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Coherent States and Uncertainty Product of the Harmonic Oscillator with Position-Dependent Mass

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Abstract

In this work, we applied the invariant method to calculate the coherent state of the harmonic oscillator with position-dependent mass, which in modern physics has great application. We also obtain the calculation of Heisenberg's uncertainty principle, and we will show that it is verified.

Keywords

Schrodinger Equation, Wave Function, Coherent State, Harmonic Oscillator, Position Dependent Mass

1. Introduction

The study of the coherent state of problems involving Harmonic oscillators with position-dependent mass has attracted considerable interest in the past few years [1]-[10]. Apart from their intrinsic mathematical interest, these problems have involked much attention because of their connections with many other problems belonging to different areas of physics, such as plasma physics, gravitation, quantum optics, quantum liquids, and nonlinear oscillator.

Moreover, one can find the formalism of position—dependent-mass in many other branches of physics, quantum information, etc.

Referring to literature, one can also see that the problem associated with position-dependent mass has attracted many researchers and still has very popular applications in various branches of physics. Much of the works on the position-dependent mass systems have been studied with a singular mass of the type [11] [12] [13]. We choose the position dependent mass under the form:

$$m(x) = m_0 e^{-ax^2} \tag{I.1}$$

The position stated above is asymmetric in nature. In fact, it has been seen that mass has an asymmetric function of position(x) in semiconductor physics. if a is negative, then the (Equation (I.1)) will be 0 at large values of x.

2. Position-Dependent Mass of the Harmonic Oscillator

The Harmonic oscillator Hamiltonien considered here is:

$$H = \frac{p_x^2}{2m(x)} + \frac{m(x)\omega_0^2}{2} x^2 \tag{II.1}$$

where x and p_x are canonically conjucate with $[x, p] = i\hbar$ and $m(x)$ and ω_0 are, respectively, the mass (position dependent) and constant frequency associated with the oscillator, and which are arbitary real function of the time. From Equation (II.1) we obtain the motion equation:

$$\ddot{x}(t) + \rho(t)\dot{x}(t) + \omega_0^2 x(t) = 0 \tag{II.2}$$

where:

$$\rho(t) = \frac{\dot{m}(x)}{m(x)} \tag{II.3}$$

The Hamiltonian of the Equation (II.1) can be transformed to H_1';

$$H_1' = \frac{P_X^2}{2m_0} + \frac{m_0\Omega^2}{2} X^2 \tag{II.4}$$

where:

$$\Omega^2(t) = \omega_0^2 - \left[\frac{1}{4}\left[\frac{\dot{m}(x)}{m(x)}\right]^2 + \frac{\ddot{m}(x)m(x) - \dot{m}(x)^2}{2m(x)^2}\right] \tag{II.5}$$

By making the following change of variable:

$$X = \left[\frac{m_0}{m(x)}\right]^{\frac{1}{2}} x \tag{II.6}$$

$$P_X = \left[\frac{m_0}{m(x)}\right]^{\frac{1}{2}} p_x + \left[m_0 m(x)\right]^{\frac{1}{2}} \frac{\dot{m}(x)}{2m(x)} x \tag{II.7}$$

where m_0 is a constant mass. Note that $[X, P] = [x, p]$, which implies that the commutation relations remain the same in both coodonates. Also, observe that the Hamiltonian Equation (II.4) is of the form of that considered by Lewis and Reissenferd [14] [15] [16].

3. Resolution of the Schrodinger Equation of the Harmonic Oscllator with Position Dependent Mass by the Invariant Method

It is well known that an exact invariant for Equation (II.4) is given by [17]:

$$I(t)=\frac{1}{2m_0}\left[m_0^2X^2\alpha^{-2}+\left(P_X\alpha-m_0\dot{\alpha}X\right)^2\right] \tag{III.1}$$

where X satisfies that equation:

$$\ddot{X}+\Omega^2(X)X=0 \tag{III.2}$$

And α function satisfies of the equation:

$$\ddot{\alpha}+\Omega^2(X)\alpha=\frac{1}{\alpha^3} \tag{III.3}$$

The invariant $I(t)$ satisfies the equation [18] [19]:

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{\partial I}{\partial t}+\frac{1}{i\hbar}\left[I,H_1'\right]=0,\ I^+=I \tag{III.4}$$

We will choose the real solutions of Equation (III.3), so that we can make $I(t)$ Hermetian. Further, the eigenfunctions $\varphi_n(X,t)$ of $I(t)$ are considered to form. Complete orthogonal set corresponding to the time and position-independent eigenvalue ϑ_n. Thus:

$$I\varphi_n(X,t)=\vartheta_n\varphi_n(X,t) \tag{III.5}$$

$$\left(\varphi_{n'}(X,t),\varphi_n(X,t)\right)=\delta_{n'n} \tag{III.6}$$

Let the time-dependant Schrödinger equation be:

$$i\hbar\frac{\partial\Psi}{\partial t}=H_1\Psi \tag{III.7}$$

$$\text{With }\ H_1=\frac{-\hbar}{2m_0}\frac{\partial^2}{\partial^2X}+m_0\frac{\Omega^2}{2}X^2 \tag{III.8}$$

with $P=-i\hbar\frac{\partial}{\partial X}$ has been used. The solutions $\Psi_n(X,t)$ of the Schrodinger Equation (III.7) are related to $\varphi_n(X,t)$ by the relation [20]:

$$\Psi_n(X,t)=\mathrm{e}^{i\mu_n(t)}\varphi_n(X,t) \tag{III.9}$$

The phase function $\mu_n(t)$ satisfy to the equation:

$$\hbar\frac{\mathrm{d}\mu_n(t)}{\mathrm{d}t}=\left\langle\varphi_n\left|i\frac{\partial}{\partial t}-H_1(t)\right|\varphi_n\right\rangle \tag{III.10}$$

knowing that each Ψ_n satisfies the Schrödinger equation, then the general solution of (III.7) verifies the following equation:

$$\Psi_n(X,t)=\sum\nolimits_n C_n\mathrm{e}^{i\mu_n(t)}\varphi_n(X,t) \tag{III.11}$$

With C_n are time-independent coffecients.

4. Solution of the Schrodinger Equation with Position Dependent Mass

In this works we are interested in solving the schrodinger equation with position dependent mass.we consider the unitary transformation:

$$V=\mathrm{e}^{\frac{-im_0\dot{\alpha}X^2}{2\hbar\alpha}} \tag{IV.1}$$

The eigenvalue Equation (III.5) becomes:

$$I'\varphi_n'(y)=\lambda_n\varphi_n'(y) \tag{IV.2}$$

where:

$$I'=VIV^{+}=\frac{-\hbar^2}{2m_0}\frac{\partial^2}{\partial^2 y}+\frac{m_0}{2}y^2 \tag{IV.3}$$

and:

$$\varphi_n'=\alpha^{1/2}U\varphi_n \tag{IV.4a}$$

$$y=\frac{X}{\alpha} \tag{IV.4b}$$

The Equation (IV.2) is an ordinary on –dimensionnal Schrodinger equation whose equation solution is given by:

$$\varphi_n'(X,t)=\left[\frac{m_0^{1/2}}{\pi^{1/2}\hbar^{1/2}n!2^n}\right]\mathrm{e}^{-\frac{m_0}{2\hbar\left(\frac{X}{\alpha}\right)}}H_n\left\{\frac{m_0X}{\hbar\alpha}\right\} \tag{IV.5}$$

For the haronic oscillator, we have

$$\lambda_n=\hbar\left(n+\frac{1}{2}\right) \tag{IV.6}$$

The Hermitian polynomial of order n is H_n, thus, by using (Equations (III.9), (VI.1), (VI.4) and (IV.5)) we find that the solution of the transformed Schrodinger Equation (III.7) is given by:

$$\Psi_n(X,t)=\mathrm{e}^{i\alpha_n(t)}\left\{\frac{m_0^{1/2}}{\pi^{\frac{1}{2}}\hbar^{\frac{1}{2}}n!2^n\alpha(t)}\right\}^{1/2}\exp\left\{\frac{im_0}{2\hbar}\left[\frac{\dot{\alpha}(t)}{\alpha(t)}+\frac{i}{\alpha(t)^2}\right]X^2\right\}\times H_n\left\{\left\{\frac{m_0}{\hbar}\right\}^{1/2}\frac{X}{\alpha(t)}\right\} \tag{IV.7}$$

Whith the phase functions $\gamma(t)$ are given by [20]:

$$\gamma(t)=-\left(n+\frac{1}{2}\right)\int_0^t\frac{\mathrm{d}t'}{\alpha^2(t')} \tag{IV.8}$$

in this work we observe that solutions IV.7 for Equation (III.7) were also obtained by Khandekar and Lawande [20] using Feynman path integrals.

By making a change of variable dependent on time [21]:

$$\alpha(t)=\left[\frac{m(x)}{m_0}\right]^{1/2}x(t) \tag{IV.9}$$

where $x(t)$ is the position, and is a real function of the time that is to dermined.

Equation of motion is giveen by:

$$\ddot{x}+\rho(x)\dot{x}+\omega_0^2x=\frac{m_0}{m(x)x^3} \tag{IV.10}$$

If we chose the position dependent mass: $m(x)=\mathrm{e}^{a_0x}$ and $m_0=\mathrm{e}^{a_0}$, where the equation of motion is given by:

$$\ddot{x}+a_0\dot{x}^2+\omega_0^2 x=\frac{1}{e^x x^3} \quad \text{(IV.11)}$$

5. Coherent States and Uncertainty Product

Coherent states play an important role in quantum optics, especially in laser physics and much work was performed in this field by Roy J. Glauber who was awarded the 2005 Nobel prize for his contribution to the quantum theory of optical coherence. We will try here to give a good overview of coherent states of laser beams. The state describing a laser beam can be briefly characterized as having by an indefinite number of photons, but a precisely defined phase, in contrast to a state with fixed particle number, where the phase is completely random. There also exists an uncertainty relation describing this contrast, which we will plainly state here but won't prove [22] [23] [24] [25]. It can be formulated for the uncertainties of amplitude and phase of the state, where the inequality reaches a minimum for coherent states, or, as we will do here, for the occupation number n and the phase φ_n [22] [26] [27]. Next the coherent states of the harmonic oscillator with position-dependent mass, we proced as follows consider the operator a and a^+ given by:

$$a=\left[\frac{1}{2m(x)\hbar}\right]^{1/2}\left[m\left[\frac{X}{\alpha(t)}\right]+i\alpha P\right] \quad \text{(V.1.a)}$$

$$a^+=\left[\frac{1}{2m(x)\hbar}\right]^{1/2}\left[m\left[\frac{X}{\alpha(t)}\right]-i\alpha P\right] \quad \text{(V.1.b)}$$

Whith $\left[a,a^+\right]=1$, in terms of a and a^+ the invariant I' can be written as:

$$I'=\hbar\left(aa^+ +\frac{1}{2}\right) \quad \text{(V.2)}$$

Now, Hartleyand and Ray [27] have shown that coherent states for I' have the form:

$$\varphi'_k(\sigma,t)=e^{-\left|k^2/2\right|}\sum_n\frac{k^n}{(n!)^{1/2}}e^{k_n(t)}\varphi'_n(\sigma) \quad \text{(V.3)}$$

where $k_n(t)$ is given by VI.7 and k is an arbitrary complex number. Note that when $\alpha(t)\to\alpha_0=\frac{1}{\omega_0^{1/2}}$ the coherent states $\varphi'_k(\sigma,t)$ became the correct coherent states for the usual time independent harmonic oscillator. The coherent states for theharmonic oscillator with position dependent-mass are obtained by the inverse transformation on $\varphi'_k(\sigma,t)$ or given by in this work with $m(x)=e^{a_0 x}$, we have:

$$\varphi_k(x,t)=a_0^{\frac{1}{2}}(\dot{x})^{-\frac{1}{2}}x e^{i\frac{\ddot{x}x^2}{2\hbar\dot{x}}}\varphi'_k(\alpha,t) \quad \text{(V.4)}$$

We can be rewite the state in the form:

$$\varphi_k(x,t)=\left\{\frac{x}{\mathrm{e}^{1/2x}}\right\}^{-1/2}\exp\left\{-i\mathrm{e}^{a_0x}\left[\dot{x}+\frac{\gamma(t)}{2}x\right]\left[\frac{x^2}{2\hbar x}\right]\right\}\varphi_k'(\sigma,t) \quad (V.5)$$

With $\varphi_k(x,t)$ is the coherent state that we have shown for the harmonic oscillator with position-dependent mass, and these states correspond to the following eigenvalue equation:

$$A\varphi_k(x,t)=k(t)\varphi_k(x,t) \quad (V.6)$$

With:

$$A=V^{+}aV=\left[\frac{1}{2m_0\hbar}\right]^{1/2}\left[m_0+i\left(xp-m(x)\dot{x}x\right]\right. \quad (V.7)$$

and:

$$k(t)=k_0\mathrm{e}^{2ik_0(t)} \quad (V.8)$$

$$k_0(t)=-\frac{1}{2}\int_0^t\frac{m_0\mathrm{d}t'}{m(x)x^2(t')} \quad (V.9)$$

In this work we choose the mass $m(x)=m_0=cte$, the states reduce to the coherent states of the time-dependent harmonic oscillator.

we calculated the product of the uncertainties of x and p in the state $\varphi_k(x,t)$, we have:

$$(\Delta x)^2=\frac{\hbar^2}{2\mathrm{e}^{a_0x}}x^2 \quad (V.10.a)$$

$$(\Delta x)^2=\frac{\hbar\mathrm{e}^{a_0x}}{2}\left[\frac{1}{x^2}+\mathrm{e}^{2x}x^2\dot{x}^2\right]^{1/2} \quad (V.10.b)$$

In this work the uncertainty product is given by the following expression:

$$\Delta x\Delta p=\frac{\hbar}{2}\left[1+\mathrm{e}^{2x}x^2\dot{x}^2\right]^{1/2}$$

6. Conclusions

The study and analysis of coherent states play an interesting role in quantum physics, in particular, in the study of quantum optics. Quantum superpositions are particularly interesting to coherent states, which exhibit various non-classical effects such as compression, phase coherence, and sub-poissonian statistics due to the quantum interference between the consistent components [28] [29].

Moreover, coherent states have been of particular interest in the context of quantum information theory whose main subjects are quantum computation, quantum teleportation and quantum cryptography [30]. In the past few years, there have been an upsurge of research about the quantum teleportation of a superposed coherent state in terms of optical elements [31] [32].

The efficiency of coherent states in various fields of semiclassical physics thanks to its interest, we describe various aspects of semiclassical coherent state propagation, ranging from recent experiments in atomic physics to the mathematical aspects of “quantum chaology” that predate the results we found.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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