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On Weak JN-Clean Rings

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Abstract

We can say for a ring R weak JN-clean ring if all elements a in R it can be written as a sum or difference of nilpotent and idempotent. Further the nilpotent element belongs to the Jacobson radical of R. The purpose of this paper is to give some characterization and basic properties of this ring. Also we will studied the relationship between weak JN-clean rings and J-reduced ring, Boolean ring, local ring and clean ring: from the main result: 1) The ring R is weak JN-clean, if and only if, $R/J(R)$ is weak JN-clean and each idempotent lifts modulo $N(R)$. 2) Let $R_1, R_2, \cdots, R_n$ be rings. Then, $R = \prod_{i=1}^{n} R_i$ is weak JN-clean if and only if each R_i for each i is weak JN-clean and at most one R_i is not nil clean. 3) Let R be weak JN-clean. Then, $2 \in J(R)$.

Subject Areas

Algebra

Keywords

Clean Rings, Local Rings, Boolean Rings and Weak JN-Clean Rings

1. Introduction

Through this paper will be an associative ring with identity. We write $U(R)$, $N(R)$, $Idem(R)$, $J(R)$ denote to group of units, the set of nilpotent elements of R, the set of idempotent elements of R, the Jacobson radical of R, respectively. A ring R is reduced if R contains no non-zero nilpotent element [1], following [2] [3]. A ring R is said to be local, if it has exactly one maximal ideal. A ring R is called Boolean ring if each element in R is idempotent [2]. A ring R is called semipotent, if each left ideal (respectively: right ideal) not contained in $J(R)$ contains a non-zero idempotent [4]. A ring R is called clean (strongly clean), if each element a in R can be written as, $a = e + u$ ($a = e + u$, $eu = ue$), where $e \in Idem(R)$ and $u \in U(R)$ [5] [6] [7] [8], similarly a nil clean ring was introduced by Diesel [7] and defined as a ring R is called nil clean, if each element a in R can be written as a sum of an idempotent and nilpotent. A ring R is called J-reduced if $a^n = 0$ for $a \in R$ and for some positive integer n. Then $a \in J(R)$ [9].

2. A Study of Some Characterization of Weak JN-Clean Ring

In this section we give the definition of weak JN-clean rings with some of its characterization and basic properties.

Definition 2.1.

An element a of a ring R is said to be weak JN-clean (resp. strongly weak JN-clean) if a can be written as $a = e + b$ or $a = b - e$ (resp. $a = e + b$ or $a = b - e$ and $be = eb$) for some nilpotent element $b \in J(R)$ and idempotent e.

Example:

1) Every local ring is weak JN-clean ring.

2) Every reduced ring is weak JN-clean ring.

3) Every field is weak JN-clean ring.

Proposition 2.2.

The homomorphic image of weak JN-clean ring element is weak JN-clean ring element.

Proof:

Let $f : R \to S$ be a ring epimorphism and suppose R is weak JN-clean. Let $s \in S$ and choose $a \in R$ such that $f(a) = s$. Then we can write $a = b + e$ or $a = b - e$ for some $b \in N(R) \subset J(R)$ and $e \in Idem(R)$. Hence $s = f(a) = f(b) + f(e)$ or $s = f(a) = f(b) - f(e)$, where clearly $f(e) \in Idem(S)$ and $f(b) \in N(S) \subset J(S)$, Thus s is weak JN-clean element.

Proposition 2.3.

Let R be a ring, then R is weak JN-clean, if and only if, $R/J(R)$ is weak JN-clean and each idempotent lifts modulo $N(R)$.

Proof:

Assume that R is weak JN-clean ring, since the homomorphic image of a nilpotent is again nilpotent and the image of idempotent is again idempotent, then $R/J(R)$ is weak JN-clean ring.

Conversely, let $a + J(R) = (b + I) + (e + I)$ or $a + J(R) = (b + I) - (e + I)$ where $b \in N(R) \subset J(R)$ and $e \in Idem(R)$. Hence, $a - b - e \in J(R)$ or $a - b + e \in J(R)$. Now, $a - e = b + j$ or $a + e = b + j$ where, $j \in J(R)$ since each idempotent lifts modulo $N(R)$. Then we have $b + j \in N(R) \subset J(R)$. Therefore R is weak JN-clean ring.

Notes:

1) Clearly every nil clean ring is weak JN-clean ring.

2) The finite products of weak JN-clean rings are not weak JN-clean for example;

If $R = \mathbb{Z}_3 \times \mathbb{Z}_3$, Then R is not weak JN-clean ring.

Now, we give the necessary condition to prove the following proposition.

Proposition 2.4.

Let $R_1, R_2, \cdots, R_n$ be rings. Then, $R = \prod_{i=1}^{n} R_i$ is weak JN-clean if and only if each R_i for each *i* is weak JN-clean and at most one R_i is not nil clean ring.

Proof:

Clearly $N\left(\prod_{i=1}^{n} R_i\right) \subseteq \prod_{i=1}^{n} N(R_i) \subseteq \prod_{i=1}^{n} J(R_i) = J\left(\prod_{i=1}^{n} R_i\right)$ and by assume *R* is weak JN-clean, then each R_i is homomorphic image of *R* is weak JN-clean, suppose for some i_1 and i_2; $i_1 \neq i_2$, R_{i_1} and R_{i_2} are not nil clean.

Now, for, R_{i_1} is not nil clean, that is not all elements $x \in R_{i_1}$ are of the form $b-e$ where $b \in N\left(R_{i_1}\right) \subset J\left(R_{i_1}\right)$ and $e \in Idem\left(R_{i_1}\right)$. But R_{i_1} is weak JN-clean, so there exists $x_{i_1} \in R_{i_1}$, with $x_{i_1} = b_{i_1} + e_{i_1}$ where, $e_{i_1} \in Idem\left(R_{i_1}\right)$ and $b_{i_1} \in N\left(R_{i_1}\right) \subset J\left(R_{i_1}\right)$, but $x_{i_1} \neq b_{i_1} - e_{i_1}$, for $e_{i_1} \in Idem\left(R_{i_1}\right)$ and $b_{i_1} \in N\left(R_{i_1}\right) \subset J\left(R_{i_1}\right)$. And also there exists $x_{i_2} \in R_{i_2}$, with $x_{i_2} = b_{i_2} - e_{i_2}$, where $e_{i_2} \in Idem\left(R_{i_2}\right)$ and $b_{i_2} \in N\left(R_{i_2}\right) \subset J\left(R_{i_2}\right)$ but, $x_{i_2} \neq b + e$ for any $e \in Idem\left(R_{i_2}\right)$ and $b \in N\left(R_{i_2}\right) \subset J\left(R_{i_2}\right)$. Now, define $a = (a_i) \in R$ by $a_{ij} = a_{i1}$ or $a_{ij} = a_{i2}$ if $i \in \{i_1, i_2\}$ and $a_i = 0$ if $i \neq i_1$ or $i \neq i_2$. Then clearly $a \neq b \mp e$ for any $b \in N(R) \subset J(R)$ and $e \in Idem(R)$, hence at most one of R_i is not nil clean.

Conversely: Assume some R_{i_0} is weak JN-clean but not nil clean that all other R_i are nil clean.

Let $a = (a_i) \in R$. In R_{i_0} we can write $a_{i_0} = b_{i_0} + e_{i_0}$ or $a_{i_0} = b_{i_0} - e_{i_0}$ where $b_{i_0} \in N\left(b_{i_0}\right) \subset J\left(b_{i_0}\right)$ and $e_{i_0} \in Idem\left(R_{i_0}\right)$

Now, if $a_{i_0} = b_{i_0} + e_{i_0}$ for $i \neq i_0$, let $a_i = b_i + e_i$ and if $a_{i_0} = b_{i_0} - e_{i_0}$ for $i \neq i_0$, let $a_i = b_i - e_i$. Then, $b = b_i \in N(R) \subset J(R)$ and $e = (e_i) \in Idem(R)$ and $a = b + e$ or $a = b - e$. Therefore *R* is weak JN-clean.

For example: $R = \mathbb{Z}_3 \times \mathbb{Z}_4$ is a weak JN-clean ring, where $\mathbb{Z}_3$ is not nil clean ring, but is weak nil clean and $\mathbb{Z}_4$ is nil clean, then is weak JN-clean.

Proposition 2.5.

Let *R* be weak JN-clean ring then, $2 \in J(R)$.

Proof:

Clearly there exist, $e \in R$ and $b \in N(R) \subset J(R)$ such that $2 = b + e$ or $2 = b - e$ if $2 = b + e$ so that, $1 - e = b - 1 \in U(R)$ then $e = 0$ thus $b = 2$ and $2 \in J(R)$ or $2 = b - e$ then $1 + e = b - 1 \in U(R)$ and this true only when $e = 0$ so that $b = 2 \in J(R)$.

Proposition 2.6.

A ring *R* is strongly weak JN-clean ring, if and only if, *eRe* is strongly weak JN-clean ring for all idempotent $e \in R$.

Proof:

Let *R* be a strongly weak JN-clean and let $e \in R$. Then, $ae \in eRe \subset R$. That is, $eae = e(b+e)e$ or $eae = e(b-e)e$ where, $b \in N(R) \subset J(R)$ and $e \in Idem(R)$. Then, $ae = ebe + e$ or $ae = ebe - e$. Now, since $b^n = 0$ for some $n \in \mathbb{Z}^+$, Then $(ebe)^n = 0$ in *eRe*, Hence $(ebe)^n = 0$ in *R*. Since *R* is strongly weak JN-clean, then $ebe \in J(R)$ and so $ebe \in eJ(R)e$. That is

$ebe \in J(eRe)$; Therefore eRe is strongly weak JN-clean. The converse is trivial.

3. The Relation between Weak JN-Clean Ring and Other Rings

In this section we give the relationship between weak JN-clean (strongly weak JN-clean) and local rings, Boolean ring, nil-clean rings, semipotent ring and J-reduced ring.

Proposition 3.1.

A ring R is weak JN-clean with $J(R)=0$. If and only if, R is Boolean ring or clean ring.

Proof:

Suppose that R is weak JN-clean ring with $J(R)=0$. Let $a \in R$. Then, a can be written as $a=b+e$ or $a=b-e$ that is, $a=(b-1)+(1-e)$ since R is weak JN-clean ring and $J(R)=0$. Then, $b=0$ thus, $a=e$ or $a=-1+(1-e)$, so $-1 \in U(R)$ and $(1-e) \in Idem(R)$.

Therefore R is Boolean ring or clean ring.

The other direction is easy to stable.

Proposition 3.2.

Let R be an abelian semipotent ring then, every element in R is weak JN-clean ring.

Proof:

Let $0 \neq a \in R$ such that $a^n=0; n \geq 2$ and let $a \in J(R)$, since R is semipotent then, there exist $0 \neq e \in aR$ such that $e=ar$, $r \in R$ thus $er=r=rar$ and we have ar is idempotent (since R is ablean). Thus, ar is central.

Hence, $e=e^n=(ar)^n=r^{n-1}a^n r=0$ and that is contradiction. Then, $a \in J(R)$, and a is a nilpotent.

Now, every element in R can be written as $k \in R$ such that $k=a+e$. Thus R is nil clean ring with every nilpotent is contained in $J(R)$. Therefore R is weak JN-clean ring.

Proposition 3.3.

Every strongly weak JN-clean element is strongly clean.

Proof:

Let $x \in R$ be strongly weak JN-clean element then, x can be written as $x=b+e$ or $x=b-e$ where $e \in Idem(R)$ and $b \in N(R) \subset J(R)$ and $eb=be$ Hence, $x=(1-e)+(2e-1+b)$, since $(2e-1)^2=1$ then, $(2e-1+b) \in U(R)$ or $x=(b-1)+(1-e)$. Hence, $(b-1) \in U(R)$ and $(1-e) \in Idem(R)$. Therefore, x is strongly clean.

Proposition 3.4.

Let R be nil clean and local ring. Then, R is strongly weak JN-clean.

Proof:

Since every nil clean ring is weak nil clean, that is each element x in R can be written as: $x=b+e$ or $x=b-e$ where, $e \in Idem(R)$ and $b \in N(R)$.

Since R is local ring then, every nilpotent element is contained in $J(R)$ and every idempotent is trivial.

Proposition 3.5.

Every weak JN-clean ring is J-reduced ring.

Proof:

Let $a^n = 0; n \geq 2$ then, there exist $b \in N(R) \subset J(R)$ and $e \in Idem(R)$ such that, $a = b + e$ or $a = b - e$. That is, $a - e = b$ or $a + e = b$. Hence $(1-a) = ((1+e) - b) \in U(R)$ or $(1-a) = ((1+e) - b) \in U(R)$ since, $(1-a^n) = 1 = (1-a)(1 + a + a^2 + \cdots + a^{n-1})$. Thus, $(1-e) \in U(R)$ or $(1+e) \in U(R)$ and that is true if $e = o$ so that $a = b \in J(R)$. Therefore R is J-reduced.

4. Conclusions

From the study on characterization and properties of weak JN-clean rings, we obtain the following results:

1) The ring R is weak JN-clean, if and only if, $R/J(R)$ is weak JN-clean and each idempotent lifts modulo $N(R)$.

2) Let $R_1, R_2, \cdots, R_n$ be rings. Then, $R = \prod_{i=1}^{n} R_i$ is weak JN-clean if and only if each R_i for each i is weak JN-clean and at most one R_i is not nil clean ring.

3) A ring R is strongly weak JN-clean ring if and only if eRe is strongly weak JN-clean ring for all idempotent $e \in R$.

4) Let R be an abelian semipotent ring then, every element in R is weak JN-clean ring.

5) Every strongly weak JN-clean element is strongly clean.

Conflicts of Interest

The authors declare no conflicts of interest.

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