eigenvalue z, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_2 < -2B$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$, and the real number $\alpha > 1$, then the operator $\tilde{H}_1$ has a unique eigenvalue z, lying the above of the continuous spectrum of the operator $\tilde{H}_1$.

F). 1). If $\varepsilon_2 > 0$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$, and the real number $\alpha > 1$, then the operator $\tilde{H}_1$ has a unique eigenvalue z_1, lying the below of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_2 < -2B$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$, and the real number $\alpha > 1$, then the operator $\tilde{H}_1$ has a unique eigenvalue z_1, lying the below of the continuous spectrum of the operator $\tilde{H}_1$.

K). 1). If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $\left(1-\frac{\alpha}{3}\right)W < E < \left(1+\frac{\alpha}{3}\right)W$, and the real number $0 < \alpha < 1$, then the operator $\tilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the above and below of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $\left(1-\frac{\alpha}{3}\right)W < E < \left(1+\frac{\alpha}{3}\right)W$, and the real number $0 < \alpha < 1$, then the operator $\tilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the above and below of the continuous spectrum of the operator $\tilde{H}_1$.

M). 1). If $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $\left(1-\frac{\alpha}{3}\right)W < E < \left(1+\frac{\alpha}{3}\right)W$, and the real number $0 < \alpha < 1$, then the operator $\tilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the above and below of the continuous spectrum of the operator $\tilde{H}_1$.

2). If $\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $\left(1-\frac{\alpha}{3}\right)W < E < \left(1+\frac{\alpha}{3}\right)W$, and the real number $0 < \alpha < 1$, then the operator $\tilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the above and below of the continuous spectrum of the operator $\tilde{H}_1$.

N). If $-2B < \varepsilon_2 < 0$, then the operator $\tilde{H}_1$ has no eigenvalues lying the outside of the continuous spectrum of the operator $\tilde{H}_1$.

Proof. In the case $\nu = 3$, the continuous spectrum of the operator $\tilde{H}_1$ coincide with segment $[m_3, M_3] = [A-6B, A+6B]$. Expressing all integrals in the equation

$$\Delta_3(z) = \left(1 + \int_{T^3} \frac{\left(\varepsilon_1 + 2\varepsilon_2 \sum_{i=1}^3 \cos s_i\right) \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3}{A + 2B\sum_{i=1}^3 \cos s_i - z}\right)\left(1 + 6\varepsilon_2 \int_{T^3} \frac{\cos s_i \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3}{A + 2B\sum_{i=1}^3 \cos s_i - z}\right)$$
$$- 6\varepsilon_2 \int_{T^3} \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3}{A + 2B\sum_{i=1}^3 \cos s_i - z} \int_{T^3} \frac{\left(\varepsilon_1 + 2\varepsilon_2 \sum_{i=1}^3 \cos s_i\right) \cos s_1 \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3}{A + 2B\sum_{i=1}^3 \cos s_i - z}$$
$$= 0$$

through the integral $J(z) = \int_{T^3} \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3}{A + 2B\sum_{i=1}^3 \cos s_i - z}$, we find that the equation $\Delta_3(z) = 0$ is equivalent to the equation

$$\left[\varepsilon_1 B^2 + \left(\varepsilon_2^2 + 2B\varepsilon_2\right)(z - A)\right] J(z) + (B + \varepsilon_2)^2 = 0. \tag{12}$$

Moreover, the function $J(z) = \int_{T^3} \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3}{A + 2B\sum_{i=1}^3 \cos s_i - z}$ is a differentiable function on the set $\mathbb{R}\setminus[m_3, M_3]$, in addition,

$J'(z) = \int_{T^3} \frac{ds_1 ds_2 ds_3}{\left[A + 2B\sum_{i=1}^{3}\cos s_i - z\right]^2} > 0$, $z \notin [m_3, M_3]$.

Thus the function $J(z)$ is an monotone increasing function on $(-\infty, m_3)$ and on $(M_3, +\infty)$. Furthermore, in the three-dimensional case $J(z) \to +0$ at $z \to -\infty$, and $J(z) = \frac{W}{6B}$ as $z = A - 6B$, and $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A + 6B$.

If $\varepsilon_1 B^2 + (\varepsilon_2^2 + 2B\varepsilon_2)(z - A) \neq 0$ then from (12) follows that

$$J(z) = -\frac{(B + \varepsilon_2)^2}{\varepsilon_1 B^2 + (\varepsilon_2^2 + 2B\varepsilon_2)(z - A)}.$$

The function $\psi(z) = -\frac{(B + \varepsilon_2)^2}{\varepsilon_1 B^2 + (\varepsilon_2^2 + 2B\varepsilon_2)(z - A)}$ has a point of asymptotic discontinuity $z_0 = A - \frac{B^2 \varepsilon_1}{\varepsilon_2^2 + 2B\varepsilon_2}$. Since $\psi'(z) = \frac{(B + \varepsilon_2)^2 (\varepsilon_2^2 + 2B\varepsilon_2)}{\left[\varepsilon_1 B^2 + (\varepsilon_2^2 + 2B\varepsilon_2)(z - A)\right]^2}$ for all $z \neq z_0$ it follows that the function $\psi(z)$ is an monotone increasing (decreasing) function on $(-\infty, z_0)$ and on $(z_0, +\infty)$ in the case $\varepsilon_2^2 + 2B\varepsilon_2 > 0$ (respectively, $\varepsilon_2^2 + 2B\varepsilon_2 < 0$), in addition, and if $\varepsilon_2 > 0$, or $\varepsilon_2 < -2B$, then $\psi(z) \to +0$ as $z \to -\infty$, $\psi(z) \to +\infty$ as $z \to z_0 - 0$, $\psi(z) \to -\infty$ as $z \to z_0 + 0$, $\psi(z) \to -0$ as $z \to +\infty$ (respectively, if $-2B < \varepsilon_2 < 0$, then $\psi(z) \to -0$ as $z \to -\infty$, $\psi(z) \to -\infty$ as $z \to z_0 - 0$, $\psi(z) \to +\infty$ as $z \to z_0 + 0$, $\psi(z) \to +0$ as $z \to +\infty$).

A). If $\varepsilon_2 = -B$ and $\varepsilon_1 < -6B$ (respectively, $\varepsilon_2 = -B$ and $\varepsilon_1 > 6B$), then the equation for eigenvalues and eigenfunctions (12) has the form:

$$\left\{\varepsilon_1 B^2 - B^2 (z - A)\right\} J(z) = 0. \tag{13}$$

It is clear, that $J(z) \neq 0$ for the values $z \notin \sigma_{cont}(\tilde{H}_1)$. Therefore, $\varepsilon_1 - z + A = 0$, *i.e.*, $z = A + \varepsilon_1$. If $\varepsilon_1 < -6B$, then this eigenvalue lying the below of the continuous spectrum of operator $\tilde{H}_1$, if $\varepsilon_1 > 6B$, then this eigenvalue lying the above of the continuous spectrum of operator $\tilde{H}_1$. If $-6B \le \varepsilon_1 < -2B$ (respectively, $2B < \varepsilon_1 \le 6B$), then this eigenvalue not lying in the outside of the continuous spectrum of operator $\tilde{H}_1$.

B). If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$ (respectively, $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$), then the equation for the eigenvalues and eigenfunctions has the form $\varepsilon_1 B^2 J(z) + B^2 = 0$, that is, $J(z) = -\frac{1}{\varepsilon_1}$. In the three-dimensional case $J(z) \to +0$ as $z \to -\infty$, and $J(z) = \frac{W}{6B}$ as $z = A - 6B$, and $J(z) \to -0$ as

$z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A + 6B$. Therefore, in order to the equation $J(z) = -\frac{1}{\varepsilon_1}$ in the below (respectively, above) of continuous spectrum of operator $\tilde{H}_1$ have the solution, one should implements the inequality $-\frac{1}{\varepsilon_1} < \frac{W}{6B}$ (respectively, $-\frac{1}{\varepsilon_1} > -\frac{W}{6B}$), *i.e.*, $\varepsilon_1 < -\frac{6B}{W}$, $\varepsilon_1 < 0$ (respectively, $\varepsilon_1 > \frac{6B}{W}$, $\varepsilon_1 > 0$). If $-\frac{6B}{W} < \varepsilon_1 < 0$ (respectively, $0 < \varepsilon_1 < \frac{6B}{W}$), then the operator $\tilde{H}_1$ has no eigenvalues the outside the continuous spectrum of operator $\tilde{H}_1$.

C). If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$ (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$), then the equation for the eigenvalues and eigenfunctions take in the form

$$\left(\varepsilon_2^2 + 2B\varepsilon_2\right)(z - A)J(z) = -(B + \varepsilon_2)^2,$$

or

$$J(z) = -\frac{(B + \varepsilon_2)^2}{\left(\varepsilon_2^2 + 2B\varepsilon_2\right)(z - A)}.$$

Denote $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. Then $J(z) = -\frac{E}{z - A}$, or $J(z) = \frac{E}{A - z}$. In the three-dimensional case $J(z) \to +0$ as $z \to -\infty$, and $J(z) = \frac{W}{6B}$ as $z = A - 6B$, and $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A + 6B$. Therefore, in order to the equation $J(z) = -\frac{E}{z - A}$ in the below (respectively, above) of continuous spectrum of operator $\tilde{H}_1$ have the solution, one should implements the inequality $\frac{E}{6B} < \frac{W}{6B}$ (respectively, $-\frac{E}{6B} > -\frac{W}{6B}$), *i.e.*, $E < W$. If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, $E > W$ (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, $E > W$), then the operator $\tilde{H}_1$ has no eigenvalues the outside the continuous spectrum of operator $\tilde{H}_1$.

D). If $\varepsilon_1 = \frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$, then the equation for eigenvalues and eigenfunctions has the form

$$\left(\varepsilon_2^2 + 2B\varepsilon_2\right)(z - A + 2B)J(z) = -(B + \varepsilon_2)^2,$$

from this we have equation in the form:

$$J(z) = -\frac{(B + \varepsilon_2)^2}{\left(\varepsilon_2^2 + 2B\varepsilon_2\right)(z - A + 2B)}. \tag{14}$$

We denote $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. In the first we consider the Equation (14) in the below of continuous spectrum of operator $\tilde{H}_1$. In the below of continuous spectrum of operator $\tilde{H}_1$, the function $\frac{E}{A-z-2B} \to +0$, as $z \to -\infty$, $\frac{E}{A-z-2B} = \frac{E}{4B}$, as $z = A-6B$, and in the three-dimensional case $J(z) \to +0$ as $z \to -\infty$, and $J(z) = \frac{W}{6B}$ as $z = A-6B$, and $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A+6B$. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the equation $J(z) = \frac{E}{A-z-2B}$ has a unique solution, if $\frac{E}{4B} > \frac{W}{6B}$, *i.e.*, $E > \frac{2W}{3}$. This inequality incorrectly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, this equation has no solution.

We now consider the equation for eigenvalues and eigenfunctions $J(z) = -\frac{E}{z-A+2B}$, in the above of continuous spectrum of operator $\tilde{H}_1$. In the above of continuous spectrum of operator $\tilde{H}_1$, the function $\frac{E}{A-z-2B} \to -0$, as $z \to +\infty$, $\frac{E}{A-z-2B} = -\frac{E}{8B}$, as $z = A+6B$, and in the three-dimensional case $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A+6B$. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the equation $J(z) = \frac{E}{A-z-2B}$ has a unique solution, if $-\frac{E}{8B} > -\frac{W}{6B}$, *i.e.*, $E < \frac{4W}{3}$. This inequality correctly. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, this equation has a unique solution *z*.

If $\varepsilon_1 = -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, then the equation for eigenvalues and eigenfunctions has the form

$$(\varepsilon_2^2 + 2B\varepsilon_2)(z-A-2B)J(z) = -(B+\varepsilon_2)^2,$$

from this we have the equation in the form (14).

We denote $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. In the first we consider the equation (14) in the below of continuous spectrum of operator $\tilde{H}_1$. In the below of continuous spectrum of operator $\tilde{H}_1$, the function $\frac{E}{A-z+2B} \to +0$, as $z \to -\infty$, $\frac{E}{A-z+2B} = \frac{E}{8B}$, as $z = A-6B$, and in the three-dimensional case $J(z) \to +0$ as $z \to -\infty$, and $J(z) = \frac{W}{6B}$ as $z = A-6B$, and $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A+6B$. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the equation $J(z)=\frac{E}{A-z+2B}$ has a unique solution, if $\frac{E}{8B}<\frac{W}{6B}$, *i.e.*, $E<\frac{4W}{3}$. This inequality correctly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, this equation has a unique solution.

We now consider the equation for eigenvalues and eigenfunctions $J(z)=-\frac{E}{z-A-2B}$, in the above of continuous spectrum of operator $\tilde{H}_1$. In the above of continuous spectrum of operator $\tilde{H}_1$, the function $\frac{E}{A-z+2B}\to -0$, as $z\to +\infty$, $\frac{E}{A-z+2B}=-\frac{E}{4B}$, as $z=A+6B$, and in the three-dimensional case $J(z)\to -0$ as $z\to +\infty$, and $J(z)=-\frac{W}{6B}$ as $z=A+6B$. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the equation $J(z)=\frac{E}{A-z+2B}$ has a unique solution, if $-\frac{E}{4B}>-\frac{W}{6B}$, *i.e.*, $E<\frac{2W}{3}$. This inequality incorrectly. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, this equation has no solution.

E). If $\varepsilon_2>0$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, (respectively, $\varepsilon_2<-2B$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$), then consider necessary, that $\varepsilon_1=\alpha\times\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, where $\alpha>1-$ real number. Then the equation for eigenvalues and eigenfunctions has the form $\left\{\alpha\times\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}\times B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)\right\}J(z)+\left(B+\varepsilon_2\right)^2=0$, or $\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A+2\alpha B)J(z)+\left(B+\varepsilon_2\right)^2=0$. From this $J(z)=-\frac{\left(B+\varepsilon_2\right)^2}{\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A+2\alpha B)}$. We denote $E=\frac{\left(B+\varepsilon_2\right)^2}{\varepsilon_2^2+2B\varepsilon_2}$, then $J(z)=-\frac{E}{z-A+2\alpha B}$. In the first we consider this equation in the below of the continuous spectrum of operator $\tilde{H}_1$. Then $J(z)\to +0$, as $z\to -\infty$, $J(z)=\frac{W}{6B}$, as $z=A-6B$, $-\frac{E}{z-A+2\alpha B}\to +0$, as $z\to -\infty$, and $-\frac{E}{z-A+2\alpha B}=\frac{E}{(6-2\alpha)B}$, as $z=A-6B$. The equation $J(z)=-\frac{E}{z-A+2\alpha B}$ have a unique solution, if $\frac{E}{(6-2\alpha)B}<\frac{W}{6B}$. From here $E<\frac{(3-\alpha)W}{3}$. This inequality is incorrect. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has no eigenvalues.

The above of continuous spectrum of operator $\tilde{H}_1$, we have the $J(z)\to -0$, if $z\to +\infty$, $J(z)=-\frac{W}{6B}$, if $z=A-6B$. Besides, $-\frac{E}{z-A+2\alpha B}\to -0$, as

$z \to +\infty$, $-\frac{E}{z-A+2\alpha B} = -\frac{E}{6B+2\alpha B}$, if $z = A+6B$.

The equation $J(z) = -\frac{E}{z-A+2\alpha B}$ have a unique solution, if $-\frac{E}{(6+2\alpha)B} > -\frac{W}{6B}$. From here $E < \frac{(3+\alpha)W}{3}$. This inequality is correctly. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues z_1.

F). If $\varepsilon_2 > 0$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$), then we assume that $\varepsilon_1 = -\alpha \times \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, where $\alpha > 1-$ real number. The equation for eigenvalues and eigenfunctions take in the form

$$(\varepsilon_2^2+2B\varepsilon_2)(z-A-2\alpha B)J(z) = -(B+\varepsilon_2)^2.$$

From here

$$J(z) = -\frac{(B+\varepsilon_2)^2}{(\varepsilon_2^2+2B\varepsilon_2)(z-A-2\alpha B)}.$$

The introduce notation $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. Then we have the equation in the form:

$$J(z) = -\frac{E}{z-A-2\alpha B}. \tag{15}$$

In the below of the continuous spectrum of operator $\tilde{H}_1$, we have the equation $J(z) = \frac{E}{A-z+2\alpha B}$. In the below of continuous spectrum of operator $\tilde{H}_1$, $-\frac{E}{z-A-2\alpha B} \to +0$, as $z \to -\infty$, $-\frac{E}{z-A-2\alpha B} = \frac{E}{6B+2\alpha B}$, as $z = A-6B$.

The equation $J(z) = -\frac{E}{z-A+2\alpha B}$ have a unique solution, if $\frac{E}{(6+2\alpha)B} < \frac{W}{6B}$. From here $E < \frac{(3+\alpha)W}{3}$. This inequality is correctly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues.

In the above of continuous spectrum of operator $\tilde{H}_1$, $-\frac{E}{z-A-2\alpha B} \to -0$, as $z \to -\infty$, $-\frac{E}{z-A-2\alpha B} = -\frac{E}{6B-2\alpha B}$, as $z = A+6B$. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues, if $-\frac{E}{6B-2\alpha B} > -\frac{W}{6B}$. From here $E < \frac{(3-\alpha)W}{3}$, what is incorrectly.

Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has no eigenvalues.

K). If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), the we take $\varepsilon_1 = \alpha \times \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, where $0 < \alpha < 1-$ positive real number. Then the equation for eigenvalues and eigenfunctions has the form:

$$(\varepsilon_2^2 + 2B\varepsilon_2)(z - A + 2\alpha B) J(z) = -(B + \varepsilon_2)^2, 0 < \alpha < 1. \quad (16)$$

We denote $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. Then the Equation (16) receive the form

$$J(z) = -\frac{E}{z - A + 2\alpha B}.$$

In the below of continuous spectrum of operator $\tilde{H}_1$, we have $-\frac{E}{z - A + 2\alpha B} \to +0$, as $z \to -\infty$, and $-\frac{E}{z - A + 2\alpha B} = \frac{E}{2B(3-\alpha)}$, as $z = A - 6B$. The equation $J(z) = -\frac{E}{z - A + 2\alpha B}$ have a unique solution the below of continuous spectrum of operator $\tilde{H}_1$, if $\frac{E}{(6-2\alpha)B} > \frac{W}{6B}$. From here $E > \frac{(3-\alpha)W}{3}$. This inequality is correctly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues z_1.

The above of continuous spectrum of operator $\tilde{H}_1$, we have $-\frac{E}{z - A + 2\alpha B} \to -0$, as $z \to +\infty$, and $-\frac{E}{z - A + 2\alpha B} = -\frac{E}{2B(3+\alpha)}$, as $z = A + 6B$. The equation $J(z) = -\frac{E}{z - A + 2\alpha B}$ have a unique solution the above of operator $\tilde{H}_1$, if $-\frac{E}{2B(3+\alpha)} > -\frac{W}{6B}$, *i.e.*, $E < \frac{(3+\alpha)W}{3}$. This inequality is correctly.

Consequently, in this case the operator $\tilde{H}_1$ have two eigenvalues z_1 and z_2, lying the below and above of continuous spectrum of operator $\tilde{H}_1$.

M). If $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ (respectively, $\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$), the we take $\varepsilon_1 = -\alpha \times \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, where $0 < \alpha < 1-$ positive real number. Then the equation for eigenvalues and eigenfunctions has the form:

$$(\varepsilon_2^2 + 2B\varepsilon_2)(z - A - 2\alpha B) J(z) = -(B + \varepsilon_2)^2, 0 < \alpha < 1. \quad (17)$$

We denote $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. Then the Equation (17) receive the form

$$J(z) = -\frac{E}{z - A - 2\alpha B}.$$

In the below of continuous spectrum of operator $\tilde{H}_1$, we have $-\frac{E}{z-A-2\alpha B} \to +0$, as $z \to -\infty$, and $-\frac{E}{z-A-2\alpha B} = \frac{E}{2B(3+\alpha)}$, as $z = A - 6B$. The equation $J(z) = -\frac{E}{z-A-2\alpha B}$ have a unique solution the below of continuous spectrum of operator $\tilde{H}_1$, if $\frac{E}{(6+2\alpha)B} < \frac{W}{6B}$. From here $E < \frac{(3+\alpha)W}{3}$. This inequality is correctly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues z_1.

The above of continuous spectrum of operator $\tilde{H}_1$, we have $-\frac{E}{z-A-2\alpha B} \to -0$, as $z \to +\infty$, and $-\frac{E}{z-A-2\alpha B} = -\frac{E}{2B(3-\alpha)}$, as $z = A + 6B$. The equation $J(z) = -\frac{E}{z-A-2\alpha B}$ have a unique solution the above of continuous spectrum of operator $\tilde{H}_1$, if $-\frac{E}{2B(3-\alpha)} < -\frac{W}{6B}$, *i.e.*, $E > \frac{(3-\alpha)W}{3}$. This inequality is correctly.

Consequently, in this case the operator $\tilde{H}_1$ have two eigenvalues z_1 and z_2, lying the below and above of $\tilde{H}_1$.

N). If $-2B < \varepsilon_2 < 0$, then $\varepsilon_2^2 + 2B\varepsilon_2 < 0$, and the function $\psi(z) = -\frac{(B+\varepsilon_2)^2}{\varepsilon_1 B + (\varepsilon_2^2 + 2B\varepsilon_2)(z-A)}$ is a decreasing function in the intervals $(-\infty, z_0)$ and $(z_0, +\infty)$; By, $z \to -\infty$ the function $\psi(z) \to -0$, and by $z \to z_0 - 0$, the function $\psi(z) \to -\infty$, and by $z \to +\infty$, $\psi(z) \to +0$, and by $z \to z_0 + 0$, $\psi(z) \to +\infty$. The function $J(z) \to +0$, by $z \to -\infty$, and by $z = A - 6B$, the function $J(z) = \frac{W}{6B}$, and by $z = A + 6B$, the function $J(z) = \frac{W}{6B}$, by $z \to +\infty$, the function $J(z) \to -0$. Therefore, the equation $\psi(z) = J(z)$, that's impossible the solutions in the outside the continuous spectrum of operator $\tilde{H}_1$. Therefore, in this case, the operator $\tilde{H}_1$ has no eigenvalues lying the outside of the continuous spectrum of the operator $\tilde{H}_1$. □

From obtaining results is obviously, that the spectrum of operator $\tilde{H}_1$ is consists from continuous spectrum and no more than two eigenvalues.

Taking into account that the function $f(\lambda, \mu, \gamma, \theta)$ is antisymmetric, and using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces [27], we can verify that the operator ${}^2\widetilde{H}_t^1$ can be represented in the form

$$ {}^{2}\tilde{H}_{t}^{1}\,{}^{2}\psi_{t}^{1} = \left\{\tilde{H}_{1}\otimes I + I\otimes\tilde{H}_{1} + K(\lambda,\mu)\right\}\otimes I\otimes I + I\otimes I\otimes\left\{\tilde{H}_{1}\otimes I + I\otimes\tilde{H}_{1}\right\} \quad (18) $$

where

$$ \left(\tilde{H}_{1}f\right)(\lambda) = \left\{A + 2B\sum_{i=1}^{\nu}\cos\lambda_{i}\right\}f(\lambda) + (A_{0}-A)\int_{T^{\nu}}f(s)\mathrm{d}s $$
$$ + 2B\int_{T^{\nu}}\sum_{i=1}^{\nu}\left[\cos\lambda_{i} + \cos s_{i}\right]f(s)\mathrm{d}s, $$

and $K(\lambda,\mu) = U\int_{T^{\nu}} f(s,\lambda+\mu-s)\mathrm{d}s + (U_{0}-U)\int_{T^{\nu}}\int_{T^{\nu}} f(s,k)\mathrm{d}s\mathrm{d}k$, and I is the unit operator in the space $\tilde{\mathcal{H}}_{1}$.

The spectrum of the operator $A\otimes I + I\otimes B$, where A and B are densely defined bounded linear operators, was studied in [28] [29] [30]. Explicit formulas were given there that express the essential spectrum $\sigma_{ess}(A\otimes I + I\otimes B)$ and discrete spectrum $\sigma_{disc}(A\otimes I + I\otimes B)$ of operator $A\otimes I + I\otimes B$ in terms of the spectrum $\sigma(A)$ and the discrete spectrum $\sigma_{disc}(A)$ of A and in terms of the spectrum $\sigma(B)$ and the discrete spectrum $\sigma_{disc}(B)$ of B:

$$ \sigma_{disc}(A\otimes I + I\otimes B) $$
$$ = \left\{\sigma(A)\setminus\sigma_{ess}(A) + \sigma(B)\setminus\sigma_{ess}(B)\right\}\setminus\left\{\left(\sigma_{ess}(A)+\sigma(B)\right)\cup\left(\sigma(A)+\sigma_{ess}(B)\right)\right\}, \quad (19) $$

$$ \sigma_{ess}(A\otimes I + I\otimes B) = \left(\sigma_{ess}(A)+\sigma(B)\right)\cup\left(\sigma(A)+\sigma_{ess}(B)\right). \quad (20) $$

It is clear that $\sigma(A\otimes I + I\otimes B) = \left\{\lambda+\mu : \lambda\in\sigma(A), \mu\in\sigma(B)\right\}$.

Consequently, we must investigate in first the spectrum of the operators $\tilde{H}_{1}$.

4. Structure of the Essential Spectrum and Discrete Spectrum of Operator ${}^{2}\tilde{H}_{t}^{1}$

Consequently, the operator represented of the form

$$ {}^{2}\tilde{H}_{t}^{1} = \left\{\tilde{H}_{2}^{t} + K(\lambda,\mu)\right\}\otimes I\otimes I + I\otimes I\otimes\tilde{H}_{2}^{t}, \quad (21) $$

where $\tilde{H}_{2}^{t} = \tilde{H}_{1}\otimes I + I\otimes\tilde{H}_{1}$ are the energy operator of two-electron systems in the impurity Hubbard model in triplet state.

We now, using the obtained results and representation (18) and (21), we first describe the structure of essential spectrum and discrete spectrum of the operator $\tilde{H}_{2}^{s} = \tilde{H}_{2}^{t} + K(\lambda,\mu)$.

From the beginning, we consider the operator $\tilde{H}(U) = \tilde{H}_{2}^{t} + K_{1}$.

Since, the family of the operators $\tilde{H}(U)$ is the family of bounded operators, that the $\tilde{H}(U)$ is the family of bounded operator valued analytical functions.

Therefore, in these family, one can the apply the Kato-Rellix theorem.

Theorem 7. *(Kato-Rellix theorem))* [27].

Let $T(\beta)$ is the analytical family in the terms of Kato. Let E_{0} is a nondegenerate eigenvalue of $T(\beta_{0})$. Then as β, near to β_{0}, the exist exactly one point $E(\beta)\in\sigma(T(\beta))$ the near E_{0} and this point is isolated and nondegenerated. $E(\beta)$ is an analytical function of β as β, the near to β_{0}, and exist the analytical eigenvector $\Omega(\beta)$ as β the near to β_{0}. If the as real $\beta-\beta_{0}$ the operator $T(\beta)$ is a self-adjoint operator, then $\Omega(\beta)$ can selected thus, that it will be normalized of real $\beta-\beta_0$.

Since, the operator $\tilde{H}_2^t$ has a nondegenerate eigenvalue, such as, the near of eigenvalue $2z_1$ of the operator $\tilde{H}_2^t$, the operator $\tilde{H}(U)$ as U, near $U_0=0$, has a exactly one eigenvalue $E(U)\in\sigma(\tilde{H}(U))$ the near $2z_1$ and this point is isolated and nondegenerated. The $E(U)$ is a analytical function of U as U, the near to $U_0=0$.

As the large values the existence no more one additional eigenvalue of the operator $\tilde{H}(U)$ is following from the same, what the perturbation $(K_1\tilde{f})(\lambda,\mu)=U\int_{T^\nu} f(s,\lambda+\mu-s)\mathrm{d}s$ is the one-dimensional operator.

A new we consider the family of operators $\tilde{H}(\varepsilon_3)=\tilde{H}(U)+K_2$.

As, the operator $\tilde{H}(U)$ has a nondegenerate eigenvalue, consequently, the near of eigenvalue $E(U)$ the operator $\tilde{H}(U)$, operator $\tilde{H}(\varepsilon_3)$ as ε_3, the near of $\varepsilon_3=0$, has a exactly one eigenvalue $E(\varepsilon_3)\in\sigma(\tilde{H}(\varepsilon_3))$ the near $E(U)$ and this point is the isolated and nondegenerated. The $E(\varepsilon_3)$ is a analytical function of ε_3, as ε_3, the near to $\varepsilon_3=0$.

Later on via z_3, and z_4 we denote the additional eigenvalues of operator $\tilde{H}_2^s$. Thus, we prove the next theorems, the described the spectra of operator $\tilde{H}_2^s$.

Now, using the obtained results (Theorem 5 and 6) and representation (18), and (21), we describe the structure of the essential spectrum and discrete spectrum of the operator ${}^2\tilde{H}_t^1$.

Theorem 8. *Let* $\nu=1$. *Then*

A). If $\varepsilon_2=-B$ and $\varepsilon_1<-2B$, or if $\varepsilon_2=-B$ and $\varepsilon_1>2B$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eight segments:

$$\begin{aligned}\sigma_{ess}({}^2\tilde{H}_t^1)=&[4A-8B,4A+8B]\cup[3A-6B+z,3A+6B+z]\\ &\cup[2A-4B+2z,2A+4B+2z]\cup[A-2B+3z,A+2B+3z]\\ &\cup[2A-4B+z_3,2A+4B+z_3]\cup[A-2B+z+z_3,A+2B+z+z_3]\\ &\cup[2A-4B+z_4,2A+4B+z_4]\cup[A-2B+z+z_4,A+2B+z+z_4]\end{aligned}$$

, and the discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}({}^2\tilde{H}_t^1)=\{4z,2z+z_3,2z+z_4\}$, where $z=A+\varepsilon_1$, and z_3 and z_4 are the additional eigenvalues of the operator $\tilde{H}_2^s$.

B). 1). If $\varepsilon_2=-2B$ or $\varepsilon_2=0$ and $\varepsilon_1<0$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eight segments:

$$\begin{aligned}\sigma_{ess}({}^2\tilde{H}_t^1)=&[4A-8B,4A+8B]\cup[3A-6B+z,3A+6B+z]\\ &\cup[2A-4B+2z,2A+4B+2z]\cup[A-2B+3z,A+2B+3z]\\ &\cup[2A-4B+z_3,2A+4B+z_3]\cup[A-2B+z+z_3,A+2B+z+z_3]\\ &\cup[2A-4B+z_4,2A+4B+z_4]\cup[A-2B+z+z_4,A+2B+z+z_4]\end{aligned}$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}({}^2\tilde{H}_t^1)=\{4z,2z+z_3,2z+z_4\}$, where $z=A-\sqrt{4B^2+\varepsilon_1^2}$.

2). If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eight segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right) &= [4A-8B, 4A+8B] \cup [3A-6B+z, 3A+6B+z] \\ &\cup [2A-4B+2z, 2A+4B+2z] \cup [A-2B+3z, A+2B+3z] \\ &\cup [2A-4B+z_3, 2A+4B+z_3] \cup [A-2B+z+z_3, A+2B+z+z_3] \\ &\cup [2A-4B+z_4, 2A+4B+z_4] \cup [A-2B+z+z_4, A+2B+z+z_4]\end{aligned}$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \{4z, 2z+z_3, 2z+z_4\}$, where $z = A + \sqrt{4B^2 + \varepsilon_1^2}$.

C). If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$ or $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of sixteen segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right) &= [4A-8B, 4A+8B] \cup [3A-6B+z_1, 3A+6B+z_1] \\ &\cup [3A-6B+z_2, 3A+6B+z_2] \cup [2A-4B+2z_1, 2A+4B+z_1] \\ &\cup [2A-4B+2z_2, 2A+4B+z_2] \cup [2A-4B+z_1+z_2, 2A+4B+z_1+z_2] \\ &\cup [A-2B+3z_1, A+2B+3z_1] \cup [A-2B+3z_2, A+2B+3z_2] \\ &\cup [A-2B+z_1+2z_2, A+2B+z_1+2z_2] \cup [A-2B+2z_1+z_2, A+2B+2z_1+z_2] \\ &\cup [2A-4B+z_3, 2A+4B+z_3] \cup [2A-4B+z_4, 2A+4B+z_4] \\ &\cup [A-2B+z_1+z_3, 2A+4B+z_1+z_3] \cup [A-2B+z_1+z_4, 2A+4B+z_1+z_4] \\ &\cup [A-2B+z_2+z_3, 2A+4B+z_2+z_3] \cup [A-2B+z_2+z_4, 2A+4B+z_2+z_4]\end{aligned},$$

and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of eleven eigenvalues:

$$\begin{aligned}\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \{&4z_1, 3z_1+z_2, 4z_2, 2z_1+2z_2, z_1+3z_2, 2z_1+z_3, z_1+z_2+z_3, \\ &2z_2+z_3, 2z_1+z_4, z_1+z_2+z_4, 2z_2+z_4\}\end{aligned},$$

where

$z_1 = A - \frac{2BE}{\sqrt{E^2-1}}$, and $z_2 = A + \frac{2BE}{\sqrt{E^2-1}}$, and $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$.

D). 1). If $\varepsilon_1 = \frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eight segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right) &= [4A-8B, 4A+8B] \cup [3A-6B+z, 3A+6B+z] \\ &\cup [2A-4B+2z, 2A+4B+2z] \cup [A-2B+3z, A+2B+3z] \\ &\cup [2A-4B+z_3, 2A+4B+z_3] \cup [A-2B+z+z_3, A+2B+z+z_3] \\ &\cup [2A-4B+z_4, 2A+4B+z_4] \cup [A-2B+z+z_4, A+2B+z+z_4]\end{aligned}$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \{4z, 2z+z_3, 2z+z_4\}$, where $z = A + \frac{2B\left(E^2+1\right)}{E^2-1}$, and

$E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$.

2). If $\varepsilon_1 = -\frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eight segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right)=[4A-8B,4A+8B]\cup[3A-6B+z,3A+6B+z]$$
$$\cup[2A-4B+2z,2A+4B+2z]\cup[A-2B+3z,A+2B+3z]$$
$$\cup[2A-4B+z_3,2A+4B+z_3]\cup[A-2B+z+z_3,A+2B+z+z_3]$$
$$\cup[2A-4B+z_4,2A+4B+z_4]\cup[A-2B+z+z_4,A+2B+z+z_4]$$
, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4z,2z+z_3,2z+z_4\}$, where $z=A-\dfrac{2B\left(E^2+1\right)}{E^2-1}$, and $E=\dfrac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$.

E). If $\varepsilon_2>0$ and $\varepsilon_1>\dfrac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, or if $\varepsilon_2<-2B$ and $\varepsilon_1>\dfrac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eight segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right)=[4A-8B,4A+8B]\cup[3A-6B+z,3A+6B+z]$$
$$\cup[2A-4B+2z,2A+4B+2z]\cup[A-2B+3z,A+2B+3z]$$
$$\cup[2A-4B+z_3,2A+4B+z_3]\cup[A-2B+z+z_3,A+2B+z+z_3]$$
$$\cup[2A-4B+z_4,2A+4B+z_4]\cup[A-2B+z+z_4,A+2B+z+z_4]$$
, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4z,2z+z_3,2z+z_4\}$, where $z=A+\dfrac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $E=\dfrac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and the real number $\alpha>1$.

F). If $\varepsilon_2>0$ and $\varepsilon_1<-\dfrac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, or if $\varepsilon_2<-2B$ and $\varepsilon_1<-\dfrac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eight segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right)=[4A-8B,4A+8B]\cup[3A-6B+z,3A+6B+z]$$
$$\cup[2A-4B+2z,2A+4B+2z]\cup[A-2B+3z,A+2B+3z]$$
$$\cup[2A-4B+z_3,2A+4B+z_3]\cup[A-2B+z+z_3,A+2B+z+z_3]$$
$$\cup[2A-4B+z_4,2A+4B+z_4]\cup[A-2B+z+z_4,A+2B+z+z_4]$$
, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4z,2z+z_3,2z+z_4\}$, where $z=A-\dfrac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $E=\dfrac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and the real number $\alpha>1$.

K). If $\varepsilon_2>0$ and $0<\varepsilon_1<\dfrac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, or if $\varepsilon_2<-2B$ and

$0<\varepsilon_1<\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of sixteen segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right)=&\left[4A-8B,4A+8B\right]\cup\left[3A-6B+z_1,3A+6B+z_1\right]\\ &\cup\left[3A-6B+z_2,3A+6B+z_2\right]\cup\left[2A-4B+2z_1,2A+4B+2z_1\right]\\ &\cup\left[2A-4B+2z_2,2A+4B+2z_2\right]\cup\left[2A-4B+z_1+z_2,2A+4B+z_1+z_2\right]\\ &\cup\left[A-2B+3z_1,A+2B+3z_1\right]\cup\left[A-2B+3z_2,A+2B+3z_2\right]\\ &\cup\left[A-2B+2z_1+z_2,A+2B+2z_1+z_2\right]\cup\left[A-2B+z_1+2z_2,A+2B+z_1+2z_2\right],\\ &\cup\left[2A-4B+z_3,2A+4B+z_3\right]\cup\left[2A-4B+z_4,2A+4B+z_4\right]\\ &\cup\left[A-2B+z_1+z_3,A+2B+z_1+z_3\right]\cup\left[A-2B+z_1+z_4,A+2B+z_1+z_4\right]\\ &\cup\left[A-2B+z_2+z_3,A+2B+z_2+z_3\right]\cup\left[A-2B+z_2+z_4,A+2B+z_2+z_4\right]\end{aligned}$$

and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of eleven eigenvalues:

$$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4z_1,3z_1+z_2,2z_1+2z_2,z_1+3z_2,4z_2,2z_1+z_3,z_1+z_2+z_3,\\ 2z_2+z_3,2z_1+z_4,2z_2+z_4,z_1+z_2+z_4\}$$, where

$z_1=A+\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $z_2=A+\frac{2B\left(\alpha-E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$, and

$E=\frac{\left(B+\varepsilon_2\right)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and the real number $0<\alpha<1$.

M). If $\varepsilon_2>0$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$, or if $\varepsilon_2<-2B$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of sixteen segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right)=&\left[4A-8B,4A+8B\right]\cup\left[3A-6B+z_1,3A+6B+z_1\right]\\ &\cup\left[3A-6B+z_2,3A+6B+z_2\right]\cup\left[2A-4B+2z_1,2A+4B+2z_1\right]\\ &\cup\left[2A-4B+2z_2,2A+4B+2z_2\right]\cup\left[2A-4B+z_1+z_2,2A+4B+z_1+z_2\right]\\ &\cup\left[A-2B+3z_1,A+2B+3z_1\right]\cup\left[A-2B+3z_2,A+2B+3z_2\right]\\ &\cup\left[A-2B+2z_1+z_2,A+2B+2z_1+z_2\right]\cup\left[A-2B+z_1+2z_2,A+2B+z_1+2z_2\right],\\ &\cup\left[2A-4B+z_3,2A+4B+z_3\right]\cup\left[2A-4B+z_4,2A+4B+z_4\right]\\ &\cup\left[A-2B+z_1+z_3,A+2B+z_1+z_3\right]\cup\left[A-2B+z_1+z_4,A+2B+z_1+z_4\right]\\ &\cup\left[A-2B+z_2+z_3,A+2B+z_2+z_3\right]\cup\left[A-2B+z_2+z_4,A+2B+z_2+z_4\right]\end{aligned}$$

and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of eleven eigenvalues:

$$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4z_1,3z_1+z_2,2z_1+2z_2,z_1+3z_2,4z_2,2z_1+z_3,z_1+z_2+z_3,\\ 2z_2+z_3,2z_1+z_4,2z_2+z_4,z_1+z_2+z_4\}$$, where

$z_1=A+\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $z_2=A+\frac{2B\left(\alpha-E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$, and

$E=\frac{\left(B+\varepsilon_2\right)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and the real number $0<\alpha<1$.

N). If $-2B < \varepsilon_2 < 0$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of three segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right) = [4A-8B, 4A+8B] \cup [2A-4B+z_3, 2A+4B+z_3] \cup [2A-4B+z_4, 2A+4B+z_4]$$, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of empty set: $\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \varnothing$.

Proof. A). From the representation (18), (21) and the formulas (19) and (20), and the Theorem 5, follow the in one-dimensional case, the continuous spectrum of the operator $\tilde{H}_1$ is consists $\sigma_{cont}\left(\tilde{H}_1\right) = [A-2B, A+2B]$, and the discrete spectrum of the operator $\tilde{H}_1$ is consists of unique eigenvalue $z = A + \varepsilon_1$. The operator K is a two-dimensional operator. Therefore, the essential spectrum of the operators $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$ and $\tilde{H}_2^s$ coincide (see. chapter XIII, paragraph 4, in [22]) and is consists from segments $[2A-4B, 2A+4B]$, and $[A-2B+z, A+2B+z]$. Of extension the two-dimensional operator K to the operator $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$ can appear no more then two additional eigenvalues z_3 and z_4. These give the statement A) of the Theorem 8.

B). In this case the operator $\tilde{H}_1$ has a one eigenvalue z_1, lying the outside of the continuous spectrum of operator $\tilde{H}_1$. Therefore, the essential spectrum of the operators $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$ is consists of the union of two segments and discrete spectrum of the operator $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$ is consists of single point. These give the statement B) of the Theorem 8. The other statements of the Theorem 8 the analogously is proved. □

The next theorems is described the structure of essential spectrum of the operator ${}^2\tilde{H}_t^1$ in the three-dimensional case.

Theorem 9. *Let* $\nu = 3$. *Then*

A).1). If $\varepsilon_2 = -B$ and $\varepsilon_1 < -6B$, or if $\varepsilon_2 = -B$ and $\varepsilon_1 > 6B$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eight segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right) = [4A-24B, 4A+24B] \cup [3A-18B+z, 3A+18B+z] \cup [2A-12B+2z, 2A+12B+2z] \cup [A-6B+3z, A+6B+3z] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [A-6B+z+z_3, A+6B+z+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+z+z_4, A+6B+z+z_4]$$, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \{4z, 2z+z_3, 2z+z_4\}$, where $z = A + \varepsilon_1$, z_3 and z_4 are the additional eigenvalues of the operator $\tilde{H}_2^s$.

2). If $\varepsilon_2 = -B$ and $-6B \le \varepsilon_1 < -2B$, or if $\varepsilon_2 = -B$ and $2B < \varepsilon_1 \le 6B$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of three segments: $$\sigma_{ess}\left({}^2\tilde{H}_t^1\right) = [4A-24B, 4A+24B] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [2A-12B+z_4, 2A+12B+z_4]$$,

and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of empty set:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \varnothing$.

B). 1). If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$, $\varepsilon_1 \le -\frac{6B}{W}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eighth segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right) = [4A-24B, 4A+24B] \cup [3A-18B+z_1, 3A+18B+z_1] \cup [2A-12B+2z_1, 2A+12B+2z_1] \cup [A-6B+3z_1, A+6B+3z_1] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [A-6B+z_1+z_3, A+6B+z_1+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+z_1+z_4, A+6B+z_1+z_4]$$

and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \{4z_1, 2z_1+z_3, 2z_1+z_4\}$, where z_1 are the eigenvalue of operator $\tilde{H}_1$.

If $-\frac{6B}{W} \le \varepsilon_1 < 0$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of three segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right) = [4A-24B, 4A+24B] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [2A-12B+z_4, 2A+12B+z_4]$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of empty set: $\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \varnothing$.

2). If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, $\varepsilon_1 \ge \frac{6B}{W}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eighth segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right) = [4A-24B, 4A+24B] \cup [3A-18B+z_2, 3A+18B+z_2] \cup [2A-12B+2z_2, 2A+12B+2z_2] \cup [A-6B+3z_2, A+6B+3z_2] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [A-6B+z_2+z_3, A+6B+z_2+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+z_2+z_4, A+6B+z_2+z_4]$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \{4z_2, 2z_2+z_3, 2z_2+z_4\}$, where z_2 are the eigenvalue of operator $\tilde{H}_1$.

If $0 < \varepsilon_1 \le \frac{6B}{W}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of three segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right) = [4A-24B, 4A+24B] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [2A-12B+z_4, 2A+12B+z_4]$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of empty set: $\sigma_{disc}\left({}^2\tilde{H}_t^1\right) = \varnothing$.

C). 1). If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, $E < W$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eighth segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right) = [4A-24B, 4A+24B] \cup [3A-18B+z, 3A+18B+z] \cup [2A-12B+2z, 2A+12B+2z] \cup [A-6B+3z, A+6B+3z] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [A-6B+z+z_3, A+6B+z+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+z+z_4, A+6B+z+z_4]$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4z,2z+z_3,2z+z_4\}$ where z $\tilde{z}$ is the eigenvalue of operator $\tilde{H}_1$, and $E=\dfrac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. If $\varepsilon_1=0$ and $\varepsilon_2>0$, $E>W$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of a union of three segment:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right)=[4A-24B,4A+24B]\cup[2A-12B+z_3,2A+12B+z_3]\cup[2A-12B+z_4,2A+12B+z_4]$$

and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of empty set: $\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\varnothing$.

2). If $\varepsilon_1=0$ and $\varepsilon_2<-2B$, $E<W$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eighth segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right)&=[4A-24B,4A+24B]\cup[3A-18B+\tilde{z},3A+18B+\tilde{z}]\\&\cup[2A-12B+2\tilde{z},2A+12B+2\tilde{z}]\cup[A-6B+3\tilde{z},A+6B+3\tilde{z}]\\&\cup[2A-12B+z_3,2A+12B+z_3]\cup[A-6B+\tilde{z}+z_3,A+6B+\tilde{z}+z_3]\\&\cup[2A-12B+z_4,2A+12B+z_4]\cup[A-6B+\tilde{z}+z_4,A+6B+\tilde{z}+z_4]\end{aligned}$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4\tilde{z},2\tilde{z}+z_3,2\tilde{z}+z_4\}$, where $\tilde{z}$ is the eigenvalue of operator $\tilde{H}_1$, and $E=\dfrac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. If $\varepsilon_1=0$ and $\varepsilon_2<-2B$, $E>W$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of a union of three segment:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right)=[4A-24B,4A+24B]\cup[2A-12B+z_3,2A+12B+z_3]\cup[2A-12B+z_4,2A+12B+z_4]$$

and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of empty set: $\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\varnothing$.

D). 1). If $\varepsilon_1=\dfrac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eighth segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right)&=[4A-24B,4A+24B]\cup[3A-18B+z,3A+18B+z]\\&\cup[2A-12B+2z,2A+12B+2z]\cup[A-6B+3z,A+6B+3z]\\&\cup[2A-12B+z_3,2A+12B+z_3]\cup[A-6B+z+z_3,A+6B+z+z_3]\\&\cup[2A-12B+z_4,2A+12B+z_4]\cup[A-6B+z+z_4,A+6B+z+z_4]\end{aligned}$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4z,2z+z_3,2z+z_4\}$, where z is the eigenvalue of operator $\tilde{H}_1$.

2). If $\varepsilon_1=-\dfrac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eighth segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right)&=[4A-24B,4A+24B]\cup[3A-18B+\tilde{z},3A+18B+\tilde{z}]\\&\cup[2A-12B+2\tilde{z},2A+12B+2\tilde{z}]\cup[A-6B+3\tilde{z},A+6B+3\tilde{z}]\\&\cup[2A-12B+z_3,2A+12B+z_3]\cup[A-6B+\tilde{z}+z_3,A+6B+\tilde{z}+z_3]\\&\cup[2A-12B+z_4,2A+12B+z_4]\cup[A-6B+\tilde{z}+z_4,A+6B+\tilde{z}+z_4]\end{aligned}$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\left\{4\tilde{z},2\tilde{z}+z_3,2\tilde{z}+z_4\right\}$, where $\tilde{z}$ is the eigenvalue of operator $\tilde{H}_1$.

E). If $\varepsilon_2>0$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1+\frac{\alpha}{3}\right)W$, or if $\varepsilon_2<-2B$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1+\frac{\alpha}{3}\right)W$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eighth segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right)&=\left[4A-24B,4A+24B\right]\cup\left[3A-18B+z_1,3A+18B+z_1\right]\\&\cup\left[2A-12B+2z_1,2A+12B+2z_1\right]\cup\left[A-6B+3z_1,A+6B+3z_1\right]\\&\cup\left[2A-12B+z_3,2A+12B+z_3\right]\cup\left[A-6B+z_1+z_3,A+6B+z_1+z_3\right]\\&\cup\left[2A-12B+z_4,2A+12B+z_4\right]\cup\left[A-6B+z_1+z_4,A+6B+z_1+z_4\right]\end{aligned}$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\left\{4z_1,2z_1+z_3,2z_1+z_4\right\}$, where z_1 is the eigenvalue of operator $\tilde{H}_1$.

F). If $\varepsilon_2>0$ and $\varepsilon_1<-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1+\frac{\alpha}{3}\right)W$, or if $\varepsilon_2<-2B$ and $\varepsilon_1<-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1+\frac{\alpha}{3}\right)W$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of eighth segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right)&=\left[4A-24B,4A+24B\right]\cup\left[3A-18B+z_1,3A+18B+z_1\right]\\&\cup\left[2A-12B+2z_1,2A+12B+2z_1\right]\cup\left[A-6B+3z_1,A+6B+3z_1\right]\\&\cup\left[2A-12B+z_3,2A+12B+z_3\right]\cup\left[A-6B+z_1+z_3,A+6B+z_1+z_3\right]\\&\cup\left[2A-12B+z_4,2A+12B+z_4\right]\cup\left[A-6B+z_1+z_4,A+6B+z_1+z_4\right]\end{aligned}$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of three eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\left\{4z_1,2z_1+z_3,2z_1+z_4\right\}$, where z_1 is the eigenvalue of operator $\tilde{H}_1$.

K). If $\varepsilon_2>0$ and $0<\varepsilon_1<\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1-\frac{\alpha}{3}\right)W$, or if $\varepsilon_2<-2B$ and $0<\varepsilon_1<\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1-\frac{\alpha}{3}\right)W$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of sixteen segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right)&=\left[4A-24B,4A+24B\right]\cup\left[3A-18B+z_1,3A+18B+z_1\right]\\&\cup\left[3A-18B+z_2,3A+18B+z_2\right]\cup\left[2A-12B+2z_1,2A+12B+2z_1\right]\\&\cup\left[2A-12B+z_1+z_2,2A+12B+z_1+z_2\right]\cup\left[2A-12B+2z_2,2A+12B+2z_2\right]\\&\cup\left[2A-12B+z_3,2A+12B+z_3\right]\cup\left[2A-12B+z_4,2A+12B+z_4\right]\\&\cup\left[A-6B+3z_1,A+6B+3z_1\right]\cup\left[A-6B+2z_1+z_2,A+6B+2z_1+z_2\right]\\&\cup\left[A-6B+z_1+2z_2,A+6B+z_1+2z_2\right]\cup\left[A-6B+3z_2,A+6B+3z_2\right]\\&\cup\left[A-6B+z_1+z_3,A+6B+z_1+z_3\right]\cup\left[A-6B+z_1+z_4,A+6B+z_1+z_4\right]\\&\cup\left[A-6B+z_2+z_3,A+6B+z_2+z_3\right]\cup\left[A-6B+z_2+z_4,A+6B+z_2+z_4\right]\end{aligned}$$

, and

discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of eleven eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4z_1, 3z_1+z_2, 2z_1+2z_2, 2z_1+z_3, 2z_1+z_4, z_1+3z_2, z_1+z_2+z_3, z_1+z_2+z_4, 4z_2, 2z_2+z_3, 2z_2+z_4\}$, where z_1 and z_2 are the eigenvalues of operator $\tilde{H}_1$.

M). If $\varepsilon_2>0$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$ and $E<\left(1+\frac{\alpha}{3}\right)W$, or if $\varepsilon_2<-2B$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$ and $E<\left(1+\frac{\alpha}{3}\right)W$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of the union of sixteen segments:

$$\begin{aligned}\sigma_{ess}\left({}^2\tilde{H}_t^1\right)=&[4A-24B,4A+24B]\cup[3A-18B+z_1,3A+18B+z_1]\\ &\cup[3A-18B+z_2,3A+18B+z_2]\cup[2A-12B+2z_1,2A+12B+2z_1]\\ &\cup[2A-12B+z_1+z_2,2A+12B+z_1+z_2]\cup[2A-12B+2z_2,2A+12B+2z_2]\\ &\cup[2A-12B+z_3,2A+12B+z_3]\cup[2A-12B+z_4,2A+12B+z_4]\\ &\cup[A-6B+3z_1,A+6B+3z_1]\cup[A-6B+2z_1+z_2,A+6B+2z_1+z_2]\\ &\cup[A-6B+z_1+2z_2,A+6B+z_1+2z_2]\cup[A-6B+3z_2,A+6B+3z_2]\\ &\cup[A-6B+z_1+z_3,A+6B+z_1+z_3]\cup[A-6B+z_1+z_4,A+6B+z_1+z_4]\\ &\cup[A-6B+z_2+z_3,A+6B+z_2+z_3]\cup[A-6B+z_2+z_4,A+6B+z_2+z_4]\end{aligned}$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of eleven eigenvalues:

$\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\{4z_1, 3z_1+z_2, 2z_1+2z_2, 2z_1+z_3, 2z_1+z_4, z_1+3z_2, z_1+z_2+z_3, z_1+z_2+z_4, 4z_2, 2z_2+z_3, 2z_2+z_4\}$, where z_1 and z_2 are the eigenvalues of operator $\tilde{H}_1$.

N). If $-2B<\varepsilon_2<0$, then the essential spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of a union of three segments:

$$\sigma_{ess}\left({}^2\tilde{H}_t^1\right)=[4A-24B,4A+24B]\cup[3A-18B+z_3,3A+18B+z_3]\cup[3A-18B+z_4,3A+18B+z_4]$$

, and discrete spectrum of the operator ${}^2\tilde{H}_t^1$ is consists of empty set: $\sigma_{disc}\left({}^2\tilde{H}_t^1\right)=\varnothing$.

Proof. A). 1). From the Theorem 6 is follows, that, if $\nu=3$ and $\varepsilon_2=-B$ and $\varepsilon_1<-6B$ (respectively, $\varepsilon_2=-B$ and $\varepsilon_1>6B$), the operator $\tilde{H}_1$ has a unique eigenvalue $z=A+\varepsilon_1$, the outside the continuous spectrum of the operator $\tilde{H}_1$. Furthermore, the continuous spectrum of the operator $\tilde{H}_1$ is consists of the segment $[A-6B,A+6B]$, therefore, the essential spectrum of the operator $\tilde{H}_2^s$ is consists of a union of two segments: $\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-12B,2A+12B]\cup[A-6B+z,A+6B+z]$. The number $2z$ is the eigenvalue for the operator $\tilde{H}_2^s$. In the representation (18) and (21) the operator K is a two-dimensional operator. Therefore, the operator $\tilde{H}_2^s$ can have two additional eigenvalues z_3 and z_4. Consequently, the operator $\tilde{H}_2^s$ can have no more than three eigenvalues $2z, z_3$ and z_4.

2). From the Theorem 6 is follows, that, if $\nu=3$ and $\varepsilon_2=-B$ and

$-6B \le \varepsilon_1 < -2B$ (respectively, $\varepsilon_2 = -B$ and $2B < \varepsilon_1 \le 6B$), then the operator $\tilde{H}_1$ has no eigenvalues, the outside the continuous spectrum of the operator $\tilde{H}_1$. Furthermore, the continuous spectrum of the operator $\tilde{H}_1$ is consists of the segment $[A-6B, A+6B]$, therefore, the essential spectrum of the operator $\tilde{H}_2^s$ is consists of a single segment: $\sigma_{ess}(H_2^s) = [2A-12B, 2A+12B]$. In the representation (18) and (21) the operator K is a two-dimensional operator. Therefore, the operator $\tilde{H}_2^s$ can have two additional eigenvalues z_3 and z_4. Consequently, the operator $\tilde{H}_2^s$ can have no more than two eigenvalues z_3 and z_4.

M). From the Theorem 6 is follows, that, if $\nu = 3$ and $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $E < \left(1 + \frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $E < \left(1 + \frac{\alpha}{3}\right)W$), the operator $\tilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the below and above of the continuous spectrum of the operator $\tilde{H}_1$. Furthermore, the continuous spectrum of the operator $\tilde{H}_1$ is consists of the segment $[A-6B, A+6B]$, therefore, then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of three segments: $\sigma_{ess}(\tilde{H}_2^s) = [2A-12B, 2A+12B] \cup [A-6B+z_1, A+6B+z_1] \cup [A-6B+z_2, A+6B+z_2]$, and point $2z_1, 2z_2$ and $z_1 + z_2$, are the eigenvalues of the operator $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$, and in the representation (18) and (21) the operator K is a two-dimensional operator. Therefore, the operator $\tilde{H}_2^s$ can have two additional eigenvalues z_3 and z_4. Consequently, the operator $\tilde{H}_2^s$ can have no more than five eigenvalues $2z_1, z_1 + z_2, 2z_2, z_3$ and z_4.

The other statements of the Theorem 9 the analogously is proved. □

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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