



Continuous Time Dynamical System with Hidden Attractors under Mathematical Control

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Abstract

In this paper, a continuous two-dimensional dynamic system is proposed. This system was analyzed by finding the equilibrium points. Also, the stability of the system was analyzed through the roots of the characteristic equation, Roth stability criteria, Hurwitz stability criteria, fractional part stability criteria, and Lyapunov function. It turns out that the system is chaotic at one point of equilibrium and stable at the other point. Also, it was found that the roots of the characteristic equation of the system were in the form of complex numbers, and the real part was relied upon in the stability analysis. And then the system was controlled using adaptive control technology.

Subject Areas

Mathematics

Keywords

Lyapunov Function, Stability, Hopf Bifurcation, Lyapunov Dimension

1. Introduction

Research on chaotic phenomena has been increasingly important in recent years because of the growing range of chaotic applications in scientific and technical systems [1]. Chaotic phenomena arise from the reactivity of adversaries to changes in the structural parameters and initial conditions of some types of dynamic systems. The aperiodicity, broad spectrum, and random-like properties of chaotic signals are characteristics of these phenomena [2]-[7]. The chaotic orbits must be packed in phase space, it is not a transitional topology, and it is sensitive to perturbations in its initial conditions, all of which should lead to unpredictable behavior over time [8] [9]. Studies claim that some of the produced chaos at-

tractors include Chen's [10], the 4-wing attractor [11], Sundarapandian V. Pehlivan [12], and the Rabinovitch system [13] [14] [15]. The fact that a chaotic system has at least one Lyapunov exponent greater than zero is one of its fundamental properties. A system becomes extremely chaotic and sensitive to even the slightest changes in its dynamics when it has a lot of positive Lyapunov exponents [16] [17]. Researchers are paying more and more attention to chaos management because of its synchronizability and controllability, which suggests that it will be helpful in a range of designs, such as biometric identification, artificial intelligence, and secure communications [18] [19]. Dissipative systems can be settled successfully using one of the Lyapunov stability [20]. A stable system will have consistent and predictable behavior, while an unstable system will have behavior that changes significantly over time [21]. If small perturbations in the initial conditions of the system result in only small changes in the long-term behavior of the variables, then the system is considered stable. Conversely, if small perturbations result in large changes in the long-term behavior of the variables, then the system is considered unstable. There are several methods for analyzing the stability of a two-dimensional continuous-time dynamical system, including linear stability analysis, eigenvalue analysis, and Lyapunov stability analysis. Each of these methods involves analyzing the properties of the system's equations and determining how the variables behave over time in response to different initial conditions [22] [23] [24].

2. System Description

Here are the equations that make up the new two-dimensional system:

$$\begin{aligned}\dot{x} &= bx - sxy \\ \dot{y} &= -dy + esxy\end{aligned}\quad (1)$$

x and y are state variables and b , d , e and s are constants.

Were

$$b = 35.5, d = 4.2, e = 31.3, s = 29.4 \quad (2)$$

3. System Analysis

When Equation (1) is set to zero, just one equilibrium point, the origin point, is produced, allowing us to examine a dynamical system's equilibrium points

$$E_0 = (0, 0), \quad E_1 = (0.041, 1.207).$$

3.1. Stability Analysis

A necessary and sufficient condition for the stability of the system is that the characteristic equation's eigenvalues have negative real components. Following is the Jacobian matrix for the new system (1) up to $E_0 = (0, 0)$:

$$J = \begin{bmatrix} 35.5 & 0 \\ 0 & -4.2 \end{bmatrix}, \quad (3)$$

The characteristic equation is:

$$\lambda^2 - 31.3\lambda - 149.1 = 0, \quad (4)$$

Roots of the characteristic equation:

$$\lambda_1 = 35.5, \lambda_2 = -4.2,$$

Thus, the system is unsteady.

3.2. Routh Stability Criterion

A system meets the Routh requirement for stability (all poles in the half-loop level), if and only if the components in the first column of the Routh row have only positive values for all of their values. The number of sign changes in the first column multiplied by the sum of the non-OLHP columns [25]. Regarding the Routh stability test, see **Table 1**.

$$a_0 = -149.1, a_1 = -31.3, a_2 = 1,$$

The system is unstable because the first column has four negative elements.

3.3. Hurwitz Stability Criteria

Determinants generated from the coefficients of the characteristic equation are used to implement this criterion. System (1) is stable if the tiny minors of its square matrix J are all positive; if not, it is unstable [25].

From Equation (3):

$$\Delta_1 = a_{n-1} = a_1 = -31.3 < 0,$$

$$\Delta_2 = \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} = \begin{vmatrix} a_1 & 0 \\ a_2 & a_0 \end{vmatrix} = \begin{vmatrix} -31.3 & 0 \\ 1 & 149.1 \end{vmatrix} = -4666.83 < 0,$$

System (1) is unstable because some of the values of the determinants are less than zero.

3.4. Lyapunov Function

Where we assume the Lyapunov function is:

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2),$$

$$\dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2} \frac{\partial x_2}{\partial t}, \quad (5)$$

The system is stable, $0 > \dot{V}$ if

We get: (5) in Equation (1) substituting.

Since $\dot{V} > 0$, as a result, new system (1) is unstable.

Table 1. Routh array.

λ^2	1	-149.1
λ^1	-31.3	0
λ^0	-149.1	0

3.5. Continued Fraction Stability Criteria

By creating a continuous fraction from the odd and even parts of the equation, the characteristic equation of a continuous system is subjected to this condition. The distinguishing equation:

$$\lambda^2 - 31.3\lambda - 149.1 = 0,$$

By taking the even terms and then the odd terms, respectively, we have:

$$Q_1(\lambda) = \lambda^2 - 149.1 \quad (6)$$

$$Q_2(\lambda) = -31.3\lambda \quad (7)$$

After dividing the even terms by the odd terms and using algebraic steps, we get the following results:

$$h_1 = -0.031, h_2 = -0.209.$$

Since some values of h are negative, the equation of the system has some positive real roots, so system (1) is chaotic.

3.6. Dissipativity

Suppose that

$$f_1 = \frac{dx}{dt}, f_2 = \frac{dy}{dt}.$$

The obtained vector field,

$$(\dot{x}, \dot{y})^T = (f_1, f_2)^T$$

$$\nabla \cdot (\dot{x}, \dot{y})^T = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = b - 1.207s - d + 0.004es = f.$$

Note that, $f = -b - 1.207s - d + 0.004es = -0.504$, for all values that are positive and greater than zero, the system (1) dissipates

Here is the exponential rate:

$$\frac{dV}{dt} = fV \Rightarrow V(t) = V_0 e^{ft} = V_0 e^{-0.504t}$$

By flowing into $(V_0 e^{-0.504t})$, the volume element (V_0) from the previous equation is condensed at the time (t) .

4. Hopf Bifurcation

One of the types of bifurcation that is recognized in mathematics occurs when a modest modification to one of the initial conditions causes a qualitative change in the behavior of the system at an equilibrium point. We take the Equation (3)

$$\lambda^2 - 31.3\lambda - 149.1 = 0,$$

The roots of Equation (3) are:

$$\lambda_1 = 35.5, \lambda_2 = -4.2,$$

Differentiate the Equation (3) and normalize it to zero to find the critical value.

$$2\lambda - 31.3 = 0,$$

So, the critical values are $\lambda = 15.65$.

Derivative at one of the eigenvalues of the equation = -8835774.285

$$\frac{\text{The critical values}}{\text{Derivative at the eigenvalue of the equation}} = \frac{15.65}{39.7} = 0.394 \neq 0$$

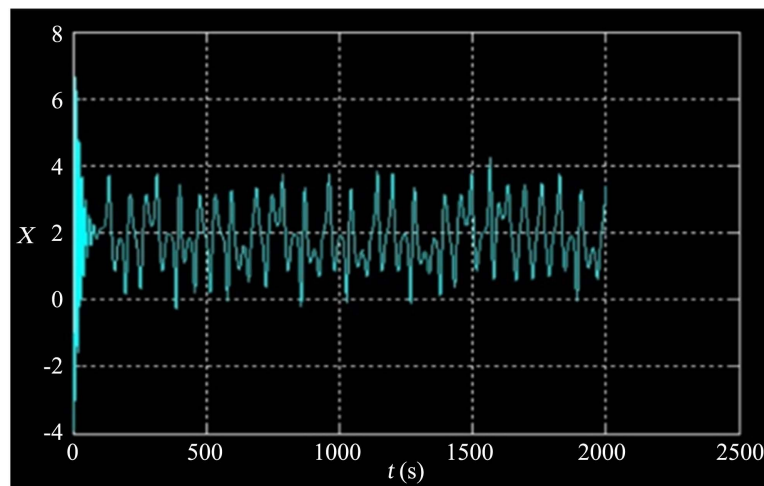
4.1. Numerical and Graphical Analysis

The fourth and fifth order Runge-Kotta method is used to solve the system (1). Initial values included

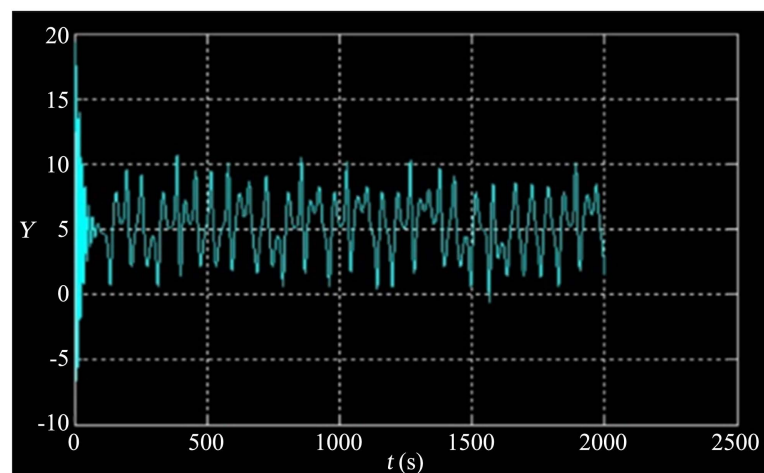
$$x | x_{(0)}, y_{(0)} = [0.5, 1]$$

4.2. Waveform of the New System (1)

The waveform exhibits aperiodic structure, the primary defining feature of chaotic systems. $x_{(t)}$ and $y_{(t)}$ for system (1) (as showed in **Figure 1**).



(a)



(b)

Figure 1. The Waveform of a new system (1). (a): time versus x ; (b): time versus y .

4.3. The System's Phase Portrait (1)

In this paragraph, the strange attractor for the system (1) in (x, y) space is shown along with the chaotic strange attractor for the system (1), shown in **Figure 2**.

Since the orbit in each graph looks to be dense, the new system exhibits a chaotic attractor.

4.4. Lyapunov Exponent and Lyapunov Dimension

The typical exponential growth rates of almost divergent trajectories in phase space are frequently referred to as the Lyapunov exponent. The new system is regarded as chaotic if it has at least one positive Lyapunov exponent. Values of the Lyapunov exponent are:

$$(L_1 = 1.523, L_2 = -2.388).$$

As a result, the system's "Kaplan-Yorke dimension" or Lyapunov dimension is as follows:

$$D_L = 1 + \frac{L_1}{|L_2|} = 1.637$$

Figure 3 shows that system (1) is very Chaotic.

5. Adaptive Controller Technique

5.1. Theoretical Results

To stabilize a chaotic system (1) use the sufficiency control law generalized with an unknown parameter c as follows:

$$\begin{aligned} \dot{x} &= 35.5x - 29.4xy + u_1 \\ \dot{y} &= -4.2y + 29.4exy + u_2 \end{aligned} \quad (8)$$

where $[u_1, u_2]^T$ are feedback controllers.

We now consider the following adaptive control procedures to make sure the managed system (7) converges asymptotically to the origin.

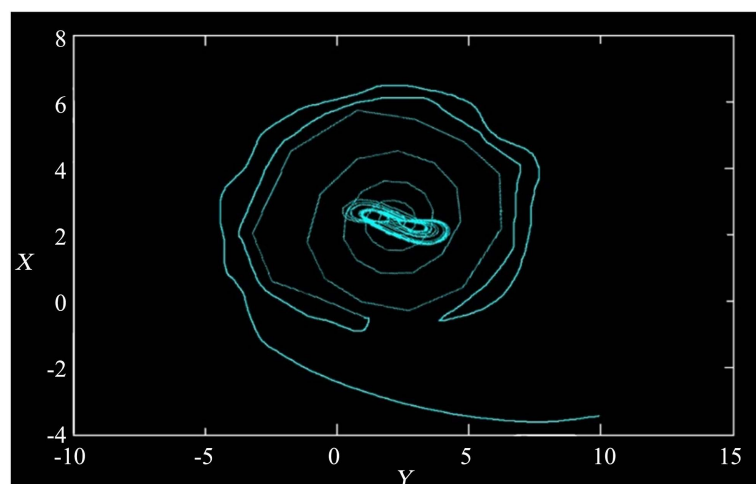


Figure 2. The system attractor in (x, y) .

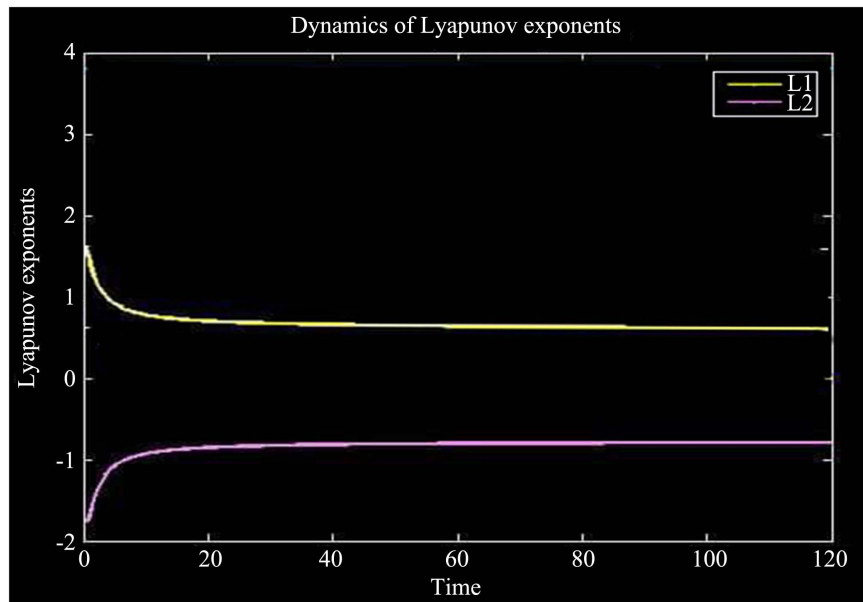


Figure 3. Lyapunov exponent of the system (1).

$$\begin{aligned} u_1 &= -35.5x + 29.4xy - \mu_1x \\ u_2 &= 4.2y - 29.4\hat{c}xy - \mu_2y \end{aligned} \quad (9)$$

where μ_1, μ_2 are constants, \hat{c} is an estimator of the parameter c .

Substituting (8) into (7), we get:

$$\begin{aligned} \dot{x} &= -\mu_1x \\ \dot{y} &= 29.4xy(e - \hat{c}) - \mu_2y \end{aligned} \quad (10)$$

Let the estimation error of the parameter be:

$$e_c = c - \hat{c} \quad (11)$$

Using (10), system (9) can be written as:

$$\begin{aligned} \dot{x} &= -\mu_1x, \\ \dot{y} &= 29.4e_cxy - \mu_2y, \end{aligned} \quad (12)$$

The parameter estimates \hat{c} is changed using the Lyapunov method of obtaining the updated law. It is thought that the quadratic Lyapunov function:

$$\mathcal{V}(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2 + e_c^2), \quad (13)$$

Which definite, positive-in \mathbb{R}^3 .

Also

$$\dot{e}_c = -\dot{\hat{c}} \quad (14)$$

Differentiate V & substituting (11) and (13), we get:

$$\dot{\mathcal{V}} = -\mu_1x^2 - \mu_2y^2 + e_c(29.4xy^2 - \dot{\hat{c}})$$

Assume that:

$$\dot{\hat{c}} = xy + \mu_3e_c \quad (15)$$

where μ_3 is higher than 0 in value.

Substitute (14) into \dot{V} , we get:

$$\dot{V} = -\mu_1 x^2 - \mu_2 y^2 - \mu_3 e_c^2 + 29.4 e_c y^2 x - e_c xy \tag{16}$$

Which is negative-definite on \mathbb{R}^3 .

The outcome is as follows because of Lyapunov stability, Eigenvalues, and the Routh array criteria.

Proposition 1. Byadaptive control (10), where $\dot{c} = xy + \mu_3 e_c$ and μ_1, μ_2, μ_3 are positive constants, The chaotic system (8) is stabilized for $x(0) \in \mathbb{R}^2$.

5.2. Simulation and Numerical Results

The controlled extremely chaotic system (8) was simulated using

$$x | x_{1(0)}, x_{2(0)} = [3, 9]$$

$$\mu_2, \mu_1 = [30, 15] \text{ and } e_c = 25.3.$$

The new system (1)'s-controlled state trajectories are displayed in **Figure 4**.

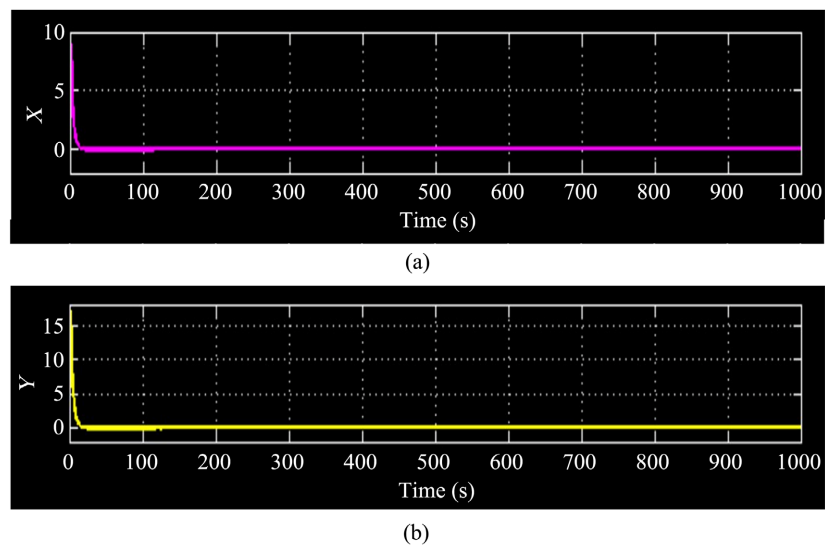


Figure 4. The behavior of state variables x, y for the controlled (8).

6. A Table of Comparisons before and after the Control

More results can be found in **Tables 2-5**. A comparison before and after control of system (1) was done, for eigenvalues given in **Table 2**, Routh array criterion values in **Table 3**, calculated values of Hurwitz stability criteria in **Table 4**, and calculated values of continued fraction in **Table 5**, all shows that system(1) is stable after control.

Table 2. Eigenvalues of a new system (1).

Equilibrium point	Before Control	After Control
(0, 0)	$\lambda_1 = 35.5$	$\lambda_1 = -57$
	$\lambda_2 = -2.617$	$\lambda_2 = -2$

Table 3. Calculated Routh array criterion values for a new system (1).

Equilibrium point	λ	Before Control		After Control	
(0, 0)	λ^2	1	-149.1	1	24
	λ^1	-31.3	0	14	0
	λ^0	-149.1	0	24	0

Table 4. Calculated values of Hurwitz stability criteria of a new system (1).

Equilibrium point	Before Control	After Control
(0, 0)	$\Delta_1 = -31.3$	$\Delta_1 = 14$
	$\Delta_2 = -4666.83$	$\Delta_2 = 366$

Table 5. Calculated values of continued fraction stability criteria of new.

Equilibrium point	Before Control	After Control
(0, 0)	$h_1 = -0.031$	$h_1 = 0.071$
	$h_2 = -0.209$	$h_2 = 0.583$

7. Conclusion

In this study, a two-dimensional model of continuous dynamical systems was taken. The permissible equilibrium points for the analysis of this system were found, and the parameters of stability were evaluated in various ways, which are:

- Roots of the characteristic equation.
- Roth stability criterion.
- The criterion of the stability of Hurwitz.
- Lyapunov function.
- Fractional stability criterion.

The Lyapunov exponentially was examined, and the system was found to be chaotic. The proposed system dissipation detected Hopf bifurcation, and then the system was regulated using an adaptive control approach. Finally, for the system under study, the numerical and morphological results before and after the control were compared.

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Conflicts of Interest

There are no conflicts of interest reported by the authors.

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