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# On Theta Transitivity in a Topological Space with Countable Base

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#### **Abstract**

In this paper, we have introduced some concepts about topological dynamical systems and proved some new corollary and theorems of transitivity of a theta irresolute function defined on topological space.

## **Subject Areas**

Mathematical Analysis

## **Keywords**

 $\theta$ -Irresolute Function, Dynamics in Topological Spaces, Transitive Functions,  $\theta$ -Adherent Points

### 1. Introduction

In this paper, we have investigated and introduced some new definitions of transitivity in topological space. To study the dynamics of a self-map  $f: X \to X$  means to study the qualitative behavior of the sequences  $\{f^n(x)\}$  as n goes to infinity when x varies in X, where  $f^n$  denotes the composition of f with itself n times:

By a topological system I mean a pair (X, f), where X is a locally compact Hausdorff topological space (the phase space), and  $f: X \to X$  is a continuous function. The dynamics of the system is given by  $x_{n+1} = f(x_n), x_0 \in X, n \in \mathbb{N}$  and the solution passing through x is the sequence  $\{f(x_n)\}$  where  $n \in \mathbb{N}$ .

Let  $x \in X$ , then the set  $\{x, f(x), f^2(x), \cdots\}$  is called an orbit of x under f and is denoted by  $O_f(x)$ , so  $O_f(x)$  is the set of points which occur on the orbit of x at some positive time, and the sequence  $x, f(x), f^2(x), \cdots$  is called the trajectory of x. Any point with dense orbit is called a transitive point. A point which is not transitive is called intransitive.

Topological dynamics is concerned with the behavior of iterations of a continuous map f from a space X into itself. Suppose for some  $x \in X$ , sequence  $x, f(x), f^2(x), \cdots$  converges to some point say  $x_0 \in X$ , then we must have  $f(x_0) = x_0$ , because f is continuous. Such points we call as *fixed points*. We say that the point x is attracted by the fixed point  $x_0$ . The set of all points in X attracted by  $x_0$  is called the *stable set* or the *basin of attraction of* the fixed point  $x_0$  and is denoted by  $W_f(x_0)$ . A fixed point  $x_0$  is said to be attracting if its stable set is a neighborhood of it.

A point  $x \in X$  is said to be *periodic* if there exists a positive integer  $n \in \mathbb{N}$  such that  $f^n(x) = x$ . The set of all periodic points of the map f is denoted by per(f).

A point  $x \in X$  is called a  $\theta$ -adherent point of A [1], if  $A \cap Cl(U) \neq \phi$  for every open set U containing x. The set of all  $\theta$ -adherent points of a subset A of X is called the  $\theta$ -closure of A and is denoted by  $Cl_{\theta}(A)$ . A subset A of X is called  $\theta$ -closed if  $A = Cl_{\theta}(A)$ . Dontchev and Maki [2] have shown that if A and B are subsets of a space X, then  $Cl_{\theta}(A \cup B) = Cl_{\theta}(A) \cup Cl_{\theta}(B)$  and that  $Cl_{\theta}(A \cap B) = Cl_{\theta}(A) \cap Cl_{\theta}(B)$ . Recall that a space  $(X, \tau)$  is Hausdorff if and only if every compact set is  $\theta$ -closed. The complement of a  $\theta$ -closed set is called a  $\theta$ -open set. The family of all  $\theta$ -open sets forms a topology on X and is denoted by  $\tau^{\theta}$ . This topology is coarser than  $\tau$  and that a space  $(X, \tau)$  is regular if and only if  $\tau = \tau^{\theta}$  [3].

#### 2. Basic Definition and Theorems

**Definition 2.1** [4] By a topological system I mean a pair (X, f), where X is a locally compact Hausdorff topological space (the phase space), and  $f: X \to X$  is a continuous function. The dynamics of the system is given by  $x_{n+1} = f(x_n), x_0 \in X, n \in \mathbb{N}$  and the solution passing through  $x_0$  is the sequence  $\{f(x_n)\}$  where  $n \in \mathbb{N}$ .

**Definition 2.2.** 1) Let  $x \in X$ , then the set  $\{x, f(x), f^2(x), \cdots\}$  is called an orbit of x under f and is denoted by  $O_f(x)$ , so  $O_f(x)$  is the set of points which occur on the orbit of x at some positive time, and the sequence  $x, f(x), f^2(x), \cdots$  is called the trajectory of x.

2) Let X be a topological space,  $f: X \to X$ ,  $\left\{f^n(x_0)\right\}_{n=0}^{\infty}$  be a sequence in X, and let  $x \in X$ . Then  $\left\{f^n(x_0)\right\}$  converges to x if for all open sets U containing x, there exists an integer N such that  $f^n(x_0) \in U$  for all n > N, Note that if this sequence is convergence then it converges to a fixed point, say y, *i.e.* f(y) = y.

Any point with dense orbit is called a transitive point. A point which is not transitive is called intransitive.

**Definition 2.3.** 1) (Transitivity) Let X be a topological space with no isolated point. Then the function  $f: X \to X$  is said to be transitive if for any two open sets U and V in X, there is a point  $x \in U$  and an n > 0 such that  $f^n(x) \in V$ . It is easily to show that if f is transitive then for every pair U, V of non-empty open

sets, there exist a positive integer n such that  $f^n(U) \cap V \neq \phi$ .

2) Let X be a topological space, the function  $f: X \to X$ , is said to be *topologically mixing* if for every pair U, V of non-empty open sets, if there exist N such that  $f^n(U) \cap V \neq \emptyset$  for all n > N.

**Definition 2.4.** (topological weak mixing) Let X has no isolated point. g is topologically *weakly mixing*, if the product of two functions  $g \times g$  is topologically transitive.

**Proposition 2.5.** Every topological mixing function implies topological weak mixing. But the converse is no necessarily true.

Proof: It is easily to prove the foregoing theorem.

**Definition 2.6.** A map f is said to be transitive (resp.,  $\theta$ -transitive [5]) if for any non-empty open (resp.,  $\theta$ -open) sets U and V in X, there exists  $n \in \mathbb{N}$  such that  $f''(U) \cap V \neq \emptyset$ .

**Theorem 2.7** [5]. Let X be a non-empty locally  $\theta$ -compact Hausdorff space. Then the intersection of a countable collection of  $\theta$ -open  $\theta$ -dense subsets of X is  $\theta$ -dense in X.

**Corollary 2.8.** A subset A of a space  $(X,\tau)$  is  $\theta$ -dense if and only if  $A \cap U \neq \phi$  for all  $U \in \tau^{\alpha}$  other than  $U = \phi$ .

Two topological spaces  $(X,\tau)$  and  $(Y,\tau_1)$  are called homeomorphic if there exists a one-to-one onto function  $f:(X,\tau)\to (Y,\tau_1)$  such that f and  $f^{-1}$  are both continuous.

Note that any homeomorphic spaces have the same dynamics, if we have any notion about first space then we have the same notion about the other one.

A map  $h: X \to Y$  is a homeomorphism if it is continuous, bijective and has a continuous inverse.

A function  $f: X \to X$  is called  $\theta$ -irresolute [6] if the inverse image of each  $\theta$ -open set is a  $\theta$ -open set in X.

A map  $h: X \to Y$  is  $\theta$ r-homeomorphism if it is bijective and thus invertible and both h and  $h^{-1}$  are  $\theta$ -irresolute.

**Theorem 2.9.** Let (X, f) be a topological system where X is a non-empty  $\theta$ -compact topological space and  $f: X \to X$  is  $\theta$ -irresolute map and that X is separable. Suppose that f is topologically  $\theta$ -transitive. Then there is an element  $x \in X$  such that the orbit  $O_f(x) = \{x, f(x), f^2(x), \dots, f^n(x), \dots\}$  is  $\theta$ -dense in X.

**Proof:** Let  $B = \{U_i\}, i = 1, 2, 3, \cdots$  be a countable basis for the  $\theta$ -topology of X. For each i, let  $O_i = \{x \in X : f^n(x) \in U_i \text{ for some } n \ge 0\}$ 

Then, clearly  $O_i$  is  $\theta$ -open and  $\theta$ -dense. It is  $\theta$ -open since f is  $\theta$ -irresolute, so,  $O_i = \bigcup_{i=1}^\infty f^{-1} \left( U_i \right) \quad \text{is } \theta \text{-open and } \theta \text{-dense since } f \text{ is topological } \theta \text{-transitive map. Further, for every } \theta \text{-open set } V \text{, there is a positive integer } n \text{ such that } f^n \left( V \right) \cap U_i \neq \emptyset \text{, since } f \text{ is } \theta \text{ transitive.}$ 

Now, apply theorem 2.7 to the countable  $\theta$ -dense set  $\{O_i\}$  to say that  $\bigcap_{i=0}^{\infty} O_i$ 

is  $\theta$ -dense and so non-empty. Let  $y \in \bigcap_{i=0}^{\infty} O_i$ . This means that, for each i, there is a positive integer n such that  $f^n(y) \in U_i$  for every i. By Corollary 2.8 this implies that  $O_f(x)$  is  $\theta$ -dense in X.

**Definition 2.10.** The function  $f: X \to X$ , is strongly transitive [7] if for any nonempty open set  $U \subset X$ ,  $X = \bigcup_{k=0}^s f^k(U)$  for some s>0. It is easily seen that  $X = \bigcup_{k=0}^\infty f^k(U)$  for any nonempty open set  $U \subset X$  if and only if  $\bigcup_{k=0}^\infty f^{-k}(x)$  is dense in X for any  $x \in X$ .

We may consider that, the last statement of the foregoing definition as lemma, because we can use this statement to prove the following corollary.

**Lemma 2.11.**  $X = \bigcup_{k=0}^{\infty} f^k(U)$  for any nonempty open set  $U \subset X$  if and only if  $\bigcup_{k=0}^{\infty} f^{-k}(x)$  is dense in X for any  $x \in X$ .

According to the definition 2.10 and lemma 2.11, we have the following important corollary.

**Corollary 2.12.** If  $\bigcup_{k=0}^{\infty} f^{-k}(x)$  is dense in X for any  $x \in X$ , then the function  $f: X \to X$ , is strongly transitive.

## 3. Conclusion:

There are the following results:

**Proposition 3.1.** Every topological mixing function implies topological weak mixing. But the converse is no necessarily true.

**Theorem 3.2.** Let (X, f) be a topological system where X is a non-empty  $\theta$ -compact topological space and  $f: X \to X$  is  $\theta$ -irresolute map and that X is separable. Suppose that f is topologically  $\theta$ -transitive. Then there is an element  $x \in X$  such that the orbit  $O_f(x) = \{x, f(x), f^2(x), \cdots, f^n(x), \cdots\}$  is  $\theta$ -dense in X.

#### **Conflicts of Interest**

The author declares no conflicts of interest.

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