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On a Nonlinear Volterra-Fredholm Integrodifferential Equation on Time Scales

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Abstract

The main aim in this work is to obtain an integral inequality with a clear estimate on time scales. The obtained inequality is used as a tool to investigate some basic qualitative properties of solutions to certain nonlinear Volterra-Fredholm integrodifferential equations on time scales.

Subject Areas

Mathematical Analysis

Keywords

Integrodifferential Equations, Time Scales, Integral Inequality, Estimate on the Solutions

1. Introduction

The theory of time scales had been begun in 1988 by Stefan Hilger [1], in order to develop a theory that can standardize a continuous and discrete analysis. Recently several authors in this field have investigated various forms of integral and integrodifferential equations under different hypotheses by using different ways, see [4] [5] [6] [7] [8]. In this article we consider the nonlinear integrodifferential equation of the following form

$$y^{\Delta}(t) = h\left(t, y(t), y^{\Delta}(t), \int_{\alpha}^{t} h_1\left(t, z, y(z), y^{\Delta}(z)\right)\Delta z, \int_{\alpha}^{\beta} h_2\left(t, z, y(z), y^{\Delta}(z)\right)\Delta z\right), \quad (1.1)$$
$$t \in J_{\mathbb{T}} \text{ with the initial condition } y(\alpha) = y_0,$$

where y is unknown function and $h : J_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $h_1, h_2 : J_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and h, h_1, h_2 are given functions, assuming them to be rd-continuous functions, $\alpha < \beta, z \le t$ and $J_{\mathbb{T}} = J \cap \mathbb{T}, J = [\alpha, \infty)$. We denote a time scale by $\mathbb{T}$ which is nonempty closed subset of $\mathbb{R}$. $\mathbb{R}^n$ denotes

Euclidean space with a suitable norm defined by $|.|$.

We can investigate the existence and uniqueness results for (1.1) by using the technique present in [6].

2. Preliminaries

The operators $\sigma(t)$ and $\rho(t)$ denote the forward and backward operators respectively which are defined by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}\in\mathbb{T}$ and $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}\in\mathbb{T}$, for all $t\in\mathbb{T}$.

For $t\in\mathbb{T}$, If $t<\sup\mathbb{T}$ and $\sigma(t)=t$, then t is said to be right-dense; while If $t>\inf\mathbb{T}$ and $\rho(t)=t$, then t is said to be left-dense. The graininess $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t)=\sigma(t)-t$. The set $\mathbb{T}^k$ is denoted by

$$\mathbb{T}^k=\begin{cases}\mathbb{T}\setminus(\rho(\sup\mathbb{T}),\sup\mathbb{T}] & \text{if } \sup\mathbb{T}<\infty\\ \mathbb{T} & \text{otherwise}\end{cases}$$

Let $z:\mathbb{T}\to\mathbb{R}$, $t\in\mathbb{T}^k$, then $z^\Delta(t)$ denotes the delta derivative of z at t which is exist with the property that given $\varepsilon>0$ there is a neighbourhood U of t such that $|z(\sigma(t),\tau)-z(s,\tau)-z^\Delta(t,\tau)(\sigma(t)-s)|\le\varepsilon|\sigma(t)-s|$ for all $s\in U$. Then $g(t)=\int_\alpha^t z(t,\tau)\Delta\tau$ implies $g^\Delta(t)=\int_\alpha^t z^\Delta(t,\tau)\Delta\tau+z(\sigma(t),t)$. If a function $g:\mathbb{T}\to\mathbb{R}$ is continuous at any right-dense point $t\in\mathbb{T}$ and the left-hand limits exists (finite) at any left-dense point $t\in\mathbb{T}$, then g is said to be rd-continuous. C_{rd} denotes the class of all rd-continuous functions. We denote the class of all regressive functions by $\mathcal{R}$ which is defined by

$$\mathcal{R}=\{p\in C_{rd}(\mathbb{T},\mathbb{R})\text{ and }1+p(t)\mu(t)\neq 0,\forall t\in\mathbb{T}\}$$

For $p\in\mathcal{R}$, we define $e_p(t,s)=\exp\left(\int_s^t \xi_\mu(\tau)(p(\tau))\Delta\tau\right)$ for $t,s\in\mathbb{T}$, with the cylinder transformation $\xi_h(\tau)=\begin{cases}\dfrac{\log(1+hz)}{h} & \text{if } h\neq 0\\ z \text{ if } h=0\end{cases}$.

For more basic information about time scales calculus, see [1] [3].

We need the following result given in [2].

Lemma 2.1. suppose $v,b\in C_{rd}$ and $a\in\mathcal{R}^+$. Then

$$v^\Delta(t)\le a(t)v(t)+b(t),\text{ for all } t\in\mathbb{T}$$

Implies $v(t)\le v(\alpha)e_a(t,\alpha)+\int_\alpha^t e_a(t,\sigma(\tau))b(\tau)\Delta\tau$, for all $t\in\mathbb{T}$.

3. Main Results

In the following result we establish an integral inequality on time scales.

Theorem 3.1. Let $v,r,b_1,b_2,p,q,g,d\in C_{rd}(J_\mathbb{T},\mathbb{R}+)$ and assume that

$$\begin{aligned}v(t)\le r(t)+b_1(t)\int_\alpha^t\Big\{&[b_2(t)p(\tau)+1]v(\tau)+b_2(\tau)\int_\alpha^\tau p(z)v(z)\Delta z\\&+q(\tau)\int_\alpha^\beta g(z)v(z)\Delta z\Big\}\Delta\tau+d(t)\int_\alpha^\beta g(z)v(z)\Delta z,\ t\in J_\mathbb{T},\end{aligned}\tag{3.1}$$

If

$$N=\int_{\alpha}^{\beta} g(\gamma) N_2(\gamma)\Delta\gamma<1, \tag{3.2}$$

Implies

$$v(t)\le N_1(t)+AN_2(t), t\in J_{\mathbb{T}}, \tag{3.3}$$

where

$$N_1(t)=r(t)+b_1(t)\int_{\alpha}^{t}\left[b_2^{\sigma}(\tau)p(\tau)+1\right]\left[r(\tau)+b(\tau)\int_{\alpha}^{\tau}e_{\left(b_2^{\sigma}p+p+1\right)b}(\tau,\sigma(z))\times\left(b_2^{\sigma}(z)p(z)+p(z)+1\right)r(z)\Delta z\right]\Delta\tau, \tag{3.4}$$

$$N_2(t)=d(t)+b_1(t)\int_{\alpha}^{t}\left\{\left[b_2^{\sigma}(\tau)p(\tau)+1\right]\left[d(\tau)+b(\tau)\int_{\alpha}^{\tau}e_{\left(b_2^{\sigma}p+p+1\right)b}(\tau,\sigma(z))\times\left(\left[b_2^{\sigma}(z)p(z)+p(z)+1\right]d(z)+q(z)\right)\Delta z\right]+q(\tau)\right\}\Delta\tau, \tag{3.5}$$

for $t\in J_{\mathbb{T}}$,

$$b_2^{\sigma}(t)=b_2(\sigma(t))=b_2\circ\sigma, b(t)=\max_{t\in J_{\mathbb{T}}}\left\{b_1(t), b_2(t)+b_2^{\Delta}(t)\right\}, \tag{3.6}$$

$$A=\frac{1}{1-N}\int_{\alpha}^{\beta} g(\gamma)N_1(\gamma)\Delta\gamma, \tag{3.7}$$

Proof. Let

$$\lambda=\int_{\alpha}^{\beta} g(z)v(z)\Delta z, \tag{3.8}$$

we shall define functions $B_1(t)$ and $B_2(t)$ by

$$B_1(t)=\int_{\alpha}^{t}\left\{\left[b_2(t)p(\tau)+1\right]v(\tau)+b_2(\tau)\int_{\alpha}^{\tau}p(z)v(z)\Delta z+q(\tau)\int_{\alpha}^{\beta}g(z)v(z)\Delta z\right\}\Delta\tau, \tag{3.9}$$

$$B_2(t)=B_1(t)+\int_{\alpha}^{t}p(z)\left[r(z)+b_1(z)B_1(z)+d(z)\lambda\right]\Delta z, \tag{3.10}$$

then $B_1(\alpha)=0$, $B_2(\alpha)=0$, $B_1(t)\le B_2(t)$ and we have

$$v(t)\le r(t)+b_1(t)B_1(t)+d(t)\lambda, \tag{3.11}$$

from (3.9), we get

$$\begin{aligned} B_1^{\Delta}(t)&=\int_{\alpha}^{t}b_2^{\Delta}(t)p(\tau)v(\tau)\Delta\tau+\left[b_2^{\sigma}(t)p(t)+1\right]v(t)\\ &\quad+b_2(t)\int_{\alpha}^{t}p(z)v(z)\Delta z+q(t)\int_{\alpha}^{\beta}g(z)v(z)\Delta z\\ &\le\left[b_2^{\sigma}(t)p(t)+1\right]r(t)+\left[b_2^{\sigma}(t)p(t)+1\right]b_1(t)B_1(t)\\ &\quad+\left[b_2^{\sigma}(t)p(t)+1\right]d(t)\lambda+\left[b_2(t)+b_2^{\Delta}(t)\right]\int_{\alpha}^{t}p(z)v(z)\Delta z+q(t)\lambda\\ &\le\left[b_2^{\sigma}(t)p(t)+1\right]r(t)+b_2^{\sigma}(t)p(t)b(t)B_1(t)\\ &\quad+b(t)\left[B_1(t)+\int_{\alpha}^{t}p(z)v(z)\Delta z\right]+\left[b_2^{\sigma}(t)p(t)+1\right]d(t)\lambda+q(t)\lambda\\ &\le\left[b_2^{\sigma}(t)p(t)+1\right]\left[r(t)+b(t)B_2(t)+d(t)\lambda\right]+q(t)\lambda, \end{aligned} \tag{3.12}$$

integrating the inequality (3.12) and using $B_1(\alpha)=0$, we have

$$B_1(t) \le \int_\alpha^t \left\{\left[b_2^\sigma(\tau)p(\tau)+1\right]\left[r(\tau)+b(\tau)B_2(\tau)+d(\tau)\lambda\right]+q(\tau)\lambda\right\}\Delta\tau, \quad (3.13)$$

therefore

$$\begin{aligned} B_2^\Delta(t) &= B_1^\Delta(t)+p(t)\left[r(t)+b_1(t)B_1(t)+d(t)\lambda\right] \\ &\le \left[b_2^\sigma(t)p(t)+1\right]\left[r(t)+b(t)B_2(t)+d(t)\lambda\right]+q(t)\lambda \\ &\quad + p(t)\left[r(t)+b(t)B_2(t)+d(t)\lambda\right] \\ &= \left[b_2^\sigma(t)p(t)+p(t)+1\right]b(t)B_2(t) \\ &\quad + \left[b_2^\sigma(t)p(t)+p(t)+1\right]\left[r(t)+d(t)\lambda\right]+q(t)\lambda, \end{aligned} \quad (3.14)$$

now applying lemma 2.1, we get

$$\begin{aligned} B_2(t) &\le \int_\alpha^t e_{\left(b_2^\sigma p+p+1\right)b}(t,\sigma(z)) \\ &\quad \times\left(\left[b_2^\sigma(z)p(z)+p(z)+1\right]\left[a(z)+d(z)\lambda\right]+q(z)\lambda\right)\Delta z, \end{aligned} \quad (3.15)$$

from (3.11), (3.13) and (3.15), we obtain that

$$v(t) \le N_1(t)+\lambda N_2(t), \quad (3.16)$$

and from (3.8) and (3.16) we observe that

$$\lambda \le A, \quad (3.17)$$

using (3.17) in (3.16) we obtain (3.3). □

We provide the result that includes the estimate on the solutions of (1.1) as follows.

Theorem 3.2. Assume that the following conditions satisfied

$$\left|h(t,v_1,v_2,v_3,v_4)\right| \le L\left[|v_1|+|v_2|+|v_3|+|v_4|\right], \quad (3.18)$$

$$\left|h_1(t,z,u,v)\right| \le c_1(t)s_1(z)\left[|u|+|v|\right], \quad (3.19)$$

$$\left|h_2(t,z,u,v)\right| \le c_2(t)s_2(z)\left[|u|+|v|\right], \quad (3.20)$$

for the functions h,h_1,h_2 in (1.1), where $0 \le L < 1$ is a constant and

$$c_1,s_1,c_2,s_2 \in C_{rd}\left(J_{\mathbb{T}},\mathbb{R}+\right)$$

If $y(t)$ is a solution of (1.1) on $J_{\mathbb{T}}$, then

$$|y(t)|+\left|y^\Delta(t)\right| \le M_1(t)+D_1M_2(t), t\in J_{\mathbb{T}}, \quad (3.21)$$

where

$$\begin{aligned} M_1(t) &= \frac{|y_0|}{1-L}+\frac{L}{1-L}\int_\alpha^t\left[c_1^\sigma(\tau)s_1(\tau)+1\right]\left[\frac{|y_0|}{1-L}+b(\tau)\int_\alpha^\tau e_{\left(c_1^\sigma s_1+s_1+1\right)b}(\tau,\sigma(z))\right. \\ &\quad \left.\times\left(c_1^\sigma(z)s_1(z)+s_1(z)+1\right)\frac{|y_0|}{1-L}\Delta z\right]\Delta\tau, t\in J_{\mathbb{T}} \end{aligned} \quad (3.22)$$

$$\begin{aligned} M_2(t) &= \frac{L}{1-L}c_2(t) \\ &\quad +\frac{L}{1-L}\int_\alpha^t\left\{\left[c_1^\sigma(\tau)s_1(\tau)+1\right]\left[\frac{L}{1-L}c_2(\tau)+b(\tau)\int_\alpha^\tau e_{\left(c_1^\sigma s_1+s_1+1\right)b}(\tau,\sigma(z))\right.\right. \\ &\quad \left.\left.\times\left(\left[c_1^\sigma(z)s_1(z)+s_1(z)+1\right]\frac{L}{1-L}c_2(z)+c_2(z)\right)\Delta z\right]+c_2(\tau)\right\}\Delta\tau, t\in J_{\mathbb{T}} \end{aligned}$$

(3.23)

Assume that

$$b(t)=\max_{t\in J_{\mathbb{T}}}\left\{\frac{L}{1-L},c_1(t)+c_1^{\Delta}(t)\right\}, \tag{3.24}$$

$$\lambda=\int_{\alpha}^{\beta}s_2(\gamma)M_2(\gamma)\Delta\gamma<1, \tag{3.25}$$

$$D_1=\frac{1}{1-\lambda}\int_{\alpha}^{\beta}s_2(\gamma)M_1(\gamma)\Delta\gamma, \tag{3.26}$$

Proof. Let $a(t)=|y(t)|+|y^{\Delta}(t)|, t\in J_{\mathbb{T}}$, since $y(t)$ is a solution of (1.1), then by using this and the hypotheses, we get

$$\begin{aligned}a(t)=&\Big|y_0+\int_{\alpha}^{t}h\Big(\tau,y(\tau),y^{\Delta}(\tau),\int_{\alpha}^{\tau}h_1\big(\tau,z,y(z),y^{\Delta}(z)\big)\Delta z,\\&\int_{\alpha}^{\beta}h_2\big(\tau,z,y(z),y^{\Delta}(z)\big)\Delta z\Big)\Delta\tau\Big|\\&+\Big|h\Big(t,y(t),y^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,y(z),y^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(t,z,y(z),y^{\Delta}(z)\big)\Delta z\Big)\Big|\\\le&|y_0|+\int_{\alpha}^{t}L\Big[a(\tau)+\int_{\alpha}^{\tau}c_1(\tau)s_1(z)a(z)\Delta z+\int_{\alpha}^{\beta}c_2(\tau)s_2(z)a(z)\Delta z\Big]\Delta\tau\\&+L\Big[a(t)+\int_{\alpha}^{t}c_1(t)s_1(z)a(z)\Delta z+\int_{\alpha}^{\beta}c_2(t)s_2(z)a(z)\Delta z\Big]\end{aligned}$$

from the above inequality, we have

$$\begin{aligned}a(t)\le&\frac{|y_0|}{1-L}+\frac{L}{1-L}\int_{\alpha}^{t}\Big\{\big[c_1(t)s_1(\tau)+1\big]a(\tau)+c_1(\tau)\int_{\alpha}^{\tau}s_1(z)a(z)\Delta z\\&+c_2(\tau)\int_{\alpha}^{\beta}s_2(z)a(z)\Delta z\Big\}\Delta\tau+\frac{L}{1-L}c_2(t)\int_{\alpha}^{\beta}s_2(z)a(z)\Delta z,\end{aligned} \tag{3.27}$$

Now applying theorem 3.1 in (3.27) we obtain (3.21). □

Remark 3.3. Since $y(t)$ is a solution of (1.1). Then (3.21) yields the bounds on $y(t)$ and $y^{\Delta}(t)$. If the estimate in (3.21) is bounded, implies the solution $y(t)$ and $y^{\Delta}(t)$ are also bounded on $J_{\mathbb{T}}$.

Consider (1.1) with the following corresponding equation

$$\begin{aligned}Y^{\Delta}(t)=H\Big(&t,Y(t),Y^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,Y(z),Y^{\Delta}(z)\big)\Delta z,\\&\int_{\alpha}^{\beta}h_2\big(t,z,Y(z),Y^{\Delta}(z)\big)\Delta z\Big),t\in J_{\mathbb{T}},\end{aligned}$$

with the initial condition

$$Y(\alpha)=Y_0 \tag{3.28}$$

where $H\in C_{rd}\left(J_{\mathbb{T}}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n\right),h_1,h_2$ as in (1.1).

The next result concerning the closeness of solution of (1.1) and (3.28).

Theorem 3.4. Suppose that the following conditions satisfied

$$\begin{aligned}&\left|h(t,v_1,v_2,v_3,v_4)-h(t,\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4)\right|\\&\le L\big[|v_1-\bar{v}_1|+|v_2-\bar{v}_2|+|v_3-\bar{v}_3|+|v_4-\bar{v}_4|\big],\end{aligned} \tag{3.29}$$

$$\left|h_1(t,z,u,v)-h_1(t,z,\bar{u},\bar{v})\right|\le c_1(t)s_1(z)\big[|u-\bar{u}|+|v-\bar{v}|\big], \tag{3.30}$$

$$\left|h_2(t,z,u,v)-h_2(t,z,\bar{u},\bar{v})\right|\le c_2(t)s_2(z)\big[|u-\bar{u}|+|v-\bar{v}|\big], \tag{3.31}$$

where the functions h, h_1, h_2 in (1.1), and $0 \le L < 1$ is a constant.

Also $c_1, s_1, c_2, s_2 \in C_{rd}(J_{\mathbb{T}}, \mathbb{R}+)$, and

$$\left| h(t, v_1, v_2, v_3, v_4) - H(t, v_1, v_2, v_3, v_4) \right| \le \varepsilon, \tag{3.32}$$

$$\left| y_0 - Y_0 \right| \le \delta, \tag{3.33}$$

where y_0 and H, Y_0 as in (1.1) and (3.28) respectively.

If $y(t)$ and $Y(t)$ be solutions of (1.1) and (3.28) on $J_{\mathbb{T}}$, then

$$\left| y(t) - Y(t) \right| + \left| y^{\Delta}(t) - Y^{\Delta}(t) \right| \le M_3(t) + D_2 M_2(t), t \in J_{\mathbb{T}}, \tag{3.34}$$

where $M_3(t)$ is described by the right side of (3.22) by substituting $m(t) = \delta + \varepsilon(1 + t - \alpha)$ instead of $|y_0|$, $M_2(t), b(t)$ and λ be as in (3.23), (3.24) and (3.25) respectively and

$$D_2 = \frac{1}{1-\lambda} \int_{\alpha}^{\beta} r_2(\gamma) M_3(\gamma) \Delta\gamma, \tag{3.35}$$

Proof. Let $w(t) = \left| y(t) - Y(t) \right| + \left| y^{\Delta}(t) - Y^{\Delta}(t) \right|, t \in J_{\mathbb{T}}$, we have

$$\begin{aligned}
w(t) &\le \left| y_0 - Y_0 \right| \\
&+ \int_{\alpha}^{t} \Big| h\Big(\tau, y(\tau), y^{\Delta}(\tau), \int_{\alpha}^{\tau} h_1\big(\tau, z, y(z), y^{\Delta}(z)\big)\Delta z, \int_{\alpha}^{\beta} h_2\big(\tau, z, y(z), y^{\Delta}(z)\big)\Delta z\Big) \\
&- h\Big(\tau, Y(\tau), Y^{\Delta}(\tau), \int_{\alpha}^{\tau} h_1\big(\tau, z, Y(z), Y^{\Delta}(z)\big)\Delta z, \int_{\alpha}^{\beta} h_2\big(\tau, z, Y(z), Y^{\Delta}(z)\big)\Delta z\Big) \Big| \Delta\tau \\
&+ \int_{\alpha}^{t} \Big| h\Big(\tau, Y(\tau), Y^{\Delta}(\tau), \int_{\alpha}^{\tau} h_1\big(\tau, z, Y(z), Y^{\Delta}(z)\big)\Delta z, \int_{\alpha}^{\beta} h_2\big(\tau, z, Y(z), Y^{\Delta}(z)\big)\Delta z\Big) \\
&- H\Big(\tau, Y(\tau), Y^{\Delta}(\tau), \int_{\alpha}^{r} h_1\big(\tau, z, Y(z), Y^{\Delta}(z)\big)\Delta z, \int_{\alpha}^{\beta} h_2\big(\tau, z, Y(z), Y^{\Delta}(z)\big)\Delta z\Big) \Big| \Delta\tau \\
&+ \Big| h\Big(t, y(t), y^{\Delta}(t), \int_{\alpha}^{t} h_1\big(t, z, y(z), y^{\Delta}(z)\big)\Delta z, \int_{\alpha}^{\beta} h_2\big(t, z, y(z), y^{\Delta}(z)\big)\Delta z\Big) \\
&- h\Big(t, Y(t), Y^{\Delta}(t), \int_{\alpha}^{t} h_1\big(t, z, Y(z), Y^{\Delta}(z)\big)\Delta z, \int_{\alpha}^{\beta} h_2\big(t, z, Y(z), Y^{\Delta}(z)\big)\Delta z\Big) \Big| \\
&+ \Big| h\Big(t, Y(t), Y^{\Delta}(t), \int_{\alpha}^{t} h_1\big(t, z, Y(z), Y^{\Delta}(z)\big)\Delta z, \int_{\alpha}^{\beta} h_2\big(t, z, Y(z), Y^{\Delta}(z)\big)\Delta z\Big) \\
&- H\Big(t, Y(t), Y^{\Delta}(t), \int_{\alpha}^{t} h_1\big(t, z, Y(z), Y^{\Delta}(z)\big)\Delta z, \int_{\alpha}^{\beta} h_2\big(t, z, Y(z), Y^{\Delta}(z)\big)\Delta z\Big) \Big| \\
&\le \delta + \int_{\alpha}^{t} L\Big[w(\tau) + \int_{\alpha}^{\tau} c_1(\tau) s_1(z) w(z)\Delta z + \int_{\alpha}^{\beta} c_2(\tau) s_2(z) w(z)\Delta z \Big]\Delta\tau \\
&\quad + \int_{\alpha}^{t} \varepsilon\Delta\tau + L\Big[w(t) + \int_{\alpha}^{t} c_1(t) s_1(z) w(z)\Delta z + \int_{\alpha}^{\beta} c_2(t) s_2(z) w(z)\Delta z \Big] + \varepsilon \\
&= m(t) + L\int_{\alpha}^{t} \Big[w(\tau) + c_1(\tau)\int_{\alpha}^{\tau} s_1(z) w(z)\Delta z + c_2(\tau)\int_{\alpha}^{\beta} s_2(z) w(z)\Delta z \Big]\Delta\tau \\
&\quad + L\Big[w(t) + \int_{\alpha}^{t} c_1(t) s_1(z) w(z)\Delta z + c_2(t)\int_{\alpha}^{\beta} s_2(z) w(z)\Delta z \Big]
\end{aligned}$$

then we get

$$\begin{aligned}
w(t) \le \frac{m(t)}{1-L} &+ \frac{L}{1-L}\int_{\alpha}^{t} \Big\{ \big[c_1(t) s_1(\tau) + 1 \big] w(\tau) + c_1(\tau)\int_{\alpha}^{\tau} s_1(z) w(z)\Delta z \\
&+ c_2(\tau)\int_{\alpha}^{\beta} s_2(z) w(z)\Delta z \Big\}\Delta\tau + \frac{L}{1-L} c_2(t)\int_{\alpha}^{\beta} s_2(z) w(z)\Delta z,
\end{aligned} \tag{3.36}$$

Now applying theorem 3.1, yields (3.34). □

The following theorem provide the continuous depends of solutions of (1.1) on given initial values.

Theorem 3.5. Assume that the conditions (3.29), (3.30) and (3.31) are satisfied for the functions h, h_1, h_2 in (1.1). Let $y_1(t)$ and $y_2(t)$ be the solutions of equation

$$y^{\Delta}(t)=h\Big(t,y(t),y^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,y(z),y^{\Delta}(z)\big)\Delta z,$$
$$\int_{\alpha}^{\beta}h_2\big(t,z,y(z),y^{\Delta}(z)\big)\Delta z\Big),t\in J_{\mathbb{T}},$$

with the given initial values

$$y_1(\alpha)=c_1 \quad\text{and}\quad y_2(\alpha)=c_2\,, \tag{3.37}$$

where h, h_1, h_2 as in (1.1), c_1 and c_2 are constants. Then

$$\left|y_1(t)-y_2(t)\right|+\left|y_1^{\Delta}(t)-y_2^{\Delta}(t)\right|\le M_4(t)+D_3M_2(t),t\in J_{\mathbb{T}}, \tag{3.38}$$

where $M_4(t)$ is described by the right side of (3.22) by substituting $|c_1-c_2|$ instead of $|y_0|$, $M_2(t), b(t)$ and λ be as in (3.23), (3.24) and (3.25) respectively and

$$D_3=\frac{1}{1-\lambda}\int_{\alpha}^{\beta}r_2(\gamma)M_4(\gamma)\Delta\gamma, \tag{3.39}$$

Proof. Let $n(t)=\left|y_1(t)-y_2(t)\right|+\left|y_1^{\Delta}(t)-y_2^{\Delta}(t)\right|,t\in J_{\mathbb{T}}$, we get

$$n(t)\le|c_1-c_2|$$
$$+\int_{\alpha}^{t}\Big|h\Big(\tau,y_1(\tau),y_1^{\Delta}(\tau),\int_{\alpha}^{\tau}h_1\big(\tau,z,y_1(z),y_1^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(\tau,z,y_1(z),y_1^{\Delta}(z)\big)\Delta z\Big)$$
$$-h\Big(\tau,y_2(\tau),y_2^{\Delta}(\tau),\int_{\alpha}^{\tau}h_1\big(\tau,z,y_2(z),y_2^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(\tau,z,y_2(z),y_2^{\Delta}(z)\big)\Delta z\Big)\Big|\Delta\tau$$
$$+\Big|h\Big(t,y_1(t),y_1^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,y_1(z),y_1^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(t,z,y_1(z),y_1^{\Delta}(z)\big)\Delta z\Big)$$
$$-h\Big(t,y_2(t),y_2^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,y_2(z),y_2^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(t,z,y_2(z),y_2^{\Delta}(z)\big)\Delta z\Big)\Big|$$
$$\le|c_1-c_2|+L\int_{\alpha}^{t}\Big[n(\tau)+c_1(\tau)\int_{\alpha}^{\tau}s_1(z)n(z)\Delta z+c_2(\tau)\int_{\alpha}^{\beta}s_2(z)n(z)\Delta z\Big]\Delta\tau$$
$$+L\Big[n(t)+\int_{\alpha}^{t}c_1(t)s_1(z)n(z)\Delta z+c_2(t)\int_{\alpha}^{\beta}s_2(z)n(z)\Delta z\Big]$$

then

$$\begin{aligned}n(t)\le{}&\frac{|c_1-c_2|}{1-L}+\frac{L}{1-L}\int_{\alpha}^{t}\Big\{\big[c_1(t)s_1(\tau)+1\big]n(\tau)+c_1(\tau)\int_{\alpha}^{\tau}s_1(z)n(z)\Delta z\\&+c_2(\tau)\int_{\alpha}^{\beta}s_2(z)n(z)\Delta z\Big\}\Delta\tau+\frac{L}{1-L}c_2(t)\int_{\alpha}^{\beta}s_2(z)n(z)\Delta z,\end{aligned} \tag{3.40}$$

Now applying theorem 3.1 in (3.40) we obtain (3.38). □

Remark 3.6. The inequality (3.38) gives the uniqueness of solutions of (3.37). If we have $c_1=c_2=0$, then we get $M_5(t)=0$ and $D_3=0$, implies the right hand side of (3.37) is equal to zero.

Now consider the initial value problems

$$Y^{\Delta}(t)=h\Big(t,Y(t),Y^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,Y(z),Y^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(t,z,Y(z),Y^{\Delta}(z)\big)\Delta z,\mu\Big),t\in J_{\mathbb{T}},Y(\alpha)=Y_0,\tag{3.41}$$

$$Y^{\Delta}(t)=h\Big(t,Y(t),Y^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,Y(z),Y^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(t,z,Y(z),Y^{\Delta}(z)\big)\Delta z,\mu_0\Big),t\in J_{\mathbb{T}},Y(\alpha)=Y_0,\tag{3.42}$$

where $h\in C_{rd}\left(J_{\mathbb{T}}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n\right)$ and μ,μ_0 are parameters.

The dependency of solutions of (3.41) and (3.42) on parameters follows in the next theorem.

Theorem 3.7. Suppose that the conditions (3.30) and (3.31) are satisfied and

$$\left|h(t,v_1,v_2,v_3,v_4,\mu)-h(t,\bar v_1,\bar v_2,\bar v_3,\bar v_4,\mu)\right|\le L\left[|v_1-\bar v_1|+|v_2-\bar v_2|+|v_3-\bar v_3|+|v_4-\bar v_4|\right],\tag{3.43}$$

$$\left|h(t,v_1,v_2,v_3,v_4,\mu)-h(t,v_1,v_2,v_3,v_4,\mu_0)\right|\le k(t)|\mu-\mu_0|,\tag{3.44}$$

where $0\le L<1$ is a constant and $k\in C_{rd}\left(J_{\mathbb{T}},\mathbb{R}+\right)$. Let $Y_1(t)$ and $Y_2(t)$ be respectively, the solutions of (3.41) and (3.42) on $J_{\mathbb{T}}$, then

$$\left|Y_1(t)-Y_2(t)\right|+\left|Y_1^{\Delta}(t)-Y_2^{\Delta}(t)\right|\le M_5(t)+D_4M_2(t),t\in J_{\mathbb{T}},\tag{3.45}$$

where $M_5(t)$ is described by the right side of (3.22) by substituting $|\mu-\mu_0|\bar k(t)$ instead of $|y_0|$, $M_2(t),b(t)$ and λ be as in (3.23), (3.24) and (3.25) respectively.

Let

$$\bar k(t)=k(t)+\int_{\alpha}^{t}k(r)\Delta\tau,\tag{3.46}$$

$$D_4=\frac{1}{1-\lambda}\int_{\alpha}^{\beta}r_2(\gamma)M_5(\gamma)\Delta\gamma,\tag{3.47}$$

Proof. Let $P(t)=\left|Y_1(t)-Y_2(t)\right|+\left|Y_1^{\Delta}(t)-Y_2^{\Delta}(t)\right|,t\in J_{\mathbb{T}}$, we have

$$\begin{aligned}
&P(t)\\
&\le\int_{\alpha}^{t}\Big|h\Big(\tau,Y_1(\tau),Y_1^{\Delta}(\tau),\int_{\alpha}^{\tau}h_1\big(\tau,z,Y_1(z),Y_1^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(\tau,z,Y_1(z),Y_1^{\Delta}(z)\big)\Delta z,\mu\Big)\\
&-h\Big(\tau,Y_2(\tau),Y_2^{\Delta}(\tau),\int_{\alpha}^{\tau}h_1\big(\tau,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(\tau,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\mu\Big)\Big|\Delta\tau\\
&+\int_{\alpha}^{t}\Big|h\Big(\tau,Y_2(\tau),Y_2^{\Delta}(\tau),\int_{\alpha}^{\tau}h_1\big(\tau,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(\tau,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\mu\Big)\\
&-h\Big(\tau,Y_2(\tau),Y_2^{\Delta}(\tau),\int_{\alpha}^{\tau}h_1\big(\tau,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(\tau,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\mu_0\Big)\Big|\Delta\tau\\
&+\Big|h\Big(t,Y_1(t),Y_1^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,Y_1(z),Y_1^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(t,z,Y_1(z),Y_1^{\Delta}(z)\big)\Delta z,\mu\Big)\\
&-h\Big(t,Y_2(t),Y_2^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(t,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\mu\Big)\Big|\\
&+\Big|h\Big(t,Y_2(t),Y_2^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(t,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\mu\Big)\\
&-h\Big(t,Y_2(t),Y_2^{\Delta}(t),\int_{\alpha}^{t}h_1\big(t,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\int_{\alpha}^{\beta}h_2\big(t,z,Y_2(z),Y_2^{\Delta}(z)\big)\Delta z,\mu_0\Big)\Big|
\end{aligned}$$

$$\leq \int_{\alpha}^{t} L\left[P(\tau)+\int_{\alpha}^{\tau} c_1(\tau)s_1(z)P(z)\Delta z+\int_{\alpha}^{\beta} c_2(\tau)s_2(z)P(z)\Delta z\right]\Delta\tau$$
$$+\int_{\alpha}^{t} k(\tau)|\mu-\mu_0|\Delta\tau$$
$$+L\left[P(t)+\int_{\alpha}^{t} c_1(t)s_1(z)P(z)\Delta z+\int_{\alpha}^{\beta} c_2(t)s_2(z)P(z)\Delta z\right]+k(t)|\mu-\mu_0|$$
$$=|\mu-\mu_0|\bar{k}(t)+L\int_{\alpha}^{t}\left[P(\tau)+c_1(\tau)\int_{\alpha}^{\tau} s_1(z)P(z)\Delta z+c_2(\tau)\int_{\alpha}^{\beta} s_2(z)P(z)\Delta z\right]\Delta\tau$$
$$+L\left[P(t)+\int_{\alpha}^{t} c_1(t)s_1(z)P(z)\Delta z+\int_{\alpha}^{\beta} c_2(t)s_2(z)P(z)\Delta z\right]$$

then we have

$$\begin{aligned} P(t) &\leq \frac{|\mu-\mu_0|\bar{k}(t)}{1-L} \\ &+\frac{L}{1-L}\int_{\alpha}^{t}\left\{\left[c_1(t)s_1(\tau)+1\right]P(\tau)+c_1(\tau)\int_{\alpha}^{\tau} s_1(z)P(z)\Delta z\right. \\ &\left.+c_2(\tau)\int_{\alpha}^{\beta} s_2(z)P(z)\Delta z\right\}\Delta\tau+\frac{L}{1-L}c_2(t)\int_{\alpha}^{\beta} s_2(z)P(z)\Delta z, \end{aligned} \tag{3.48}$$

Now applying theorem 3.1, we have (3.45). □

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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