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Norm Inequality for Intrinsic Square Functions in a Generalized Hardy-Morrey Space

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Abstract

We define a generalized Hardy-Morrey space and give an atomic characterization whenever we are in some appropriate range of index. This atomic decomposition helps us to give a control of some intrinsic square functions and their commutators in the above mentioned spaces.

Subject Areas

Mathematics

Keywords

Amalgam Spaces, Hardy-Amalgam Spaces, Hardy-Morrey Space, Intrinsic Square Function, Commutator

1. Introduction

For $0 < p, q \leq \infty$, the Hardy-amalgam space $\mathcal{H}^{(q,p)}$ was defined by Ablé and Feuto in [1] by taking in the maximal characterizations of classical Hardy spaces the Wiener amalgam quasi-norm $\|\cdot\|_{q,p}$ instead of the Lebesgue one. The authors gave an atomic decomposition theorem of resultant spaces and norm inequalities for some classical operators. But it seems difficult to establish, as is the case in the spaces of Lebesgue and Fofana, inequalities in norm for the commutators associated with these operators. We think that this is due to the fact that Hardy-amalgam space is too big.

We recall that a locally integrable function f belongs to the amalgam space $\left(L^q, L^p\right)$ if $\left\| y \mapsto \left\| f \chi_{B(y,1)} \right\|_q \right\|_p < \infty$, where $\|\cdot\|_q$ stands for the classical Lebesgue quasi-norm and $\chi_{B(y,r)}$ the characteristic function of the ball centered at y with radius $r>0$. We put

$$\|f\|_{q,p}=\left\|y\mapsto\left\|f\chi_{B(y,1)}\right\|_q\right\|_p. \tag{1}$$

Notice that for $f\in\left(L^q,L^p\right)$, $\alpha>0$ and $r>0$, the dilated function $St_r^{(\alpha)}f$ defined by $\left(St_r^{(\alpha)}f\right)(x)=r^{-\frac{d}{\alpha}}f\left(r^{-1}x\right)$, belongs to $\left(L^q,L^p\right)$ and $\left\|St_r^{(\alpha)}f\right\|_{q,p}$ is equivalent to $\|f\|_{q,p}$, but with the equivalence constants depending on r and α. These operators are linear and bounded on $\left(L^q,L^p\right)$.

An important subspace of $\left(L^q,L^p\right)$ space when dealing with classical operators such as Riesz potential and the fractional maximal operators (see in [2] [3] [4]), is the Fofana space $\left(L^q,L^p\right)^\alpha$ which is a subspace of $\left(L^q,L^p\right)$ defined for $0<q,p,\alpha\leq\infty$ by

$$\left(L^q,L^p\right)^\alpha=\left\{f\in\left(L^q,L^p\right)/\|f\|_{q,p,\alpha}<\infty\right\}, \tag{2}$$

where

$$\|f\|_{q,p,\alpha}=\sup_{r>0}\left\|St_r^{(\alpha)}f\right\|_{q,p}. \tag{3}$$

These sub-spaces, introduced by Fofana in [5], are non-trivial only if $q\leq\alpha\leq p$. Thus, we will always assume that this condition is fulfilled. They can be viewed as some generalized Morrey spaces, since for $q<\alpha$, the space $\left(L^q,L^\infty\right)^\alpha\left(\mathbb{R}^d\right)$ is exactly the classical Morrey space $L^{q,d\frac{q}{\alpha}}\left(\mathbb{R}^d\right)$.

In this work, we consider the subspace of Hardy-amalgam space defined by taking in the maximal characterization of classical Hardy space, the quasi-norm $\|\cdot\|_{q,p,\alpha}$ instead of the one of Lebesgue and we prove that the resulting space has an atomic decomposition once $0<q\leq\alpha\leq p<\infty$ with $0<q\leq1$. We also give norm inequalities for Wilson intrinsic square functions [6], and their commutators. This paper is organized as follows.

The next section is devoted to some properties of Fofana's spaces. In Section 3, we give the definition of our generalized Hardy-Morrey space and some relationships between this space and some existing one. In Section 4, we deal with the atomic decomposition of our spaces, and we give a norm inequality for Wilson intrinsic square functions and their commutators in the last section.

In this work, $\mathcal{S}$ will denote the Schwartz class of rapidly decreasing smooth functions equipped with its usual topology. The dual space of $\mathcal{S}$ is the space of tempered distributions denoted by $\mathcal{S}'$. The pairing between $\mathcal{S}'$ and $\mathcal{S}$ is denoted by $\langle\cdot,\cdot\rangle$.

The letter C will be used for non-negative constants independent of the relevant variables that may change from one occurrence to another. When a constant depends on some important parameters $\alpha,\gamma,\cdots$, we denote it by $C(\alpha,\gamma,\cdots)$. Constants with subscript such as $C_{\alpha,\gamma,\cdots}$, do not change in different occurrences and depend on the parameters mentioned in them. We adopt the following abbreviation $\mathbf{A} \lesssim \mathbf{B}$ for the inequalities $\mathbf{A} \leq C\mathbf{B}$, where C is a non-negative constant independent of the main parameters, and $\mathbf{A} \leq_{\alpha,\gamma,\cdots} \mathbf{B}$ for the inequalities $\mathbf{A} \leq C_{\alpha,\gamma,\cdots}\mathbf{B}$. If $\mathbf{A} \lesssim \mathbf{B}$ and $\mathbf{B} \lesssim \mathbf{A}$, then we write $\mathrm{A} \approx \mathrm{B}$.

For a real number $\lambda > 0$ and a cube $Q \subset \mathbb{R}^d$ (by cube we mean a bounded cube whose edges are parallel to the coordinate axes), we write λQ for the cube with same center as Q and side-length λ times side-length of Q, while $\lfloor \lambda \rfloor$ stands for the greatest integer less than or equal to λ. Also, for $x \in \mathbb{R}^d$ and a real number $\ell > 0$, $Q(x,\ell)$ will denote the cube centered at x and side-length ℓ. We use the same notations for balls. For a measurable set $E \subset \mathbb{R}^d$, we denote by χ_E its characteristic function and by $|E|$ its Lebesgue measure. We adopt the notation $\operatorname{supp} f$, to designate the support of a complex-valued function f defined in $\mathbb{R}^d$.

2. Basic Facts about Fofana's Spaces

Fofana's spaces have among others, the following properties (see for example [5] and [3]):

1) Let $0 < q, p, \alpha \leq \infty$. The space $\left(\left(L^q, L^p\right)^\alpha\left(\mathbb{R}^d\right), \|\cdot\|_{q,p,\alpha}\right)$ is a Banach space if $1 \leq q \leq \alpha \leq p$ and a quasi-Banach space if $0 < q < 1$.

2) If $\alpha \in \{p,q\}$ then $\left(L^q, L^p\right)^\alpha\left(\mathbb{R}^d\right) = L^\alpha\left(\mathbb{R}^d\right)$ with equivalent quasi-norms.

3) If $q < \alpha < p$ then $L^\alpha\left(\mathbb{R}^d\right) \subsetneq L^{\alpha,\infty}\left(\mathbb{R}^d\right) \subsetneq \left(L^q, L^p\right)^\alpha\left(\mathbb{R}^d\right) \subsetneq \left(L^q, L^p\right)\left(\mathbb{R}^d\right)$, where $L^{\alpha,\infty}\left(\mathbb{R}^d\right)$ is the weak Lebesgue space on $\mathbb{R}^d$ defined by

$$L^{\alpha,\infty}\left(\mathbb{R}^d\right) = \left\{ f \in L^1_{loc}\left(\mathbb{R}^d\right) / \|f\|^*_{\alpha,\infty} < \infty \right\},$$

with $\|f\|^*_{\alpha,\infty} := \sup_{\lambda>0} \left|\left\{x \in \mathbb{R}^d / \left|f(x)\right| > \lambda\right\}\right|^{\frac{1}{\alpha}}$.

4) Let f and g be two measurable functions on $\mathbb{R}^d$. If $|f| \leq |g|$, then $\|f\|_{q,p,\alpha} \leq \|g\|_{q,p,\alpha}$.

5) For every measurable complex-valued function f on $\mathbb{R}^d$, we have

$$\|f\|_{q,p,\alpha} \approx \sup_{r>0} r^{\frac{d}{\alpha}-\frac{d}{q}-\frac{d}{p}} \left[\int_{\mathbb{R}^d} \left\|f \chi_{B(y,r)}\right\|_q^p \mathrm{d}y\right]^{\frac{1}{p}}$$

with the usual modification when $p = \infty$.

It is proved in ([7], Proposition 4.2) that the Hardy-Littlewood maximal operator is bounded in $\left(L^q, L^p\right)^\alpha\left(\mathbb{R}^d\right)$ whenever $1 < q \leq \alpha \leq p \leq \infty$. We recall that for a locally integrable function f, the Hardy-Littlewood maximal function $\mathfrak{M}(f)$ is defined by

$$\mathfrak{M}(f)(x) := \sup_{r>0} \left|B(x,r)\right|^{-1} \int_{B(x,r)} \left|f(y)\right| \mathrm{d}y, \ \forall\, x \in \mathbb{R}^d.$$

The following result which is more general than the above, is just an adaptation of ([8], Proposition 11.12). The proof is given just for the sake of completeness.

Proposition 1. *Let* $1 < q \leq \alpha \leq p < \infty$ *and* $1 < u \leq \infty$. *For all sequences*

$\{f_n\}_{n\geq 0}$ *of measurable functions, we have*

$$\left\|\left(\sum_{n\geq 0}\left|\mathfrak{M}(f_n)\right|^u\right)^{\frac{1}{u}}\right\|_{q,p,\alpha}\approx\left\|\left(\sum_{n\geq 0}\left|f_n\right|^u\right)^{\frac{1}{u}}\right\|_{q,p,\alpha},$$

with the equivalence constants not depending on the sequence $\{f_n\}_{n\geq 0}$.

Proof. Let $r>0$, $1<u<\infty$, $1<q\leq\alpha\leq p<\infty$ and $\{f_n\}_{n\geq 0}$ be a sequence of measurable functions. It is well known that

$$\left\|\left(\sum_{n\geq 0}\left|\mathfrak{M}(f_n)\right|^u\right)^{\frac{1}{u}}\right\|_{p}\approx\left\|\left(\sum_{n\geq 0}\left|f_n\right|^u\right)^{\frac{1}{u}}\right\|_{p}. \tag{4}$$

It is also easy to see that for $y\in\mathbb{R}^d$ and $x\in B(y,r)$ we have $B(x,r)\subset B(y,2r)$ so that

$$\mathfrak{M}(f)(x)\lesssim\mathfrak{M}(f)(y) \tag{5}$$

for all measurable functions *f*. It follows that

$$\left\|\left\|\left(\sum_{n\geq 0}\left|\mathfrak{M}(f_n)\right|^u\right)^{\frac{1}{u}}\chi_{B(y,r)}\right\|_q\right\|_p\lesssim r^{\frac{d}{q}}\left[\int_{\mathbb{R}^d}\left(\sum_{n\geq 0}(\mathfrak{M}f_n)^u(y)\right)^{\frac{p}{u}}\mathrm{d}y\right]^{\frac{1}{p}}$$
$$\lesssim r^{\frac{d}{q}}\left\|\left(\sum_{n\geq 0}\left|f_n\right|^u\right)^{\frac{1}{u}}\right\|_p.$$

But then,

$$r^{\frac{d}{q}}\left\|\left(\sum_{n\geq 0}\left|f_n\right|^u\right)^{\frac{1}{u}}\right\|_p\lesssim\left[\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(\sum_{n\geq 0}\left|f_n(x)\right|^u\right)^{\frac{q}{u}}\chi_{B(x,r)}(y)\mathrm{d}y\right)^{\frac{p}{q}}\mathrm{d}x\right]^{\frac{1}{p}}$$
$$\lesssim\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(\mathfrak{M}\left(\sum_{n\geq 0}\left|f_n\right|^u\right)^{\frac{1}{u}}\right)^q(y)\,\chi_{B(x,r)}(y)\mathrm{d}y\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}}$$
$$\lesssim\left\|\left\|\mathfrak{M}\left(\sum_{n\geq 0}\left|f_n\right|^u\right)^{\frac{1}{u}}\chi_{B(x,r)}\right\|_q\right\|_p.$$

So, for $r>0$, we have

$$\left\|\left\|\left(\sum_{n\geq 0}\left|\mathfrak{M}(f_n)\right|^u\right)^{\frac{1}{u}}\chi_{B(y,r)}\right\|_q\right\|_p\lesssim\left\|\left\|\mathfrak{M}\left(\sum_{n\geq 0}\left|f_n\right|^u\right)^{\frac{1}{u}}\chi_{B(y,r)}\right\|_q\right\|_p.$$

Multiplying both sides of the above inequality by $r^{\frac{1}{\alpha}-\frac{1}{p}-\frac{1}{q}}$ and taking the supremum over all $r>0$ yields

$$\left\|\left(\sum_{n\geq 0}\left|\mathfrak{M}(f_n)\right|^u\right)^{\frac{1}{u}}\right\|_{q,p,\alpha} \lesssim \left\|\mathfrak{M}\left(\sum_{n\geq 0}\left|f_n\right|^u\right)^{\frac{1}{u}}\right\|_{q,p,\alpha}.$$

Hence

$$\left\|\left(\sum_{n\geq 0}\left|\mathfrak{M}(f_n)\right|^u\right)^{\frac{1}{u}}\right\|_{q,p,\alpha} \lesssim \left\|\left(\sum_{n\geq 0}\left|f_n\right|^u\right)^{\frac{1}{u}}\right\|_{q,p,\alpha} \lesssim \left\|\left(\sum_{n\geq 0}\left|\mathfrak{M}(f_n)\right|^u\right)^{\frac{1}{u}}\right\|_{q,p,\alpha}$$

thanks to ([7], Proposition 4.2) and the fact that $u>1$ and $\left|f_n\right|^u \leq \left|\mathfrak{M}(f_n)\right|^u$ for all $n\geq 0$.

The case $u=\infty$ follows immediately from the fact that

$$\left\|\sup_{\mathbb{R}^d}\left|\mathfrak{M}(f_n)\right|\right\|_p \approx \left\|\sup_{\mathbb{R}^d}\left|f_n\right|\right\|_p$$

for $p>1$. □

3. Generalized Hardy-Morrey Spaces

Let $0<q<\infty$. The classical Hardy space $\mathcal{H}^q$ is defined as the space of all tempered distributions f satisfying $\|f\|_{\mathcal{H}^q} := \int_{\mathbb{R}^d}\left|\mathcal{M}f(x)\right|^q \mathrm{d}x < \infty$, where

$$\mathcal{M}f(x) = \mathcal{M}_\varphi f(x) := \sup_{t>0}\left|(f*\varphi_t)(x)\right|, \quad (6)$$

with φ in the Schwartz class $\mathcal{S}$ having non vanish integral, and $\varphi_t(x) = t^{-d}\varphi\left(t^{-1}x\right)$ for all $t>0$.

It is well known that the space $\mathcal{H}^q$ doesn't depend on the function φ. Hence we will consider through this paper, $\varphi\in\mathcal{S}$ having its support in the unit ball and such that $\int_{\mathbb{R}^d}\varphi(x)\mathrm{d}x = 1$. The associate maximal function will be denoted $\mathcal{M}_0$ or $\mathcal{M}_\varphi$ while $\mathcal{M}$ or $\mathcal{M}_\phi$ will be used for an arbitrary $\phi\in\mathcal{S}$ having non vanishing integral.

Let $0<q,p,\alpha\leq\infty$. We define the space $\mathcal{H}^{(q,p,\alpha)} := \mathcal{H}^{(q,p,\alpha)}\left(\mathbb{R}^d\right)$ of Hardy type, by

$$\mathcal{H}^{(q,p,\alpha)} := \left\{f\in\mathcal{S}' : \|\mathcal{M}f\|_{q,p,\alpha} < \infty\right\}. \quad (7)$$

We accordingly define the local version of these spaces by replacing in (7) the maximal function $\mathcal{M}f$ by its local version $\mathcal{M}_{\mathrm{loc}}f$. We recall that $\mathcal{M}_{\mathrm{loc}}f$ is defined as $\mathcal{M}f$, but with the supremum taken only on the interval $(0,1]$. We will refer to $\mathcal{H}^{(q,p,\alpha)}$ spaces as generalized Hardy-Morrey spaces; in fact for $p=\infty$ and $q<\alpha$, we recovered the Hardy-Morrey space defined by Jia and Wang in [9]. We can also call these spaces Hardy-Fofana spaces given their definition.

It is clear that

$$\mathcal{H}^{(q,p,\alpha)} \subset \mathcal{H}_{\mathrm{loc}}^{(q,p,\alpha)}. \quad (8)$$

Hardy spaces are translations and dilations invariant, in the sense that for $f\in\mathcal{H}^p$, $p>0$,

$$\|\tau_x f\|_{\mathcal{H}^p} = \|f\|_{\mathcal{H}^p} \text{ and } \left\|St_r^{(p)} f\right\|_{\mathcal{H}^p} = \|f\|_{\mathcal{H}^p},$$

where for $x \in \mathbb{R}^d$, $r > 0$ and $f \in \mathcal{S}'$, $\tau_x f$ and $St_r^{(\alpha)} f$ are defined as tempered distributions whose actions on Schwartz function φ are given respectively by

$$\langle \tau_x f, \varphi \rangle = \langle f, \tau_{-x}\varphi \rangle \text{ and } \left\langle St_r^{(\alpha)} f, \varphi \right\rangle = \left\langle f, St_{r^{-1}}^{(\alpha')} \varphi \right\rangle.$$

These are immediate consequences of the invariance properties of Lebesgue spaces for translation and dilation, and the fact that these operators commutes with the maximal operator define by (6). It follows that

$$\|f\|_{\mathcal{H}^{(q,p,\alpha)}} = \sup_{r>0} \left\|St_r^{(\alpha)} f\right\|_{\mathcal{H}^{(q,p)}}, \text{ for all } f \in \mathcal{H}^{(q,p,\alpha)}, \tag{9}$$

where $\|f\|_{\mathcal{H}^{(q,p)}} := \|\mathcal{M}f\|_{q,p}$.

The following relationship between our spaces and the classical Hardy, Hardy-amalgam and the weak Hardy spaces as defined by Grafakos and He in [10], are immediate consequences of Fofana's spaces properties. The proofs are omitted.

Proposition 2. *Let* $0 < q \le \alpha \le p < \infty$.

$$\mathcal{H}^\alpha \subset \mathcal{H}^{(q,p,\alpha)} \subset \mathcal{H}^{(q,p)}. \tag{10}$$

Furthermore, we have

$$\mathcal{H}^{(q,p,\alpha)} = \mathcal{H}^\alpha \text{ if } \alpha \in \{p, q\}, \tag{11}$$

and

$$\mathcal{H}_{\text{weak}}^\alpha \subset \mathcal{H}^{(q,p,\alpha)} \text{ if } q < \alpha < p. \tag{12}$$

The Proposition is still valid if we replace all the spaces by their local versions.

For the relation between these Hardy type spaces and Fofana's spaces, we have the following proposition which is an extension of a well-known result in classical Hardy spaces.

Proposition 3 *Let* $1 \le q \le \alpha \le p < \infty$.

1) *If* $1 < q$ *then the spaces* $\mathcal{H}^{(q,p,\alpha)}$, $\mathcal{H}_{\text{loc}}^{(q,p,\alpha)}$ *and* $(L^q, L^p)^\alpha$ *are equal with equivalence norms.*

2) *The space* $\mathcal{H}^{(1,p,\alpha)}$ *is continuously embedded in* $(L^1, \ell^p)^\alpha$.

Proof Let $1 \le q \le \alpha \le p < \infty$. For $f \in \mathcal{H}^{(q,p,\alpha)} \subset \mathcal{H}^{(q,p)}$, we have $f \in (L^q, L^p)$ with $\lim_{t\to 0} f * \varphi_t(x) = f(x)$ for almost every $x \in \mathbb{R}^d$, according to ([11], Theorem 3.2). Now, for all $x \in \mathbb{R}^d$, we have

$$|f * \varphi_t(x)| \le \mathcal{M}_\varphi(f)(x)$$

so that letting t tends to 0, yields $|f(x)| \le \mathcal{M}_\varphi(f)(x)$ for almost every $x \in \mathbb{R}^d$. Hence

$$\|f\|_{q,p,\alpha} \le \|\mathcal{M}_\varphi(f)\|_{q,p,\alpha} = \|f\|_{\mathcal{H}^{(q,p,\alpha)}}.$$

We suppose now that $1 < q \le \alpha \le p < \infty$ and $f \in (L^q, L^p)^\alpha$. We have

$$\mathcal{M}_\varphi f(x) \lesssim \mathfrak{M}(f)(x), \quad x \in \mathbb{R}^d,$$

where $\mathfrak{M}(f)$ is the classical Hardy-Littlewood maximal function. It follows that

$$\|f\|_{\mathcal{H}_{\text{loc}}^{(q,p,\alpha)}} = \left\|\mathcal{M}_{\varphi_{\text{loc}}} f\right\|_{q,p,\alpha} \leq \left\|\mathcal{M}_{\varphi} f\right\|_{q,p,\alpha} \lesssim \|f\|_{q,p,\alpha},$$

thanks to ([7], Proposition 2.4). □

Just like classical Hardy and Hardy-amalgam spaces, the spaces $\mathcal{H}^{(q,p,\alpha)}$ are quasi-Banach once we have $q<1$. More precisely, we have the following result.

Proposition 4 *Let* $0<q\leq\alpha\leq p\leq\infty$ *with* $q<1$.

1) *For* $f,g\in\mathcal{H}^{(q,p,\alpha)}$,

$$\|f+g\|_{\mathcal{H}^{(q,p,\alpha)}}^{q} \leq \|f\|_{\mathcal{H}^{(q,p,\alpha)}}^{q} + \|g\|_{\mathcal{H}^{(q,p,\alpha)}}^{q} \tag{13}$$

2) *The space* $\mathcal{H}^{(q,p,\alpha)}$ *is a quasi-Banach space, when it is equipped with the quasi-norm* $\|\cdot\|_{\mathcal{H}^{(q,p,\alpha)}}$.

Proof The Relation (13) follows immediately from the fact that L^q and ℓ^q are completed quasi-normed spaces for $0<q<1$, with

$$\|f+g\|_q^q \leq \|f\|_q^q + \|g\|_q^q.$$

For the second assertion, we adapt the proof of ([1], Proposition 3.8). Let $(f_n)_{n\geq 0}$ be a sequence in $\mathcal{H}^{(q,p,\alpha)}$ satisfying

$$\sum_{k\geq 0}\|f_k\|_{\mathcal{H}^{(q,p,\alpha)}}^{q} < \infty.$$

The sequence $\left\{\sum_{k=0}^{n} f_k\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}^{(q,p,\alpha)}$ and consequently in $\mathcal{H}^{(q,p)}$. Hence it converges in $\mathcal{H}^{(q,p)}$ and consequently in $\mathcal{S}'$. Let f be its limit. Since

$$\mathcal{M}_{\varphi}(f) = \mathcal{M}_{\varphi}\left(\sum_{k\geq 0} f_k\right) \leq \sum_{k\geq 0}\mathcal{M}_{\varphi}(f_k)$$

and

$$\left\|f - \sum_{k=0}^{n} f_k\right\|_{\mathcal{H}^{(q,p,\alpha)}}^{q} \leq \sum_{k=n+1}^{+\infty}\|f_k\|_{\mathcal{H}^{(q,p,\alpha)}}^{q}$$

tends to zero as n goes to infinity, the series $\sum_{k\geq 0} f_k$ converges to f in $\mathcal{H}^{(q,p,\alpha)}$ so that $\left(\mathcal{H}^{(q,p,\alpha)}, \|\cdot\|_{\mathcal{H}^{(q,p,\alpha)}}\right)$ is completed. □

The space $\mathcal{H}^{(q,p,\alpha)}$ doesn't depend on the test function ϕ use in its definition. In fact, we can replace the maximal function by grand-maximal function.

Let N be a positive integer and $\mathcal{F}_N = \{\psi\in\mathcal{S} : \mathfrak{N}_N(\psi)\leq 1\}$, where

$$\mathfrak{N}_N(\psi) = \int_{\mathbb{R}^d}(1+|x|)^N\left(\sum_{|\beta|\leq N+1}\left|\partial^{\beta}\psi(x)\right|\right)\mathrm{d}x,$$

with $|\beta| = \beta_1+\cdots+\beta_d$ for a multi-index $\beta=(\beta_1,\cdots,\beta_d)$. We recall that the radial grand-maximal function $\mathcal{M}_{\mathcal{F}_N}^{0} f$ and its non-tangential version $\mathcal{M}_{\mathcal{F}_N} f$ are defined respectively by

$$\mathcal{M}_{\mathcal{F}_N}^{0} f(x) = \sup_{\psi\in\mathcal{F}_N}\mathcal{M}_{\psi}(f)(x) \text{ and } \mathcal{M}_{\mathcal{F}_N} f(x) = \sup_{\psi\in\mathcal{F}_N}\sup_{t>0}\left\{\max_{|x-y|\leq 1}\left|f*\psi_t(y)\right|\right\},$$

for all $x \in \mathbb{R}^d$. An immediate consequence of the relation (9) is that for $N \geq \left\lfloor \frac{d}{q} \right\rfloor + 1$ and $a > 0$ we have

$$\left\| \mathcal{M}^0_{\mathcal{F}_N} f \right\|_{q,p,\alpha} \approx \left\| \mathcal{M}_{\mathcal{F}_N} f \right\|_{q,p,\alpha} \approx \left\| \mathcal{M} f \right\|_{q,p,\alpha} \approx \left\| \mathcal{M}^*_{\varphi,a} f \right\|_{q,p,\alpha}, \tag{14}$$

with the equivalence constants depending only on q, p, α and φ. The maximal function $\mathcal{M}^*_{\varphi,a} f$ is defined by

$$\mathcal{M}^*_{\varphi,a} f(x) = \sup_{t>0} \left\{ \sup_{|x-y| \leq at} \left| \left(f * \varphi_t \right)(y) \right| \right\}.$$

Relation (14) follows from ([1], Theorem 3.7) and the fact that the operators $\mathcal{M}^0_{\mathcal{F}_N}$, $\mathcal{M}_{\mathcal{F}_N}$ and $\mathcal{M}$ commute with St^{α}_r for $\alpha, r > 0$. We can also take in the definition of $\mathcal{H}^{(q,p,\alpha)}$ the Poisson kernel instead of Schwartz function. More precisely, we have

$$\| f \|_{\mathcal{H}^{(q,p,\alpha)}} \approx \left\| \sup_{t>0} \left| f * P_t \right| \right\|_{q,p,\alpha}, \quad f \in \mathcal{H}^{(q,p,\alpha)}. \tag{15}$$

4. Atomic Decomposition of $\mathcal{H}^{(q,p,\alpha)}$ Spaces

Throughout this paragraph, we assume that $0 < q \leq \alpha \leq p < \infty$, $q \leq 1 < r \leq \infty$ and $\alpha \leq r$. We also assume that s is an integer greater or equal to $\left\lfloor d\left(\frac{1}{q} - 1\right) \right\rfloor$.

A function $\mathfrak{a} : \mathbb{R}^d \to \mathbb{C}$ is called (q,α,r,s)-atom if it satisfies the following conditions:

1) There exists a cube Q such that $\operatorname{supp}(\mathfrak{a}) \subset Q$,

2) $\|a\|_r \leq |Q|^{\frac{1}{r} - \frac{1}{\alpha}}$;

3) $\int_{\mathbb{R}^d} x^{\beta} \mathfrak{a}(x) \mathrm{d}x = 0$, for all multi-index β such that $|\beta| \leq s$.

We denote by $\mathcal{A}(q,\alpha,r,s)$, the set of all couples $(\mathfrak{a}, Q)$ such that $\mathfrak{a}$ and Q satisfy conditions (1)-(3).

Notice that the generalized Hölder inequality can be stated as follows. Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p$ and $g \in L^q$ then $fg \in Ł^r$. Moreover, we have

$$\| fg \|_r \leq \| f \|_p \| g \|_q.$$

Proposition 5. *For all* (q,α,r,s)*-atoms*

$$\| \mathfrak{a} \|_{\mathcal{H}^{(q,p,\alpha)}} \leq C$$

where C is a constant not depending on $\mathfrak{a}$.

Proof. Let a be a (q,α,r,s)-atom. We have

$$\left\| \mathcal{M}_0(\mathfrak{a}) \right\|_{q,p,\alpha} \leq \left\| \mathcal{M}_0(\mathfrak{a}) \right\|_{\alpha} \tag{16}$$

according to Relation (3). We assume that $\alpha > 1$, since otherwise, the result follows from the classical case and (16). Thanks to the definition of (q,α,r,s)

-atom and Hölder's inequality, we have $\|a\|_\alpha \leq 1$. The result is just a consequence of the size condition on the atom and the boundedness of the Hardy-Littlewood maximal operator on L^α. □

Notice that (q,q,r,s)-atom is exactly the atom for classical Hardy space $\mathcal{H}^q$ denoted (q,r,s)-atom. It is also the atom for the Hardy-amalgam space $\mathcal{H}^{(q,p)}$ and in this case, the collection of $(\mathfrak{a},Q)$ satisfying conditions (1)-(3) is denoted $\mathcal{A}(q,r,s)$ as we can see in [1].

It is easy to see that $(\mathfrak{a},Q)\in\mathcal{A}(q,\alpha,r,s)$ if and only if $\left(|Q|^{\frac{1}{\alpha}-\frac{1}{q}}\mathfrak{a},Q\right)\in\mathcal{A}(q,r,s)$. As a consequence of this relation, we have the following result.

Theorem 6. *Let* $0<\eta\leq 1$. *For all sequences* $\{(\mathfrak{a}_n,Q_n)\}_{n\geq 0}$ *in* $\mathcal{A}(q,\alpha,\infty,s)$ *and all sequences of scalars* $\{\lambda_n\}_{n\geq 0}$ *such that*

$$\left\|\sum_{n\geq 0}\left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_\alpha}\right)^\eta \chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}<\infty, \tag{17}$$

the series $f:=\sum_{n\geq 0}\lambda_n\mathfrak{a}_n$ *converges in the sense of distribution and in* $\mathcal{H}^{(q,p,\alpha)}$. *Moreover, we have*

$$\|f\|_{\mathcal{H}^{(q,p,\alpha)}}\lesssim_{\varphi,d,q,p,s}\left\|\sum_{n\geq 0}\left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_\alpha}\right)^\eta \chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}}. \tag{18}$$

Proof. Fix a sequence $\{(\mathfrak{a}_n,Q_n)\}_{n\geq 0}$ in $\mathcal{A}(q,\alpha,\infty,s)$ and a sequence of scalars $\{\lambda_n\}_{n\geq 0}$ satisfying (17). Since $\sum_{n\geq 0}\lambda_n\mathfrak{a}_n=\sum_{n\geq 0}\frac{\lambda_n}{|Q_n|^{\frac{1}{\alpha}-\frac{1}{q}}}\left(|Q_n|^{\frac{1}{\alpha}-\frac{1}{q}}\mathfrak{a}_n\right)$ and $\left\{\left(|Q_n|^{\frac{1}{\alpha}-\frac{1}{q}}\mathfrak{a}_n,Q_n\right)\right\}$ is a sequence of elements of $\mathcal{A}(q,\infty,s)$ such that

$$\begin{aligned}\left\|\sum_{n\geq 0}\left(\frac{|\lambda_n||Q_n|^{\frac{1}{q}-\frac{1}{\alpha}}}{\|\chi_{Q_n}\|_q}\right)^\eta \chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta}}^{\frac{1}{\eta}}&=\left\|\sum_{n\geq 0}\left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_\alpha}\right)^\eta \chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta}}^{\frac{1}{\eta}}\\&\leq\left\|\sum_{n\geq 0}\left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_\alpha}\right)^\eta \chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}}<\infty,\end{aligned}$$

the series $f:=\sum_{n\geq 0}\lambda_n\mathfrak{a}_n$ converges in the sense of $\mathcal{S}'$ and $\mathcal{H}^{(q,p)}$ according to ([1], Theorem 4.3). It Remains to prove that $f\in\mathcal{H}^{(q,p,\alpha)}$. The series $f:=\sum_{n\geq 0}\lambda_n\mathfrak{a}_n$ converges in the sense of $\mathcal{S}'$, implies that

$$\mathcal{M}_0(f)=\mathcal{M}_0\left(\sum_{n\geq 0}\lambda_n\mathfrak{a}_n\right)\lesssim\sum_{n\geq 0}|\lambda_n|\mathcal{M}_0(\mathfrak{a}_n).$$

Since

$$\mathcal{M}_0(\mathfrak{a}_n)(x) \lesssim_{\varphi,d,s} \left(\mathfrak{M}(\chi_{Q_n})(x)\right)^{\frac{d+s+1}{d}} \left\|\chi_{Q_n}\right\|_\alpha^{-1},$$

it comes that

$$\|f\|_{\mathcal{H}^{(q,p,\alpha)}} \lesssim_{\varphi,d,s} \left\|\sum_{n\geq 0}\left[\frac{|\lambda_n|}{\left\|\chi_{Q_n}\right\|_\alpha}\right]^\eta \chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} < \infty,$$

thanks to Proposition 1. □

The next result is an immediate consequence of Proposition 3.4 [11] and the relation between amalgam space and that of Fofana. The proof is omitted.

Proposition 7. *Let* $0 < q \leq \alpha \leq p < \infty$. *Then* $L^\infty \cap \mathcal{H}^{(q,p,\alpha)}$ *is a dense sub-space of* $\mathcal{H}^{(q,p,\alpha)}$.

Theorem 8. *Let* $s \geq \left\lfloor d\left(\frac{1}{q}-1\right)\right\rfloor$ *be an integer. For all* $f \in L^\infty \cap \mathcal{H}^{(q,p,\alpha)}$, *there exists a sequence* $\{(\mathfrak{a}_n, Q_n)\}_{n\geq 0}$ *in* $\mathcal{A}(q,\alpha,\infty,s)$ *and a sequence of scalars* $\{\lambda_n\}_{n\geq 0}$ *such that*

$$f = \sum_{n\geq 0}\lambda_n \mathfrak{a}_n \text{ in the sense of } \mathcal{S}' \text{ and } \mathcal{H}^{(q,p,\alpha)},$$

and for all $\eta > 0$,

$$\left\|\sum_{n\geq 0}\left(\frac{|\lambda_n|}{\left\|\chi_{Q_n}\right\|_\alpha}\right)^\eta \chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} \lesssim_{\varphi,d,s,q,p,\eta} \|f\|_{\mathcal{H}^{(q,p,\alpha)}}. \tag{19}$$

Proof. Let $f \in L^\infty \cap \mathcal{H}^{(q,p,\alpha)} \subset L^1_{loc} \cap \mathcal{H}^{(q,p)}$. According to ([11], Theorem 3.1.14) there exists a sequence of functions $\{A_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ and a sequence of closed cubes $\{Q^*_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{N}}$ such that:

1) For all $j \in \mathbb{Z}$, we have $\sum_{k\geq 0}\chi_{Q^*_{j,k}} \lesssim_d 1$ and $\mathcal{O}^j := \left\{x \in \mathbb{R}^d : \mathcal{M}^0(f)(x) > 2^j\right\} = \bigcup_{k\geq 0} Q^*_{j,k}$.

2) If $Q^*_{j,k} \cap Q^*_{j+1,\ell} \neq \varnothing$ there exist $C_0 > C > 1$ such that: $\mathrm{diam}Q^*_{j+1,\ell} \leq C\mathrm{diam}Q^*_{j,k}$, and $Q^*_{j+1,\ell} \subset C_0 Q^*_{j,k}$.

3) $\mathrm{supp}(A_{j,k}) \subset \tilde{Q}_{j,k} := C_0 Q^*_{j,k}$, $|A_{j,k}| \leq C_1 2^j$, a.e. and $\int_{\mathbb{R}^d} A_{j,k}(x)\mathfrak{p}(x)\mathrm{d}x = 0$ for all polynomials $\mathfrak{p}$ of degree less or equal to *s*. The positive constant C_1 is independent of f, j and *k*.

4) $f = \sum_{j=-\infty}^{+\infty}\sum_{k\geq 0} A_{j,k}$ almost everywhere and in $\mathcal{S}'$.

Let us put

$$\lambda_{j,k} := C_1 2^j \left|\tilde{Q}_{j,k}\right|^{\frac{1}{\alpha}} \text{ and } \mathfrak{a}_{j,k} := \lambda_{j,k}^{-1} A_{j,k}.$$

It comes that

$$\operatorname{supp}(\mathfrak{a}_{j,k}) \subset \tilde{Q}_{j,k},\ \|\mathfrak{a}_{j,k}\|_\infty \le |\tilde{Q}_{j,k}|^{-\frac{1}{\alpha}} \text{ and } \int_{\mathbb{R}^d} x^\beta \mathfrak{a}_{j,k}(x)\,dx = 0,$$

for all $\beta \in \mathbb{N}^d$ such that $|\beta| \le s$. It follows that, $(\mathfrak{a}_{j,k}, \tilde{Q}_{j,k}) \in \mathcal{A}(q,\alpha,\infty,s)$ and

$$f = \sum_{j=-\infty}^{+\infty} \sum_{k\ge 0} \lambda_{j,k} \mathfrak{a}_{j,k} \tag{20}$$

almost everywhere and in $\mathcal{S}'$. It remains to prove that this series converges in the sense of $\mathcal{H}^{(q,p,\alpha)}$ and that Relation (19) is satisfied.

Notice that for all non negative integers *j* and all integers *k*, we have

$$\chi_{\tilde{Q}_{j,k}}(x) \lesssim_{d,C_0,\gamma} \left[\mathfrak{M}\left(\chi_{Q^*_{j,k}}\right)(x)\right]^\gamma, \gamma > 0, \tag{21}$$

for all $x \in \mathbb{R}^d$; with $\tilde{Q} := C_0 Q^*$. It comes from Relation (21) and Proposition 1 that for $\eta > 0$, we have

$$\begin{aligned}\left\|\sum_{j=-\infty}^{+\infty}\sum_{k\ge 0}\left(\frac{|\lambda_{j,k}|}{\|\chi_{\tilde{Q}_{j,k}}\|_\alpha}\right)^\eta \chi_{\tilde{Q}_{j,k}}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} &\le A\left\|\sum_{j=-\infty}^{+\infty}\sum_{k\ge 0} 2^{j\eta}\left[\mathfrak{M}\left(\chi_{Q^*_{j,k}}\right)\right]^\gamma\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} \\ &\le A\left\|\left[\sum_{j=-\infty}^{+\infty}\sum_{k\ge 0}\left(2^{j\frac{\eta}{\gamma}}\chi_{Q^*_{j,k}}\right)^\gamma\right]^{\frac{1}{\gamma}}\right\|_{\frac{q\gamma}{\eta},\frac{p\gamma}{\eta},\frac{\gamma\alpha}{\eta}}^{\frac{\gamma}{\eta}}\end{aligned}$$

with $\gamma := 1 + \frac{\eta}{q}$, and $A = C_{C_1,q,p,\alpha,\eta,d,C_0}$. But then,

$$\left\|\left[\sum_{j=-\infty}^{+\infty}\sum_{k\ge 0}\left(2^{j\frac{\eta}{\gamma}}\chi_{Q^*_{j,k}}\right)^\gamma\right]^{\frac{1}{\gamma}}\right\|_{\frac{q\gamma}{\eta},\frac{p\gamma}{\eta},\frac{\gamma\alpha}{\eta}}^{\frac{\gamma}{\eta}} \lesssim_d \left\|\sum_{j=-\infty}^{+\infty} 2^{j\eta}\chi_{\mathcal{O}^j}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} \lesssim_{d,\eta} \left\|\mathcal{M}^0_{\mathcal{F}_N}(f)\right\|_{q,p,\alpha}$$

thanks to the estimation $\sum_{j=-\infty}^{+\infty} 2^{j\eta}\chi_{\mathcal{O}^j} \le C(\eta)\left[\mathcal{M}^0_{\mathcal{F}_N}(f)\right]^\eta$ given in ([11], Relation (4.18)). It follows that

$$\left\|\sum_{j=-\infty}^{+\infty}\sum_{k\ge 0}\left(\frac{|\lambda_{j,k}|}{\|\chi_{\tilde{Q}_{j,k}}\|_\alpha}\right)^\eta \chi_{\tilde{Q}_{j,k}}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} \lesssim_{C_1,q,p,\alpha,\eta,C_0} \|f\|_{\mathcal{H}^{(q,p,\alpha)}}. \tag{22}$$

Hence the series $\sum_{j=-\infty}^{+\infty}\sum_{k=0}^{+\infty}\lambda_{j,k}\mathfrak{a}_{j,k}$ converges in $\mathcal{H}^{(q,p,\alpha)}$, thanks to Theorem 6. □

Theorem 9. *For all* $f \in \mathcal{H}^{(q,p,\alpha)}$, *there exists a sequence* $\{(\mathfrak{a}_n, Q_n)\}_{n\ge 0}$ *in* $\mathcal{A}(q,\alpha,\infty,s)$ *and a sequence* $\{\lambda_n\}_{n\ge 0}$ *of scalars such that*

$$f = \sum_{n\ge 0}\lambda_n \mathfrak{a}_n \text{ in the sense of } \mathcal{S}' \text{ and } \mathcal{H}^{(q,p,\alpha)}$$

and, for all $\eta > 0$,

$$\left\| \sum_{n\geq 0} \left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_\alpha} \right)^\eta \chi_{Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} \lesssim_{d,s,q,p,\eta,\alpha} \|f\|_{\mathcal{H}^{(q,p,\alpha)}}.$$

Proof. Let $f \in \mathcal{H}^{(q,p,\alpha)}$. There exists a sequence $\{f_n\}_{n\geq 0}$ in $L^\infty \cap \mathcal{H}^{(q,p,\alpha)}$ which converges to f in $\mathcal{H}^{(q,p,\alpha)}$ and such that:

$$\|f_n - f_{n-1}\|_{\mathcal{H}^{(q,p,\alpha)}}^q \leq \left(\frac{1}{2}\right)^n \|f\|_{\mathcal{H}^{(q,p,\alpha)}}^q, \quad n \geq 1. \tag{23}$$

We put

$$g_0 := f_0 \text{ and } g_n := f_n - f_{n-1}, \quad n \geq 1. \tag{24}$$

Let $n \geq 0$. Since g_n belongs to $L^\infty \cap \mathcal{H}^{(q,p,\alpha)}$, it comes from Theorem 8 that there exists a sequence $\left\{\left(\mathfrak{a}_{j,k}^n, \tilde{Q}_{j,k}^n\right)\right\}_{j\in\mathbb{Z},k\in\mathbb{N}} \subset \mathcal{A}(q,\alpha,\infty,s)$, such that

$$g_n = \sum_{j=-\infty}^{+\infty} \sum_{k\geq 0} \lambda_{j,k}^n \mathfrak{a}_{j,k}^n \text{ in the sense of } \mathcal{S}' \text{ and } \mathcal{H}^{(q,p,\alpha)}$$

and

$$\left\| \sum_{j=-\infty}^{+\infty} \sum_{k\geq 0} \left(\frac{|\lambda_{j,k}^n|}{\|\chi_{\tilde{Q}_{j,k}^n}\|_\alpha} \right)^\eta \chi_{\tilde{Q}_{j,k}^n} \right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} \lesssim_{\alpha,d,s,q,p,\eta} \|g_n\|_{\mathcal{H}^{(q,p,\alpha)}}, \quad \eta > 0. \tag{25}$$

From Relation (24) we have

$$\lim_{n\to+\infty} \left\| f - \sum_{m=0}^{n} g_m \right\|_{\mathcal{H}^{(q,p,\alpha)}} = \lim_{n\to+\infty} \|f - f_n\|_{\mathcal{H}^{(q,p,\alpha)}} = 0,$$

which allows us to say that

$$f = \sum_{n\geq 0} g_n = \sum_{n\geq 0} \sum_{j=-\infty}^{+\infty} \sum_{k\geq 0} \lambda_{j,k}^n \mathfrak{a}_{j,k}^n \text{ in the sense of } \mathcal{S}' \text{ and } \mathcal{H}^{(q,p,\alpha)}.$$

Let $\eta > 0$. If $\eta > q$ we have

$$\begin{aligned}
\left\| \sum_{n\geq 0} \left(\sum_{j=-\infty}^{+\infty} \sum_{k\geq 0} \left(\frac{|\lambda_{j,k}^n|}{\|\chi_{\tilde{Q}_{j,k}^n}\|_\alpha} \right)^\eta \chi_{\tilde{Q}_{j,k}^n} \right) \right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{q}{\eta}} &\leq \sum_{n\geq 0} \left\| \sum_{j=-\infty}^{+\infty} \sum_{k\geq 0} \left(\frac{|\lambda_{j,k}^n|}{\|\chi_{\tilde{Q}_{j,k}^n}\|_\alpha} \right)^\eta \chi_{\tilde{Q}_{j,k}^n} \right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{q}{\eta}} \\
&\lesssim_{\alpha,d,s,q,p,\eta} \sum_{n\geq 0} \|g_n\|_{\mathcal{H}^{(q,p,\alpha)}}^q \\
&\lesssim_{\alpha,d,s,q,p,\eta} \sum_{n\geq 1} \left(\frac{1}{2}\right)^n \|f\|_{\mathcal{H}^{(q,p,\alpha)}}^q
\end{aligned}$$

thanks to Relations (23), (24) and (25). If $\eta \leq q$, then

$$\left\|\sum_{n\geq 0}\left(\sum_{j=-\infty}^{+\infty}\sum_{k\geq 0}\left(\frac{\left|\lambda_{j,k}^{n}\right|}{\left\|\chi_{\tilde{Q}_{j,k}^{n}}\right\|_{\alpha}}\right)^{\eta}\chi_{\tilde{Q}_{j,k}^{n}}\right)\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{q}{\eta}} \leq \left[\sum_{n\geq 0}\left\|\sum_{j=-\infty}^{+\infty}\sum_{k\geq 0}\left(\frac{\left|\lambda_{j,k}^{n}\right|}{\left\|\chi_{\tilde{Q}_{j,k}^{n}}\right\|_{\alpha}}\right)^{\eta}\chi_{\tilde{Q}_{j,k}^{n}}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}\right]^{\frac{q}{\eta}}$$

$$\lesssim_{\alpha,d,s,q,p,\eta}\left[\sum_{n\geq 0}\left\|g_{n}\right\|_{\mathcal{H}^{(q,p,\alpha)}}^{\eta}\right]^{\frac{q}{\eta}}$$

$$\lesssim_{\alpha,d,s,q,p,\eta}\left(\sum_{n\geq 1}\left[\left(\frac{1}{2}\right)^{n}\left\|f\right\|_{\mathcal{H}^{(q,p,\alpha)}}^{q}\right]^{\eta}\right)^{\frac{1}{\eta}}$$

once according to Relations (24) and (25). Thus,

$$\left\|\sum_{n\geq 0}\sum_{j=-\infty}^{+\infty}\sum_{k\geq 0}\left(\frac{\left|\lambda_{j,k}^{n}\right|}{\left\|\chi_{\tilde{Q}_{j,k}^{n}}\right\|_{\alpha}}\right)^{\eta}\chi_{\tilde{Q}_{j,k}^{n}}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} \lesssim_{\varphi,q,p,d,s,\eta}\left\|f\right\|_{\mathcal{H}^{(q,p,\alpha)}},$$

which ends the proof. □

Notice that Theorem 9 stills valid if we replace the family $\mathcal{A}(q,\alpha,\infty,s)$ by $\mathcal{A}(q,\alpha,r,s)$ since $\mathcal{A}(q,\alpha,\infty,s)\subset\mathcal{A}(q,\alpha,r,s)$. To prove the converse of Theorem 9, we need the following result which is an adaptation of ([11], Proposition 3.1.4). We omit the proof.

Proposition 10. *Let* $1<u\leq w\leq v<s\leq\infty$ *and* $r>0$.

If $\{\delta_n\}_{n\geq 0}$ *is a sequence of scalars and* $\{b_n\}_{n\geq 0}$ *a sequence of elements of* L^s *such that for* $n\geq 0$, *there exists a cube* Q_n *satisfying:*

1) $\operatorname{supp}(b_n)\subset Q_n$
2) $\left\|b_n\right\|_s\leq\left|Q_n\right|^{\frac{1}{s}-\frac{1}{w}}$,

then

$$\left\|\sum_{n\geq 0}\left|\delta_n St_r^{(\alpha)}b_n\right|\right\|_{u,v}\lesssim\left\|\sum_{n\geq 0}\frac{\left|\delta_n\right|}{\left\|\chi_{Q_n}\right\|_w}St_r^{(\alpha)}\chi_{Q_n}\right\|_{u,v},\quad r,\alpha>0$$

where the implicit constant doesn't depend on *r*, $\{\delta_n\}$ and $\{Q_n\}$.

Theorem 11. *Let* $\max\{p;1\}<r<+\infty$, $0<\eta<q$ *and* $s\geq\left\lfloor d\left(\frac{1}{q}-1\right)\right\rfloor$ *be an integer. For all sequences* $\{(\mathfrak{a}_n,Q_n)\}_{n\geq 0}$ *in* $\mathcal{A}(q,\alpha,r,s)$ *and all sequences* $\{\lambda_n\}_{n\geq 0}$ *of scalars such that*

$$\left\|\sum_{n\geq 0}\left(\frac{\left|\lambda_n\right|}{\left\|\chi_{Q_n}\right\|_{\alpha}}\right)^{\eta}\chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}<\infty, \tag{26}$$

the series $f:=\sum_{n\geq 0}\lambda_n\mathfrak{a}_n$ *converges in the sense of* $\mathcal{S}'$ *and* $\mathcal{H}^{(q,p,\alpha)}$, *with*

$$\|f\|_{\mathcal{H}^{(q,p,\alpha)}} \lesssim_{\varphi,d,q,p,s} \left\| \sum_{n\geq 0} \left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_{\alpha}} \right)^{\eta} \chi_{Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} \tag{27}$$

Proof. The proof is just an adaptation of the one of ([1], Theorem 4.6).

Let $\{(\mathfrak{a}_n, Q_n)\}_{n\geq 0}$ be a sequence of elements of $\mathcal{A}(q,\alpha,r,s)$ and $\{\lambda_n\}_{n\geq 0}$ a sequence of scalars such that relation (26) is satisfied. Put $\left\| \sum_{n\geq 0} \left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_{\alpha}} \right)^{\eta} \chi_{Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}} = A$. For $\rho > 0$, we have

$$\left\| \sum_{n\geq 0} \left(\frac{|\lambda_n| |\rho Q_n|^{\frac{1}{q}-\frac{1}{\alpha}}|}{\|\chi_{\rho Q_n}\|_{q}} \right)^{\eta} \chi_{\rho Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta}} = \left\| \sum_{n\geq 0} \left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_{\alpha}} \right)^{\eta} St_{\rho}^{\frac{\alpha}{\eta}} \chi_{Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta}} \leq A.$$

Since $\{(\mathfrak{a}_n, Q_n)\}_{n\geq 0} \subset \mathcal{A}(q,\alpha,r,s)$ implies that $\{(St_{\rho}^{\alpha}\mathfrak{a}_n, \rho Q_n)\}_{n\geq 0} \subset \mathcal{A}(q,\alpha,r,s)$ which is equivalent to $\left\{\left(St_{\rho}^{\alpha}\mathfrak{a}_n |\rho Q_n|^{\frac{1}{\alpha}-\frac{1}{q}}, \rho Q_n\right)\right\}_{n\geq 0} \subset \mathcal{A}(q,r,s)$, we have that $g := \sum \lambda_n St_{\rho}^{\alpha}\mathfrak{a}_n$ converges in the sense of $\mathcal{S}'$ and $\mathcal{H}^{(q,p)}$, and

$$\|g\|_{\mathcal{H}^{(q,p)}} \lesssim \left\| \sum_{n\geq 0} \left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_{\alpha}} \right)^{\eta} St_{\rho}^{\frac{\alpha}{\eta}} \chi_{Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta}},$$

thanks to ([1], Theorem 4.6). Let $f = St_{\rho^{-1}}^{\alpha}(g)$. We have $f := \sum \lambda_n \mathfrak{a}_n$ in the sense of $\mathcal{H}^{(q,p)}$, and

$$\|St_{\rho}^{\alpha} f\|_{\mathcal{H}^{(q,p)}} \lesssim \left\| \sum_{n\geq 0} \left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_{\alpha}} \right)^{\eta} St_{\rho}^{\frac{\alpha}{\eta}} \chi_{Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta}}.$$

Estimate (27) follows from the definition of $\mathcal{H}^{(q,p,\alpha)}$ and the convergence of the series $f := \sum \lambda_n \mathfrak{a}_n$ in $\mathcal{H}^{(q,p,\alpha)}$ is obtained as in the proof of Theorem 6. ☐

Remark 12. *Let* $1 < r \leq +\infty$, $s \geq \left\lfloor d\left(\frac{1}{q}-1\right) \right\rfloor$ *an integer and* $\alpha \leq r$.

1) *If* $\max\{p,1\} < r \leq \infty$ *then for* $f \in \mathcal{H}^{(q,p,\alpha)}$

$$\|f\|_{\mathcal{H}^{(q,p,\alpha)}} \approx \inf \left\{ \left\| \sum_{n\geq 0} \left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_{\alpha}} \right)^{\eta} \chi_{Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} : f = \sum_{n\geq 0} \lambda_n \mathfrak{a}_n \right\},$$

where the infimum is taken over all atomic decompositions of f using (q,α,r,s) *-atoms,* $0 < \eta \leq 1$ *if* $r = \infty$ *and* $0 < \eta < q$ *if not.*

2) *Let* $\mathcal{H}_{\text{fin},r,s}^{(q,p,\alpha)}$ *consist of finite linear combinations of* (q,α,r,s) *-atoms of* $\mathbb{R}^d$. *The space* $\mathcal{H}_{\text{fin},\infty,s}^{(q,p,\alpha)} \cap \mathcal{C}$ *is a dense subspace of* $\mathcal{H}^{(q,p,\alpha)}$, *where* $\mathcal{C}$ *stands for the space of continuous functions on* $\mathbb{R}^d$.

Proof. These are immediate consequences of ([1], Proposition 4.9 and Lemma 4.10), Relation (9) and the fact that if f is a finite linear combinations of (q,r,s)-atoms, then

$$\sup_{\rho>0}\left\|St_\rho^\alpha f\right\|_{\mathcal{H}_{\text{fin},r,s}^{(q,p)}} = \inf\left\{\left\|\sum_{n=0}^{N}\left(\frac{|\lambda_n|}{\left\|\chi_{Q_n}\right\|_\alpha}\right)^\eta \chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} : f=\sum_{n=0}^{N}\lambda_n\mathfrak{a}_n\right\},$$

where the infimum is taken over all finite atomic decompositions of f using (q,α,r,s)-atoms and

$$\|f\|_{\mathcal{H}_{\text{fin},r,s}^{(q,p)}} := \inf\left\{\left\|\sum_{n=0}^{N}\left(\frac{|\lambda_n|}{\left\|\chi_{Q_n}\right\|_q}\right)^\eta \chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta}}^{\frac{1}{\eta}} : f=\sum_{n=0}^{N}\lambda_n\mathfrak{a}_n\right\},$$

where the infimum is taken over all finite atomic decompositions of f using (q,r,s)-atoms and $0<\eta\le 1$ if $r=\infty$ and $0<\eta\le q$ if $\max\{p,1\}<r<\infty$. □

5. Intrinsic Square Function and Its Commutator

Let $0<\gamma\le 1$. We denote by C_γ the family of functions ψ defined on $\mathbb{R}^d$ such that $\text{supp}(\psi)\subset \bar{B}(0,1)$, $\int_{\mathbb{R}^d}\psi(x)\text{d}x=0$ and

$$\forall x,x'\in\mathbb{R}^d,\quad \left|\psi(x)-\psi(x')\right|\le\left|x-x'\right|^\gamma,$$

where $\psi_t(x)=t^{-d}\psi(t^{-1}x)$ and $\mathbb{R}_+^{d+1}=\mathbb{R}^d\times\left]0,+\infty\right[$. The notation $\bar{B}$ stands for the closure set of B.

Let f be a measurable function. The intrinsic square function $S_\gamma f$ (of order γ) of f is defined by:

$$S_\gamma(f)(x)=\left[\int_{\Gamma(x)}\left(\sup_{\psi\in C_\gamma}\left|f*\psi_t(y)\right|\right)^2\frac{\text{d}y\text{d}t}{t^{d+1}}\right]^{\frac{1}{2}}$$

for all $x\in\mathbb{R}^d$, where $\Gamma(x)=\left\{(y,t)\in\mathbb{R}_+^{d+1}:|y-x|<t\right\}$ is the cone of aperture 1. We have the following result.

Proposition 13. *Let* $0<\gamma\le 1$, $0<q<1$ *and* $q\le\alpha\le p<\infty$. *The operator* S_γ *can be extended into a bounded operator from* $\mathcal{H}^{(q,p,\alpha)}$ *to* $\left(L^q,\ell^p\right)^\alpha$.

Proof. Let $s\ge\left\lfloor d\left(\frac{1}{q}-1\right)\right\rfloor$ be an integer, $\phi\in\mathcal{S}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}\phi(x)\text{d}x=1$. We consider $f=\sum_{n=0}^{j}\lambda_n\mathfrak{a}_n\in\mathcal{H}_{\text{fin},\infty,s}^{(q,p,\alpha)}\cap C(\mathbb{R}^d)$, where $\left\{(\mathfrak{a}_n,Q_n)\right\}_{n=0}^{j}$ is a sequence of elements of $\mathcal{A}(q,\alpha,\infty,s)$ and $\{\lambda_n\}_{n=0}^{j}$ a sequence of scalars. We put $\tilde{Q}_n := 2\sqrt{d}Q_n$ for all $n \in \{0,1,\cdots,j\}$ and denote by x_n and ℓ_n respectively the center and the side-length of Q_n. We have

$$\left\|S_\gamma(f)\right\|_{q,p,\alpha}^q \le \left\|\sum_{n=0}^{j}|\lambda_n|\left|S_\gamma(\mathfrak{a}_n)\right|\chi_{\tilde{Q}_n}\right\|_{q,p,\alpha}^q + \left\|\sum_{n=0}^{j}|\lambda_n|\left|S_\gamma(\mathfrak{a}_n)\right|\chi_{\mathbb{R}^d\setminus\tilde{Q}_n}\right\|_{q,p,\alpha}^q \tag{28}$$

For the first term on the right hand side of (28), we notice that for $0<\eta<q$,

$$\left\|\left(S_\gamma(a_n)\chi_{\tilde{Q}_n}\right)^\eta\right\|_{\frac{r}{\eta}} \le C(r,\gamma,\eta)\|a_n\|_r^\eta \le C(r,\gamma,\eta,d)\left\|\tilde{Q}_n\right\|^{\frac{\frac{1}{r}-\frac{1}{\alpha}}{\eta}}$$

since $\mathcal{A}(q,\alpha,\infty,s)\subset\mathcal{A}(q,\alpha,r,s)$ and S_γ is bounded on $L^r(\mathbb{R}^d)$, for $r>1$ (see [12]). Therefore

$$\left\|\sum_{n=0}^{j}|\lambda_n|^\eta\left(S_\gamma(a_n)\chi_{\tilde{Q}_n}\right)^\eta\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}} \le C(r,d,\eta,\gamma)\left\|\sum_{n=0}^{j}\left(\frac{|\lambda_n|}{\left\|\chi_{Q_n}\right\|_\alpha}\right)^\eta\chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}}$$

thanks to Proposition 10, since $1<\frac{q}{\eta}\le\frac{\alpha}{\eta}\le\frac{p}{\eta}\le\frac{r}{\eta}<\infty$. It remains to estimate the second term. Let $x\in\mathbb{R}^d\setminus\tilde{Q}_n$ and $\psi\in C_\gamma$. We have

$$\left|S_\gamma(a_n)(x)\right| \lesssim_\gamma \left(\int_{\Gamma(x)}\left(\int_{Q_n\cap\bar{B}(y,t)}|a_n(z)|\right)^2\frac{\mathrm{d}y\mathrm{d}t}{t^{3d+1}}\right)^{\frac{1}{2}}.$$

Let $(y,t)\in\Gamma(x)$. We assume that $Q_n\cap\bar{B}(y,t)\ne\varnothing$ since otherwise $S_\gamma(a_n)(x)=0$. It is easy to see that $\bar{B}(y,t)\subset\bar{B}(x,2t)$ and $t\ge\frac{\sqrt{d}}{4}\ell_n$. Since $|a_n(z)|\le\mathcal{M}_\phi a_n(z)$ and $\mathcal{M}_\phi a_n(z)\le\mathcal{M}_{\phi,2}^* a_n(x)$ for $z\in\bar{B}(x,2t)$, we have

$$\begin{aligned}\left|S_\gamma(a_n)(x)\right| &\lesssim_\gamma \left(\int_{\frac{\sqrt{d}}{4}\ell_n}^{+\infty}\int_{B(x,t)}\left(\int_{Q_n\cap\bar{B}(y,t)}\mathcal{M}_\phi a_n(z)\mathrm{d}z\right)^2\frac{\mathrm{d}y\mathrm{d}t}{t^{3d+1}}\right)^{\frac{1}{2}}\\ &\lesssim_\gamma \left(\int_{\frac{\sqrt{d}}{4}\ell_n}^{+\infty}\int_{B(x,t)}\left(\int_{Q_n\cap\bar{B}(y,t)}\mathcal{M}_{\phi,2}^* a_n(x)\mathrm{d}z\right)^2\frac{\mathrm{d}y\mathrm{d}t}{t^{3d+1}}\right)^{\frac{1}{2}}\\ &\lesssim_\gamma \mathcal{M}_{\phi,2}^* a_n(x)\ell_n^d\left(\int_{\frac{\sqrt{d}}{4}\ell_n}^{+\infty}\int_{B(x,t)}\frac{\mathrm{d}y\mathrm{d}t}{t^{3d+1}}\right)^{\frac{1}{2}}\\ &\lesssim_{\gamma,d} \mathcal{M}_{\phi,2}^* a_n(x).\end{aligned}$$

Additionally, we have the following estimate

$$\mathcal{M}_{\phi,2}^* a_n(x) \le C(d,s,\phi)\frac{\left(\mathfrak{M}\left(\chi_{Q_n}\right)(x)\right)^v}{\left\|\chi_{Q_n}\right\|_\alpha}, \tag{29}$$

where $v=\frac{d+s+1}{d}$. This allows us to say that

$$\left|S_\gamma(a_n)(x)\right|\chi_{\mathbb{R}^d\setminus\tilde{Q}_n}(x) \le C(d,s,\phi,\gamma)\frac{\left(\mathfrak{M}\left(\chi_{Q_n}\right)(x)\right)^v}{\left\|\chi_{Q_n}\right\|_\alpha}$$

for all $n\in\{0,1,\cdots,j\}$, so that

$$\begin{aligned}\left\|\sum_{n=0}^{j}|\lambda_n||S_\gamma(a_n)|\chi_{\mathbb{R}^d\setminus\tilde{Q}_n}\right\|_{q,p,\alpha} &\le D\left\|\sum_{n=0}^{j}|\lambda_n|\frac{\left(\mathfrak{M}(\chi_{Q_n})\right)^{\nu}}{\|\chi_{Q_n}\|_\alpha}\right\|_{q,p,\alpha}\\ &\le D\left\|\left(\sum_{n=0}^{j}\left(\mathfrak{M}\left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_\alpha}\right)^{\frac{1}{\nu}}\chi_{Q_n}\right)^{\nu}\right)^{\frac{1}{\nu}}\right\|_{q\nu,p\nu,\nu\alpha}^{\nu}\\ &\le D\left\|\left(\sum_{n=0}^{j}\left(\left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_\alpha}\right)^{\frac{1}{\nu}}\chi_{Q_n}\right)^{\nu}\right)^{\frac{1}{\nu}}\right\|_{q\nu,p\nu,\nu\alpha}^{\nu}\\ &\le D\left\|\sum_{n=0}^{j}\left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_\alpha}\right)^{\eta}\chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}},\end{aligned}$$

where $D=C(d,s,\phi,\gamma,\eta)$. Finally we have

$$\|S_\gamma(f)\|_{q,p,\alpha}\lesssim_{r,d,\eta,\gamma,s,\phi}\|f\|_{\mathcal{H}_{\text{fin},\infty,s}^{(q,p,\alpha)}}\lesssim_{r,d,\eta,\gamma,s,\phi}\|f\|_{\mathcal{H}^{(q,p,\alpha)}}.$$

The density of $\mathcal{H}_{\text{fin},\infty,s}^{(q,p,\alpha)}\cap C(\mathbb{R}^d)$ in $\mathcal{H}^{(q,p,\alpha)}$ gives the result. □

This result generalized the analogue established in the context of Fofana's spaces in [13] when $1<q$.

Our next result deals with the boundedness of the commutator operator associated to this intrinsic square function. Let b be a locally integrable function. The commutator of b and S_γ is defined by

$$[b,S_\gamma](f)(x)=\left[\int_{\Gamma(x)}\sup_{\psi\in\mathcal{C}_\gamma}\left|\int_{\mathbb{R}^d}(b(x)-b(z))\psi_t(y-z)f(z)\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{d+1}}\right]^{\frac{1}{2}}.$$

For the case of Lebesgue spaces and Fofana spaces, the boundedness of the commutator has been obtained under the assumption that b belongs to *BMO*.

We recall that the space *BMO* consists of functions $f\in L^1_{\text{loc}}$ satisfying $\|f\|_{BMO}<\infty$ where

$$\|f\|_{BMO}=\sup_{B:ball}\frac{1}{|B|}\int_B|f(x)-f_B|\mathrm{d}x$$

with f_B denoting the average over B of f, *i.e.* $f_B=\frac{1}{|B|}\int_B f(x)\mathrm{d}x$.

We say that a locally integrable function b belongs to $BMO^d:=BMO^d(\mathbb{R}^d)$ if there exists $0<\mu_b<d$ such that for all cubes $Q:=Q(x_Q,\ell_Q)$ of $\mathbb{R}^d$,

$$|\mathfrak{b}(x)-\mathfrak{b}_Q|\le C\left(\ell_Q^{-1}|x-x_Q|\right)^{\mu_b} \tag{30}$$

for all $x \notin Q$ (C is a positive constant which does not depend on Q and x).

Proposition 14. *Let* $0<\gamma\le 1$, $0<q<1$, $q\le\alpha\le p<\infty$, $s\ge\left\lfloor d\left(\frac{1}{q}-1\right)\right\rfloor$ *be an integer and* $\mathfrak{b}\in BMO^d$. *Then* $\left[\mathfrak{b},S_\gamma\right]$ *is extended to a bounded operator from* $\mathcal{H}^{(q,p,\alpha)}$ *to* $\left(L^q,\ell^p\right)^\alpha$.

Proof. Let $\phi\in\mathcal{S}\left(\mathbb{R}^d\right)$ such that $\int_{\mathbb{R}^d}\phi(x)\mathrm{d}x=1$. Let $f=\sum_{n=0}^{j}\lambda_n a_n\in\mathcal{H}_{\mathrm{fin},\infty,s}^{(q,p,\alpha)}\cap C\left(\mathbb{R}^d\right)$, with $\left\{\left(a_n,Q_n\right)\right\}_{n=0}^{j}$ a sequence of elements of $\mathcal{A}(q,\alpha,\infty,s)$ and $\left\{\lambda_n\right\}_{n=0}^{j}$ a sequence of scalars. We put $\tilde{Q}_n:=2\sqrt{d}Q_n$ for all $n\in\{0,1,\cdots,j\}$ and denote by x_n and ℓ_n respectively the center and the side-length of Q_n. We have:

$$\left\|\left[\mathfrak{b},S_\gamma\right](f)\right\|_{q,p,\alpha}^{q}\le\left\|\sum_{n=0}^{j}\left|\lambda_n\right|\left|\left[\mathfrak{b},S_\gamma\right]\left(a_n\right)\chi_{\tilde{Q}_n}\right|\right\|_{q,p,\alpha}^{q}+\left\|\sum_{n=0}^{j}\left|\lambda_n\right|\left|\left[\mathfrak{b},S_\gamma\right]\left(a_n\right)\chi_{\mathbb{R}^d\setminus\tilde{Q}_n}\right|\right\|_{q,p,\alpha}^{q} \tag{31}$$

Fix $0<\eta<q$. Since $\left[\mathfrak{b},S_\gamma\right]$ is bounded on L^r for $r>\max\{1,p\}$, we have

$$\left\|\sum_{n=0}^{j}\left|\lambda_n\right|^\eta\left(\left[\mathfrak{b},S_\gamma\right]\left(a_n\right)\chi_{\tilde{Q}_n}\right)^\eta\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}}\le C\left(r,\gamma,\eta,d,\|\mathfrak{b}\|_{BMO}\right)\left\|\sum_{n=0}^{j}\left(\frac{\left|\lambda_n\right|}{\left\|\chi_{Q_n}\right\|_\alpha}\right)^\eta\chi_{Q_n}\right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}},$$

thanks to Proposition 10.

Let us now estimate the second term on the right hand side of (5). Let $x\in\mathbb{R}^d\setminus\tilde{Q}_n$. Using the same arguments as in the proof of Proposition 13, we obtain

$$\begin{aligned}\left|\left[\mathfrak{b},S_\gamma\right]\left(a_n\right)(x)\right|&\lesssim_\gamma\left(\int_{\frac{\sqrt{d}}{4}\ell_n}^{+\infty}\int_{B(x,t)}\left(\int_{Q_n\cap\bar{B}(y,t)}\left|\mathfrak{b}(x)-\mathfrak{b}(z)\right|\mathcal{M}_{\phi,2}^{*}a_n(x)\mathrm{d}z\right)^2\frac{\mathrm{d}y\mathrm{d}t}{t^{3d+1}}\right)^{\frac{1}{2}}\\&\lesssim_\gamma\left(\int_{\frac{\sqrt{d}}{4}\ell_n}^{+\infty}\int_{B(x,t)}\left(\int_{Q_n\cap\bar{B}(y,t)}\left|\mathfrak{b}(x)-\mathfrak{b}_{Q_n}\right|\mathcal{M}_{\phi,2}^{*}a_n(x)\mathrm{d}z\right)^2\frac{\mathrm{d}y\mathrm{d}t}{t^{3d+1}}\right)^{\frac{1}{2}}\\&\quad+\left(\int_{\frac{\sqrt{d}}{4}\ell_n}^{+\infty}\int_{B(x,t)}\left(\int_{Q_n\cap\bar{B}(y,t)}\left|\mathfrak{b}(z)-\mathfrak{b}_{Q_n}\right|\mathcal{M}_{\phi,2}^{*}a_n(x)\mathrm{d}z\right)^2\frac{\mathrm{d}y\mathrm{d}t}{t^{3d+1}}\right)^{\frac{1}{2}}\\&=E+F.\end{aligned}$$

Since $\mu_b<d$, $\mathfrak{b}\in BMO^d$ and $t\ge\frac{\sqrt{d}}{4}\ell_n$, it comes that $\ell_n^{-1}\left|x-x_n\right|<4t\ell_n^{-1}$ and then

$$\begin{aligned}E&\le\left(\int_{\frac{\sqrt{d}}{4}\ell_n}^{+\infty}\int_{B(x,t)}\left(\ell_n^{-1}\left|x-x_n\right|\right)^{2\mu_b}\left(\int_{Q_n}\mathcal{M}_{\phi,2}^{*}a_n(x)\mathrm{d}z\right)^2\frac{\mathrm{d}y\mathrm{d}t}{t^{3d+1}}\right)^{\frac{1}{2}}\\&\le C\left(d,\mu_b\right)\mathcal{M}_{\phi,2}^{*}a_n(x)\ell_n^{d-\mu_b}\left(\int_{\frac{\sqrt{d}}{4}\ell_n}^{+\infty}\frac{\mathrm{d}t}{t^{2d-2\mu_b+1}}\right)^{\frac{1}{2}}\\&\le C\left(d,\mu_b\right)\mathcal{M}_{\phi,2}^{*}a_n(x).\end{aligned}$$

With regard to the term F, we have

$$F \le \|\mathfrak{b}\|_{BMO} |B(0,1)| \mathcal{M}^{*}_{\phi,2} a_n(x) \ell_n^d \left(\int_{\frac{\sqrt{d}}{4}\ell_n}^{+\infty} \frac{\mathrm{d}t}{t^{2d+1}} \right)^{\frac{1}{2}}$$
$$\le \|\mathfrak{b}\|_{BMO} C(d) \mathcal{M}^{*}_{\phi,2} a_n(x).$$

Therefore,

$$\left| [\mathfrak{b}, S_\gamma](a_n)(x) \right| \le C\left(d, \gamma, \mu_b, \|\mathfrak{b}\|_{BMO}\right) \mathcal{M}^{*}_{\phi,2} a_n(x).$$

Now

$$\mathcal{M}^{*}_{\phi,2} a_n(x) \le C(d,s,\phi) \frac{\left(\mathfrak{M}\left(\chi_{Q_n}\right)(x)\right)^{v}}{\|\chi_{Q_n}\|_{\alpha}},$$

where $v = \dfrac{d+s+1}{d}$ according to relation (29); so

$$\left| [\mathfrak{b}, S_\gamma](a_n) \chi_{\mathbb{R}^d \setminus \tilde{Q}_n}(x) \right| \le C\left(d, \mu_b, s, \phi, \gamma, \|\mathfrak{b}\|_{BMO}\right) \frac{\left(\mathfrak{M}\left(\chi_{Q_n}\right)(x)\right)^{v}}{\|\chi_{Q_n}\|_{\alpha}}$$

for all $n \in \{0,1,\cdots,j\}$. Hence there exists $\mathfrak{c} := C\left(d, \mu_b, s, \phi, \gamma, \|\mathfrak{b}\|_{BMO}\right)$, such that

$$\left\| \sum_{n=0}^{j} |\lambda_n| \left| [\mathfrak{b}, S_\gamma](a_n) \right| \chi_{\mathbb{R}^d \setminus \tilde{Q}_n} \right\|_{q,p,\alpha} \le \mathfrak{c} \left\| \sum_{n=0}^{j} |\lambda_n| \frac{\left(\mathfrak{M}\left(\chi_{Q_n}\right)\right)^{v}}{\|\chi_{Q_n}\|_{\alpha}} \right\|_{q,p,\alpha}$$
$$\le \mathfrak{c} \left\| \sum_{n=0}^{j} \frac{|\lambda_n|}{\|\chi_{Q_n}\|_{\alpha}} \chi_{Q_n} \right\|_{q,p,\alpha}$$

thanks to Proposition 1. It comes that

$$\left\| \sum_{n=0}^{j} |\lambda_n| \left| [\mathfrak{b}, S_\gamma](a_n) \right| \chi_{\mathbb{R}^d \setminus \tilde{Q}_n} \right\|_{q,p,\alpha} \le \mathfrak{c} \left\| \sum_{n=0}^{j} \left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_{\alpha}} \right)^{\eta} \chi_{Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}}.$$

Finally

$$\left\| [\mathfrak{b}, S_\gamma](f) \right\|_{q,p,\alpha} \le C(r, \mu_b, \phi, \gamma, \eta, d, b) \left\| \sum_{n=0}^{j} \left(\frac{|\lambda_n|}{\|\chi_{Q_n}\|_{\alpha}} \right)^{\eta} \chi_{Q_n} \right\|_{\frac{q}{\eta},\frac{p}{\eta},\frac{\alpha}{\eta}}^{\frac{1}{\eta}}$$
$$\le C(r, \mu_b, \phi, \gamma, \eta, d, b) \|f\|_{\mathcal{H}^{(q,p,\alpha)}}.$$

The result follows from the density of $\mathcal{H}^{(q,p,\alpha)}_{\mathrm{fin},\infty,s} \cap C\left(\mathbb{R}^d\right)$ in $\mathcal{H}^{(q,p,\alpha)}$. □

6. Conclusion

In this article, we have defined Fofana spaces of Hardy type and given their atomic decompositions. These decompositions allowed us to control some intrinsic square functions as well as their commutators with functions in BMO^d, a proper subspace of *BMO*. We are certain that this subspace can be improved if we do not consider the *BMO* space. Moreover, we assert that a similar control can be given for the commutators of the Calderon-zygmund operators and the elements of BMO^d. This is a work in progress and will be the subject of a forthcoming article.

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Conflicts of Interest

The authors declare no conflicts of interest.

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