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Structure of the Star with Ideal Gases

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Abstract

In this paper, we calculate some structure functions of an idealized stellar model, which can be solved by the total mass and radius of a star. These functions have enlightening and pedagogical significance. We find that the equation of state of matter is decisive to the fate of a star. Only if the equation of state includes the driving effect of gravity on particles, then it satisfies some increasing and causal conditions and is compatible with Einstein's field equation. In this case, we always have singularity-free balanced star, no matter how heavy the star is. Usually, we believe that the main factor determining the stellar structure is the pressure equilibrium of the thermonuclear reaction against gravity. But this opinion is inadequate. The calculation of this paper shows that, the pressure generated by the driving effect of gravity on particles is dominant.

Keywords

Equation of State, Stellar Structure, Neutron Star, Singularity-Free Solution

1. Introduction

The spherical symmetric metric for a static star is described by Schwarzschild metric [1] [2]

$$g_{\mu\nu} = \mathrm{diag}\left(b(r), -a(r), -r^2, -r^2\sin^2\theta\right). \tag{1.1}$$

For the energy momentum tensor of perfect fluid

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - Pg_{\mu\nu} = \mathrm{diag}\left(b\rho, aP, r^2P, r^2\sin^2\theta P\right), \tag{1.2}$$

where $\rho(r), P(r)$ are proper mass energy density and pressure, $U_\mu = \left(\sqrt{b}, 0, 0, 0\right)$, we have the independent equations as follow

$$\left(\frac{r}{a}\right)' = 1 - 8\pi G\rho r^2, \quad \left(G_{00} = -8\pi G T_{00}\right), \tag{1.3}$$

$$\frac{b'}{b} = \frac{a-1}{r} + 8\pi GPar, \quad (G_{11} = -8\pi GT_{11}), \tag{1.4}$$

$$P' = -(\rho + P)\frac{b'}{2b}, \quad (T^{1\nu}_{;\nu} = 0). \tag{1.5}$$

To determine the stellar structure of an irrotational star, we solve these equations. However (1.3) - (1.5) is not a closed system, the solution depends on the equation of state (EOS) $P = P(\rho)$. For polytropes, we take $P = P_0\rho^\gamma$ [1] [2]. For the compact stars, there are a lot of EOS derived from particle models [3]-[8], which provide the structural information and parameters such as the maximum mass for neutron stars. For realistic stellar models, some simulating calculations can be found in [9].

However, in the equations of state in all the above literatures, the driving effect of gravity on the particles is ignored. Originally, if a set of ideal gas constrained by its own gravity is taken as a system, the particles move along the geodesic. Under given initial velocity distribution, the evolution of the system is fully determined. But, after the system is simplified to a perfect fluid model, the dynamical equation of the system becomes incomplete and an equation of state should be introduced additionally [10]. The energy-momentum conservation law of the fluid, namely the static equilibrium Equation (1.5), is insufficient to describe the dynamical effect of gravity on fluid particles. This implies that some information of system is lost during the simplification process. Only the equation of state including the driving effect of gravity is compatible with relativity. In the paper, we derive the valid equations of stellar structure. The calculations show that the driving effect of gravity plays a dominant role in the stellar structure.

2. Phenomenological Analysis for the Behavior of a Star

At first, we make a few simple calculations and examine the behavior of the metric and particles inside a star to get some intuition. The first phenomenon is that, the temporal singularity and spatial singularity occur at different time and place if the space-time becomes singular, and the temporal one occurs firstly.

Denoting the mass distribution by

$$M(r) = 4\pi G\int_0^r \rho r^2 \mathrm{d}r, \quad \mathcal{R} = 2M(r). \tag{2.1}$$

Then by (1.3) we have solution

$$a = \begin{cases} \left(1 - \dfrac{\mathcal{R}(r)}{r}\right)^{-1}, & \text{if } r < R, \\ \left(1 - \dfrac{R_s}{r}\right)^{-1}, & \text{if } r \geq R, \end{cases} \tag{2.2}$$

where R is the radius of the star, and the Schwarzschild radius becomes

$$R_s = 2M(R) = \mathcal{R}(R) = 8\pi G\int_0^R \rho r^2 \mathrm{d}r. \tag{2.3}$$

For any normal star with $R > R_s$. From the above solution we learn $a(r)$ is a continuous function and

$$a \geq 1, \quad a(0) = \lim_{r\to\infty} a = 1, \quad a_{\max} = a(r_m), \; (0 < r_m \leq R). \tag{2.4}$$

So the spatial singularity $a \to \infty$ does not appear at the center of the star when the singularity begins to form.

On the other hand, by (1.4) and $P \geq 0$, we find $b'(r)$ is a continuous function satisfying

$$b'(0) = 0, \quad b'(r) > 0, \; (\forall r > 0), \quad b = 1 - \frac{R_s}{r}, \; (r \geq R). \tag{2.5}$$

(1.9) shows $b(r)$ is a monotonically increasing function of *r* with smoothness at least $C^1([0,\infty))$. Consequently, the temporal singularity $b \to 0$ should take place at the center.

The trends of $a(r), b(r)$ are shown in **Figure 1**, where we take the Schwarzschild radius $R_s = 1$ as length unit. From **Figure 1**, we find $b(1) \to 0$ and $a(1) \to \infty$ occur simultaneously. Noting $b(r)$ is a monotonically increasing function of *r*, so $b(0) \to 0$ certainly occurs before $a(1) \to \infty$.

The second phenomenon is that, the particles near the center of a star are unbalanced, and violent explosion takes place inside the star before the temporal singularity occurs. When $b(0) \to 0$, by (2.5) we have

$$b \to b_0 r^\alpha, \; (\alpha > 1), \quad \text{if } r \ll R. \tag{2.6}$$

Substituting it into (1.5), we find

$$-P' \to (\rho + P)\frac{\alpha}{2r} \to +\infty, \quad (r \to 0). \tag{2.7}$$

According to fluid mechanics, $-\partial_r P$ corresponds to the radial boosting force, so (2.7) means violent explosion.

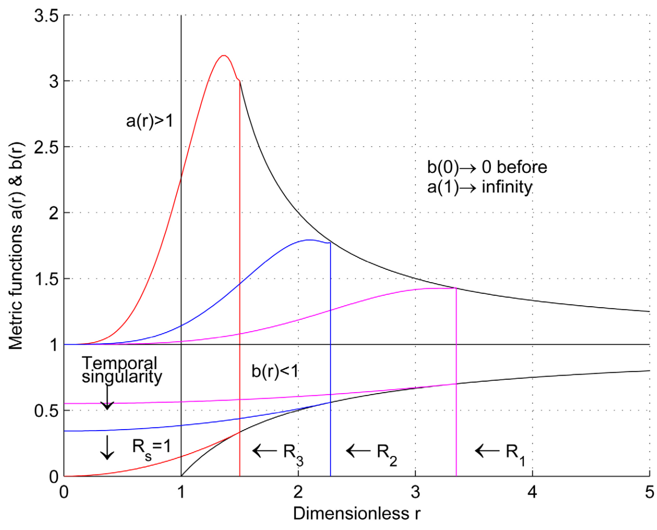

Figure 1. The trends of (a,b) as $R \to R_s$, which show the spatial singularity does not occur at the center, and the temporal singularity occurs at the center before $a \to \infty$.

More clearly, we examine the motion of a particle inside the star. Solving the geodesic in the orthogonal subspace (t,r,θ), we get the first integrals [10]

$$\dot{t}=\frac{1}{C_1 b},\ \dot{\theta}=\frac{C_2}{r^2},\ \dot{\varphi}=0,\ \dot{r}^2=\frac{1}{r}\left(\frac{1}{C_1^2 b}-\frac{C_2^2}{r^2}-1\right), \tag{2.8}$$

where C_1, C_2 are constants, $\dot{t}=\frac{\mathrm{d}t}{\mathrm{d}s}$. The normal velocity of the particle is given by

$$v_r^2=\frac{a\mathrm{d}r^2}{b\mathrm{d}t^2}=1-C_1^2 b\left(1+\frac{C_2^2}{r^2}\right),\ v_\theta^2=\frac{r^2\mathrm{d}\theta^2}{b\mathrm{d}t^2}=\frac{C_1^2 C_2^2 b}{r^2},\ v_\varphi^2=0. \tag{2.9}$$

The sum of the speeds provides the mechanical energy conservation law of a particle,

$$v^2=1-C_1^2 b(r),\quad \text{with } v^2\equiv v_r^2+v_\theta^2+v_\varphi^2. \tag{2.10}$$

(2.10) holds for all particles with $v_\varphi\neq 0$ due to the symmetry of the space-time.

From (2.10) we learn $v\to 1$ as $b\to 0$, this means all particles escape at light velocity when the temporal singularity occurs. So instead of a final collapse, the fate of a star with heavy mass should be explosion and disintegration. The gravity of a star drives the inside particles to move rapidly and leads to high temperature. The driving force in a star is tremendous and cannot be overlooked. By simple calculation we find that, even at 20 km below the earth surface, the pressure can destroy steel. How the particles to react to the collapse of a star should be strictly researched with dynamics [11].

There are also different opinions upon the gravitational collapse and effect. A heuristic computation for axisymmetrical collapse is presented in [12], which reveals that the fate of a collapsing star sensitively depends on the parameters in the EOS. In a full quantum treatment, [13] shows that the traditional classical singularity in the core of the Schwarzschild black hole is replaced. The two-particle system seems to be non-singular from the quantum point of view. In [14], the authors investigate the theoretical implications of the constraint that the graviton is massless to an Einstein-Gauss-Bonnet theory with linear coupling of the scalar field to the 4-d Gauss-Bonnet invariant. It is shown that the constraint of having gravitational wave speed of the primordial gravitational waves equal to the light velocity, severely restricts the dynamics of the scalar field, imposing a direct constant-roll evolution on it. The spectral index of the primordial scalar perturbations for the GW170817-compatible Einstein-Gauss-Bonnet theory with linear coupling is different in comparison to the same theory with non-linear coupling.

3. Equations of State of Ideal Gas

If all other interactions among particles are ignored, then the star consists of ideal gas. We consider the structure of such star of ideal gases. Next we derive EOS of ideal gas in gravity and show its asymptotic properties [10]. The gas satisfies the following assumptions:

(**A1**) All particles are classical ones only driven by the gravity, namely, they are characterized by 4-vector momentum p_k^μ and move along geodesic.

(**A2**) The collisions among particles are elastic, so the process is adiabatic.

In microscopic view, the energy momentum tensor of ideal gas can be expressed by [1]

$$T^{\mu\nu} = \sum_n m_n u_n^\mu u_n^\nu \delta^3\left(\vec{x} - \vec{X}_n\right)\sqrt{1-v_n^2}, \tag{3.1}$$

where m_n is the proper mass of the *n*-th particle, u_n^μ 4-vector velocity, $\vec{v}_n$ the usual 3-d speed, $\vec{X}_n(t)$ the central coordinate. The macroscopic energy-momentum tensor is given by (1.2). The functions of state in (3.1) including relativistic factor $\sqrt{1-v_n^2}$, whose statistical expectation cannot be calculated directly. To research the thermodynamic properties of gas, we use piston and cylinder to drive gas. In astrophysics, we have more ideal piston and cylinder that is the space-time with Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, which is absolutely adiabatic and reversible. The FLRW metric drives the gases homogeneously expanding and contracting as the scale factor a varies, and the results have general meanings according to the principle of equivalence.

In the microscopic view, for particles and photons, they are only driven by average gravity and move along geodesics, and the collisions between particles can be treated as instantaneous behavior. So all thermodynamic functions can be rigorously solved according to dynamics and statistics. Assuming that N particles occupy volume V, which is equivalent to the Einstein's lift. However, this lift is not rigid, as a thermodynamic function of state, the volume V varies under the compression and drive of gravity on the particles. Since the thermodynamic equations, such as equation of state and energy conservation laws, are locally valid statistical laws, which have little to do with the macroscopic shape of piston-cylinder system. Therefore, we can derive the equations of state in a symmetrical FLRW space-time, so that the treatment is relatively simple and clear. It can be proven that the results hold generally in comoving coordinate system. According to equivalence principles, these results should also hold generally.

For FLRW space-time, we have the line element in conformal coordinate system

$$\mathrm{d}s^2 = a^2(t)\left(\mathrm{d}t^2 - \mathrm{d}r^2 - \mathcal{S}^2(r)\mathrm{d}\theta^2 - \mathcal{S}(r)^2\sin^2\theta\mathrm{d}\varphi^2\right), \tag{3.2}$$

where

$$\mathcal{S} = \begin{cases} \sin r & \text{if } \kappa = 1, \\ r & \text{if } \kappa = 0, \\ \sinh r & \text{if } \kappa = -1. \end{cases} \tag{3.3}$$

The energy conservation law $T^{\mu\nu}_{;\nu}$ in this case is given by

$$\mathrm{d}\left(\rho a^3\right) = -3Pa^2\mathrm{d}a. \tag{3.4}$$

For a given equation $\rho=\rho(a)$, we can solve for the pressure $P=P(a)$ from (3.4).

In the FLRW space-time (3.2), for a particle moving along r, we have the first integral of geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}s}r=\frac{C}{a^2},\quad \frac{\mathrm{d}}{\mathrm{d}s}t=\frac{1}{a^2}\sqrt{a^2+C^2}, \tag{3.5}$$

where C is a constant only depends on the initial data. By (3.5) we get the drifting speed of a particle in usual sense

$$v_n\equiv\frac{a\mathrm{d}r}{a\mathrm{d}t}=\frac{b_n}{\sqrt{a^2+b_n^2}},\quad \sqrt{1-v_n^2}=\frac{a}{\sqrt{a^2+b_n^2}}. \tag{3.6}$$

So the momentum of a particle $p=\frac{m_n v}{\sqrt{1-v^2}}$ satisfies

$$p(t)a(t)=p(t_0)a(t_0), \tag{3.7}$$

where m_n is the proper mass of the particle. For the massless photons, we can check that the wavelength $\lambda(t)$ satisfies $\frac{\lambda(t)}{a(t)}\equiv\frac{\lambda_0}{a_0}$, so their momentum p also satisfy (3.7). Although (3.7) is derived in subspace-time (t,r), but it is suitable for all particles due to the symmetry of the FLRW metric.

The relation between momentum p and the kinetic energy K is given by

$$p^2=K(K+2m). \tag{3.8}$$

By (3.7) we have $p_n^2=\frac{C_n}{a^2}$, where C_n are constants only depending on the initial data at $t=t_0$. Then on the one hand, for all particles we have the average square momentum directly

$$\overline{p}^2=\frac{C_0}{a^2}, \tag{3.9}$$

where C_0 is a constant only determined by initial data at t_0. One may argue that (3.9) is probably broken by the collision of the particles. The following Lemma shows that (3.9) holds in statistical sense.

Lemma *The average square momentum of the ideal gas is independent of the elastic collision of the particles.*

Proof. For any elastic collision, we have momentum conservation law $\vec{p}_1+\vec{p}_2=\vec{P}_1+\vec{P}_2$, and then

$$p_1^2+p_2^2=P_1^2+P_2^2+2\left(\vec{P}_1\cdot\vec{P}_2-\vec{p}_1\cdot\vec{p}_2\right). \tag{3.10}$$

Taking average for (3.10), we have

$$\overline{p}^2=\overline{P}^2+\Delta. \tag{3.11}$$

Since the elastic collision is a reversible process, in statistical sense, we have the exactly equal numbers of reversible process, so we also have

$$\overline{P}^2=\overline{p}^2+\Delta. \tag{3.12}$$

Comparing (3.11) with (3.12), we have $\Delta = 0$ and $\overline{p}^2 = \overline{P}^2$. Since collision is finished instantaneously, (3.9) holds for all time t.

On the other hand, $\overline{p}^2$ can be calculated according to statistical principle. Assuming the distribution of kinetic energy K of the particles is given by

$$\mathrm{d}\mathcal{P} = \mathcal{F}\left(K\right)\mathrm{d}K, \tag{3.13}$$

then we have moment function,

$$\int_0^\infty \mathrm{d}\mathcal{P} = 1, \quad \int_0^\infty K\mathrm{d}\mathcal{P} = \frac{3}{2}kT, \quad \int_0^\infty K^2\mathrm{d}\mathcal{P} = \frac{3}{2\sigma}\left(kT\right)^2, \tag{3.14}$$

where the second formula can be regarded as definition of temperature, σ is a constant reflecting the concrete distribution function of particles. In statistical mechanics, we usually use the distribution functions of momentum, which is inconvenient for calculation in the case of relativistic gases. The following discussions have nothing to do with explicit function $\mathcal{F}\left(K\right)$, and at most uses the second order moment. In the case of Maxwell distribution, we have

$$\mathrm{d}\mathcal{P} = \exp\left(-\frac{K}{kT}\right)\sqrt{\frac{4K}{\pi kT}}\frac{\mathrm{d}K}{kT}, \quad \sigma = \frac{2}{5}. \tag{3.15}$$

By the moments (3.14) we have

$$\begin{aligned}\overline{p}^2 &= \sum_n \int_0^\infty \frac{1}{N} p_n^2 \mathcal{F}\left(K_n\right)\mathrm{d}K_n \\ &= \sum_n \int_0^\infty \frac{1}{N} K_n\left(K_n + 2m_n\right)\mathcal{F}\left(K_n\right)\mathrm{d}K_n \\ &= \sum_n kT\frac{1}{N}\left(\frac{3}{2\sigma}kT + 3m_n\right),\end{aligned} \tag{3.16}$$

where N is a given number of particles, which occupy volume $V = \Omega a^3$. Obviously, V is unnecessary to be the whole universe, so the derivation is independent of curvature κ. For a given number N of particles, the angular volume Ω is a constant. The scale factor a acts as a piston, which varies and drives the particles. Comparing (3.16) with (3.9), we get the relation between T and a as follows,

$$kT = \frac{\sigma\overline{m}b^2}{a\left(a + \sqrt{a^2 + b^2}\right)}, \quad a = \frac{\sigma\overline{m}b}{\sqrt{kT\left(kT + 2\sigma\overline{m}\right)}}, \tag{3.17}$$

where $\overline{m} = \frac{1}{N}\sum_n m_n$ is the average mass of all particles, and b is a constant only depending on initial data.

By (1.2) we have $\rho = T_0^0$. On the other hand, in average sense, by (3.1) we have [10]

$$T_0^0 = \sum_n m_n g_{00} u_n^0 u_n^0 \sqrt{1 - v_n^2}\,\delta^3\left(\vec{x} - \vec{X}_n\right) = \sum_n \frac{m_n}{\sqrt{1 - v_n^2}}\delta^3\left(\vec{x} - \vec{X}_n\right). \tag{3.18}$$

We take a as the independent variable in the statistical calculation. By (3.14) and (3.17), we have

$$\begin{aligned}\rho &= \frac{1}{V}\int \sum_{X_n\in V}\frac{m_n}{\sqrt{1-v_n^2}}\mathrm{d}\mathcal{P} = \frac{1}{a^3\Omega}\sum_{X_n\in\Omega}\int\left(m_n+K_n\right)\mathrm{d}\mathcal{P} \\ &= \frac{1}{a^3\Omega}\sum_{X_n\in\Omega}\left(\bar{m}+\frac{3}{2}kT\right) = \frac{\varrho}{a^3}\left(1+\frac{3\sigma}{2a}\left(\sqrt{a^2+b^2}-a\right)\right),\end{aligned} \tag{3.19}$$

where $\varrho = \frac{1}{\Omega}\sum_{X_n\in\Omega} m_n$ is the angular density of proper mass, which is a constant. Substituting (3.19) into energy conservation law (3.4) we obtain

$$P \equiv \frac{1}{3}\sum_n \frac{m_n v_n^2}{\sqrt{1-v_n^2}}\delta^3\left(\vec{x}-\vec{X}_n\right) = \frac{\sigma\varrho b^2}{2a^4\sqrt{a^2+b^2}} = \frac{NkT}{V}\left(1-\frac{kT}{2\left(\sigma\bar{m}+kT\right)}\right). \tag{3.20}$$

In the above derivation, the FLRW metric (3.2) is only used as a piston-cylinder system to drive the ideal gas, so the results are actually valid in general cases. Using (3.17), we have relation

$$a = \frac{\sigma b}{\sqrt{J\left(J+2\sigma\right)}}, \quad J \equiv \frac{kT}{\bar{m}c^2} = \frac{\sigma}{a}\left(\sqrt{a^2+b^2}-a\right), \tag{3.21}$$

where J is the dimensionless temperature. The above results conclude the following theorem.

Theorem *For relativistic ideal gases, we have the equation of state as*

$$\mathcal{N} = \mathcal{N}_0\left[J\left(J+2\sigma\right)\right]^{\frac{3}{2}}, \quad J \equiv \frac{kT}{\bar{m}c^2}, \tag{3.22}$$

$$\rho = \mathcal{N}\left(\bar{m}c^2+\frac{3}{2}kT\right) = \varrho\left[J\left(J+2\sigma\right)\right]^{\frac{3}{2}}\left(1+\frac{3}{2}J\right)c^2, \tag{3.23}$$

$$P = \mathcal{N}kT\frac{2\sigma\bar{m}c^2+kT}{2\left(\sigma\bar{m}c^2+kT\right)} = \varrho\left[J\left(J+2\sigma\right)\right]^{\frac{5}{2}}\frac{c^2}{2\left(\sigma+J\right)}, \tag{3.24}$$

where J is dimensionless temperature, which act as independent variable and parameter, $\mathcal{N}$ is the number density of particles, $\mathcal{N}_0 = \mathcal{N}_0\left(\bar{m},\sigma\right)$ is related to property of the particles but independent of J, $c = 2.99\times 10^8\ \mathrm{m/s}$ the light velocity, σ a factor reflecting the energy distribution function, for Maxwell distribution $\sigma = \frac{2}{5}$. $\bar{m} = \frac{1}{N}\sum_{n=1}^N m_n$ is the average static mass of all particles, $\left(\rho, P\right)$ are usual energy density and pressure, ϱ a mass density defined by $\varrho \equiv \mathcal{N}_0\bar{m}$.

In the above derivation, the equations of motion of particles are geodesics, hence the above conclusions are compatible with relativity and includes the driving effect of gravity. Together with (1.3) - (1.5), we get a closed and consistent dynamics for stellar interior structure. By (3.23) and (3.24), we get the polytropic index γ is not a constant for large range of T satisfying $1<\gamma<\frac{5}{3}$. We have $\gamma\to\frac{5}{3}$ as $\left(T\to 0\right)$, which is caused by inertia of particles and leads to finite radius of a star. The velocity of sound

$$C_s \equiv c\sqrt{\frac{\mathrm{d}P}{\mathrm{d}\rho}} = \frac{\sqrt{3}}{3}\left(\frac{c^2 J(2\sigma+J)(5\sigma^2+8\sigma J+4J^2)}{(\sigma+J)^2\left[2\sigma+(2+5\sigma)J+4J^2\right]}\right)^{\frac{1}{2}} < \frac{\sqrt{3}}{3}c. \tag{3.25}$$

By (3.22) - (3.25), we have asymptotic properties of EOS for the particles

$$P \doteq \begin{cases} P_0\rho^{\frac{5}{3}}\left(1-\frac{1}{2\sigma}(5\sigma+2)J\right), & \text{if } T\to 0, \\ \frac{1}{3}\rho\left(1+\left(\sigma-\frac{2}{3}\right)J^{-1}\right), & \text{if } T\to\infty. \end{cases} \tag{3.26}$$

The above characteristics (3.25) and (3.26) of EOS are the conditions for a singularity-free star. More clearly, if the EOS of the material of a star satisfies

$$0 < C_s \le \frac{1}{\sqrt{3}}, \quad P\to P_0\rho^\gamma, \text{ if } (\gamma>1,\rho\to 0), \; P\to\frac{1}{3}\rho, \text{ if } (\rho\to\infty), \tag{3.27}$$

then we have a singularity-free solution for a static star, no matter how heavy the star is. The EOS (3.22) - (3.24) is derived from the coupling system of particles and space-time, so it is compatible with Einstein's field equation. The singular solutions are usually caused by incompatible EOS. An EOS ignoring the driving effect of gravity is invalid in general relativity.

For the realistic particles with other interactions, the reasonable EOS should have asymptotic properties of the following model equation

$$P = \frac{C_0^2\rho^{1+n}}{k+\rho^n}, \quad \left(0 < C_0 \le \frac{1}{\sqrt{3}}, k>0, n>0\right). \tag{3.28}$$

The star becomes larger as $k\to 0$ or $n\to 0$ or $C_0\to 1$. This conclusion can be checked as follows. The realistic static asymptotically flat space-time with spherical symmetry can be generally solved by the following procedure. The dynamics (1.3) - (2.1) can be reduced to the following initial problem of an ordinary differential equation system [1] [10],

$$M'(r) = 4\pi G\rho r^2, \quad M(0)=0, \tag{3.29}$$

$$\rho'(r) = -\frac{(\rho+P)(4\pi GPr^3+M)}{C_s^2(r-2M)r}, \quad \rho(0)=\rho_0, \tag{3.30}$$

in which $P=P(\rho)$ can be (3.28). The interior solution ends at $C_s\to 0$ or $\rho\to 0$ as $r\to R$, where $R<\infty$ is the radius of the star. For any given $\rho_0>0$ we get a unique singularity-free balanced solution. The interior metric components are given by

$$a = \left(1-\frac{2M}{r}\right)^{-1}, \quad b = \exp\left(-\int_r^R \frac{2(4\pi GPr^3+M)}{r(r-2M)}\mathrm{d}r\right). \tag{3.31}$$

Noticing $M(r)=\frac{1}{2}\mathcal{R}$ has length dimension, Equations (3.29) and (3.30) have scale invariance, so the total mass is proportional to ρ_0 if other conditions are the same. Outside the star, we have $\rho(r)=P(r)=0$ in the region

$r \geq R$. The total mass of the star (2.3) can reaches any large value for adequate parameters (k, n, C_0) and large enough $0 < \rho_0 < \infty$. So the asymptotic properties of EOS is decisive for the fate of a star.

4. The Equations for Stellar Structure

Now we take the EOS (3.22) - (3.24) of ideal gas as example to show the concrete structure of a star. For this EOS, the structural functions of a star have simple form and can be completely solved. Equation (1.5) can be rewritten as

$$\frac{\mathrm{d}b}{\mathrm{d}J} = -\frac{2b}{\rho + P}\frac{\mathrm{d}P}{\mathrm{d}J}. \tag{4.1}$$

Substituting (3.23) and (3.24) into (4.1), we get

$$b = \frac{4(R - R_s)(\sigma + J)^2}{R\left[2\sigma + (2 + 5\sigma)J + 4J^2\right]^2}, \tag{4.2}$$

where $R_s = 2M_{\mathrm{tot}}$ the Schwarzschild radius. Substituting (2.2) and (3.23) - (4.2) into (1.3) and (1.4), we get the following dimensionless equations for stellar structure,

$$\mathcal{R}'(r) = \left(\frac{r}{\chi}\right)^2 (2 + 3J)\left[J(J + 2\sigma)\right]^{\frac{3}{2}}, \tag{4.3}$$

$$J'(r) = \frac{-\xi(J)}{2(r - \mathcal{R})}\left(\left(\frac{r}{\chi}\right)^2 \left[J(J + 2\sigma)\right]^{\frac{5}{2}} + \frac{\mathcal{R}}{r}(\sigma + J)\right), \tag{4.4}$$

where χ is a constant with length dimension which acts as length scale,

$$\chi = c(4\pi G\varrho)^{-\frac{1}{2}} = c(4\pi G\mathcal{N}_0\bar{m})^{-\frac{1}{2}}, \tag{4.5}$$

$$\xi \equiv \frac{2\sigma + (2 + 5\sigma)J + 4J^2}{5\sigma^2 + 8\sigma J + 4J^2}, \quad (\xi \approx 1). \tag{4.6}$$

The rigorous solution to (4.3) and (4.4) is absent. However, they are dimensionless equations convenient for numeric simulation. If we take $\chi = 1$ as the unit of length, the solution can be uniquely determined by the following boundary conditions

$$\mathcal{R}(0) = 0, \quad J(0) = J_0 > 0, \quad J(r) = 0, \quad (\forall r \geq R). \tag{4.7}$$

By adjusting J_0, we get different solution. The solutions are displayed in **Figure 2** and **Figure 3**. The total mass of the star is proportional to any given parameter χ, so it can be arbitrarily large.

In practical observation, the radius R and the total mass or equivalent Schwarzschild radius R_s of a star can be easily measured. Then the other structural parameters can be determined by this radii pair (R_s, R). In what follows, we show the procedure how to use **Figure 2** and **Figure 3** solve practical problems.

Taking the sun as example, we have the radii pair as

$$R_s = 2.96 \times 10^3\ \mathrm{m}, \quad R_\odot = 6.96 \times 10^8\ \mathrm{m}. \tag{4.8}$$

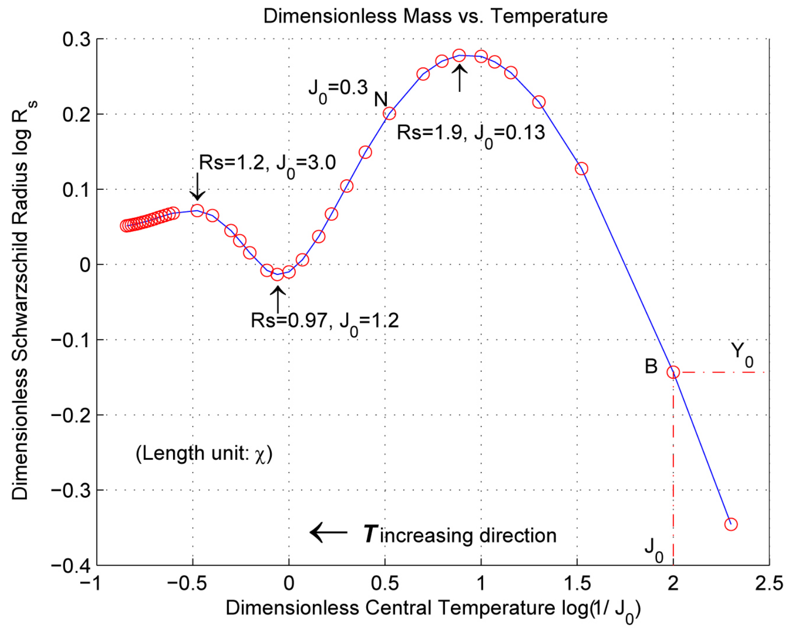

Figure 2. Relation between mass and central temperature similar to H-R diagram, where the curves are concentrated by the scale χ and dimensional temperature *J*.

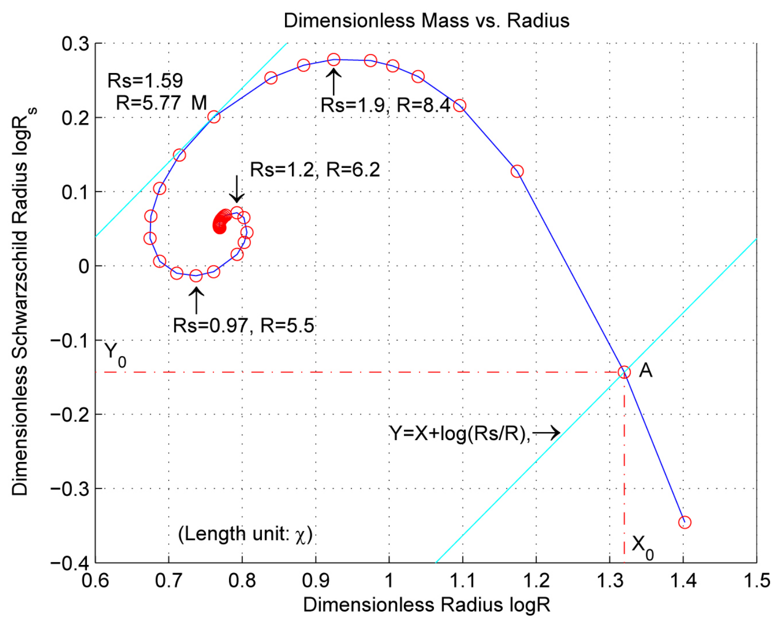

Figure 3. Relation between mass and radius. All structural information of a star are determined by a given radii pair (R_s, R).

By $R_s/R_\odot = 4.25\times10^{-6}$, according to the relation shown in **Figure 3**, we can solve the intersection A and get the dimensionless radii pair

$$X_0 = \log(R/\chi) = 2.320,\quad Y_0 = \log(R_s/\chi) = -3.052. \tag{4.9}$$

Consequently, we have

$$\chi = 10^{-2.32} R = 3.335\times10^{6}\ \text{m}. \tag{4.10}$$

By Y_0 we get intersection B in **Figure 2**, and then get the central dimensionless temperature $J_0 = 1.145\times10^{-6}$ for the sun. Taking it as initial value we can solve (4.3) and (4.4), and then get detailed structural information for the sun.

By (4.5) and (4.10), we get

$$\varrho = \frac{c^2}{4\pi G\chi^2} = 9.640\times10^{12}\left(\text{kg}/\text{m}^3\right). \tag{4.11}$$

Then by (3.23) and (3.24) we solve the mass density and pressure at the center

$$\rho(0) = \varrho\left[J_0\left(J_0+2\sigma\right)\right]^{\frac{3}{2}}\left(1+\frac{3}{2}J_0\right) = 8.45\times10^3\left(\text{kg}/\text{m}^3\right), \tag{4.12}$$

$$P(0) = \frac{1}{2}\varrho\left[J_0\left(J_0+2\sigma\right)\right]^{\frac{5}{2}}\left(\sigma+J_0\right)^{-1}c^2 = 8.70\times10^8\left(\text{MPa}\right). \tag{4.13}$$

The temperature depends on the average mass $\bar{m}$. By (3.22), we have

$$T_c = \bar{m}c^2 J_0/k = n_p\left(m_p c^2 J_0/k\right) = 1.247\times10^7 n_p\left(\text{K}\right), \tag{4.14}$$

where $m_p = 1.673\times10^{-27}\,\text{kg}$ is the static mass of proton, n_p is the equivalent proton number for the particles.

If the ionization in the sun is about $\text{H}^+ + \text{N}^+ + 2\text{e}^-$, then we have

$$n_p = \left(70\%\times1+30\%\times14\right)/4 = 1.23, \tag{4.15}$$

$$\bar{m} = n_p m_p = 2.06\times10^{-27}\left(\text{kg}\right), \tag{4.16}$$

$$T_c = 1.247\times10^7 n_p = 1.53\times10^7\left(\text{K}\right), \tag{4.17}$$

$$\mathcal{N}_0 = c^2/\left(4\pi G\bar{m}\chi^2\right) = 4.68\times10^{39}\left(\text{m}^{-3}\right). \tag{4.18}$$

In contrast with (4.12) and (4.13), we find the central density and pressure in the sun are about one order of magnitude less than the current data

$$\rho(0) = 1.6\times10^5\,\text{kg}/\text{m}^3,\quad P(0) = 2.5\times10^{10}\,\text{MPa}.$$

This difference should be caused by the dynamical effect of gravity.

The compactest stars (with the maximum R_s/R) correspond to the points M, N in **Figure 2** and **Figure 3**. The radii pair is $R_s : R = 1.59 : 5.77$ and the central temperature $J_0 = 0.30$. For a compact star with the solar mass $M_\odot$, we get the length unit and radius as

$$\chi = R_s/1.59 = 1.86\,\text{km},\quad R = 5.77\chi = 10.7\,\text{km}. \tag{4.19}$$

Along the above procedure, we solve

$$\rho(0) = 8.50\times10^{18}\,\text{kg}/\text{m}^3, \tag{4.20}$$

$$P(0) = 1.24\times10^{29}\,\text{MPa}, \tag{4.21}$$

$$T(0) = 3.27\times10^{12} n_p\,\text{K}. \tag{4.22}$$

From **Figure 4** we find that, the kinetic energy distribution function of the gas has only a small influence on the solution, and the stellar structure functions are not sensitive to the values of the parameter Σ.

These are typical data for a neutron star [1]-[8]. The metric functions (a,b) and the mass, temperature distributions $(\mathcal{R},J)$ for this star are displayed in **Figure 5**.

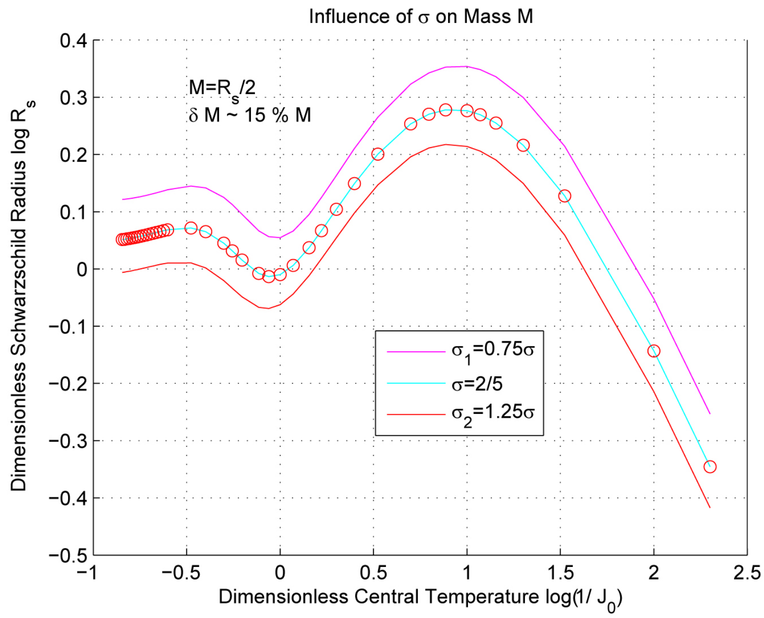

Figure 4. The influence of energy distribution on solutions. The results are not sensitive to σ.

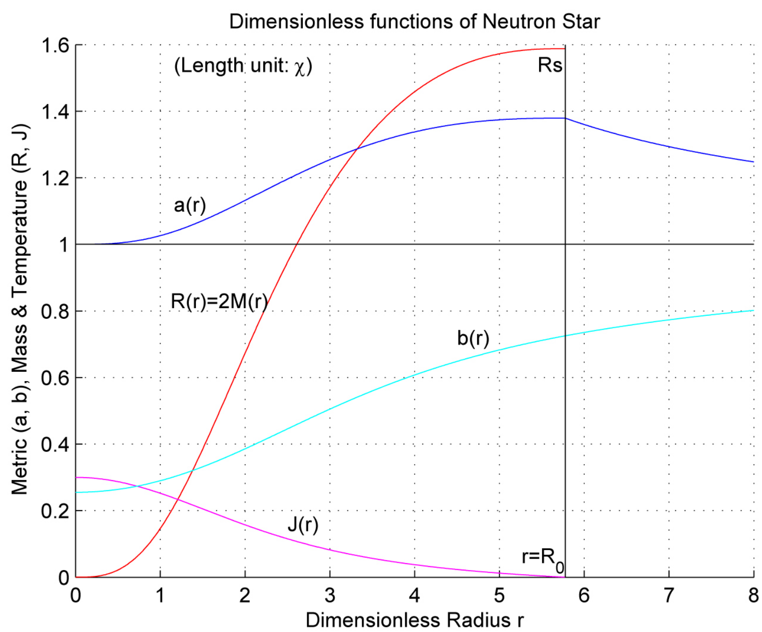

Figure 5. Structural functions for the compactest stars. The trends are typical for all stars.

5. Discussion and Conclusion

From the above calculation and analysis, we find the EOS of matter is decisive to the fate of a star. If EOS satisfies (3.27), then the EOS is compatible with Einstein's filed equation, and we always have singularity-free balanced star, no matter how heavy the star is. In this case, the driving effect of the gravity is dominated comparing with other interaction. Gravity is a conservative force, and the kinetic energy and the gravitational potential energy of particles inside interconvert into each other. The powerful gravity of a massive star leads to extreme high temperature inside the star. Instead of suddenly stopping at the center to wait for collapse into singularity, the falling particles will move quickly across the center and continue outward. From **Figure 3** we find that, for the most compact stars, the ratio of the stellar radius and the Schwarzschild radius $R/R_s \geq 5.77/1.59 = 3.63$. Therefore, the trapped surface inside the Schwarzschild horizon of singularity theorem is a concept of illusion, which cannot dynamically form. The collapse is a dynamical process which should be analyzed by detailed dynamics [11].

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Weinberg, S.L. (1972) Gravitation and Cosmology (Ch.11). Wiley, New York.

[2] Chandrasekhar, S. (1967) An Introduction to the Study of Stellar Structure. Dover Publications, New York.

[3] Lattimera, J.M. and Prakash, M. (2007) Neutron Star Observations: Prognosis for Equation of State Constraints. *Physics Reports*, **442**, 109-165. https://doi.org/10.1016/j.physrep.2007.02.003

[4] Blaschke, D., Klahn, T. and Sandin, F. (2007) Equation of State at High Densities and Modern Compact Star Observations. arXiv:0708.4216

[5] Schaab, C., *et al*. (1996) Thermal Evolution of Compact Stars. *Nuclear Physics A*, **605**, 531-565. https://doi.org/10.1016/0375-9474(96)00164-9

[6] Grigorian, H. (2007) Structure and Cooling of Compact Stars. arXiv:astro-ph/0702235

[7] Ma, C.Q. and Gao, C.Y. (2006) Phase Structures of Compact Stars in the Modified Quark-Meson Coupling Model. arXiv:nucl-th/0612107v3

[8] Krastev, P. G., Li, B.A. and Worley, A. (2007) Constraining Properties of Rapidly Rotating Neutron Stars Using Data from Heavy-Ion Collisions. arXiv:0709.3621v1

[9] Broomhall, A.-M., Chatterjee, P., Howe, R., Norton, A.A. and Thompson, M.J. (2014) The Sun's Interior Structure and Dynamics, and the Solar Cycle. arXiv:1411.5941v2

[10] Gu, Y.Q. (2020) Clifford Algebra and Unified Field Theory (Ch.7.2, Ch.9.2). LAP LAMBERT Academic Publishing, Chisinau.

[11] Gu, Y.Q. (2020) Functions and Relations for an Evolving Star with Spherical Symmetry. *Advances in Astrophysics*, **5**, 37-42. https://doi.org/10.22606/adap.2020.52001

[12] Sekiguchi, Y. and Shibata, M. (2004) Axisymmetric Collapse Simulations of Rotating Massive Stellar Cores in Full General Relativity: Numerical Study for Prompt Black Hole Formation. *Physical Review D*, **71**, Article ID: 084013. https://doi.org/10.1103/PhysRevD.71.084013

[13] Corda, C. and Feleppa, F. (2018) The Quantum Black Hole as a Gravitational Hydrogen Atom. Preprints. To Appear in *Advances in Theoretical and Mathematical Physics* (2022). https://doi.org/10.20944/preprints201810.0413.v3

[14] Oikonomou, V.K. and Fronimos, F.P. (2020) A Nearly Massless Graviton in Einstein-Gauss-Bonnet Inflation with Linear Coupling Implies Constant-Roll for the Scalar Field. *Europhysics Letters*, **131**, No. 3. https://doi.org/10.1209/0295-5075/131/30001