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# Considerable Development of the Type Additive-Quadratic $g(\lambda)$-Functional Inequalities with 3$k$-Variable in $(\alpha_1,\alpha_2)$-Homogeneous *F*-Spaces

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## Abstract

In this article, I use the direct method to study two general functional inequalities with multivariables. First, I prove that the $g(\lambda)$-function inequalities (1) and (2) are additive in $(\alpha_1;\alpha_2)$-homogeneous *F*-spaces. After that, I continue to prove that the $g(\lambda)$-function inequality (1) and (2) are quadratic in the $(\alpha_1;\alpha_2)$-homogeneous *F*-space. That is the main result in this paper.

## Subject Areas

Mathematics

## Keywords

Additive $g(\lambda)$-Functional Inequality, $(\alpha_1;\alpha_2)$-Homogeneous F-Space, Additive-Quadratic $g(\lambda)$-Functional Inequality, $(\alpha_1,\alpha_2)$-Homogeneous F-Space

## 1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be a normed spaces on the same field $\mathbb{K}$, and $f:\mathbf{X}\to\mathbf{Y}$. I use the notation $\|\cdot\|$ for all the norm on both $\mathbf{X}$ and $\mathbf{Y}$. In this paper, I investisgate some additive-quadraic $\lambda$-functional inequality in $(\alpha_1;\alpha_2)$-homogeneous *F*-spaces.

In fact, when $\mathbf{X}$ is a $\alpha_1$-homogeneous *F*-spaces and that $\mathbf{Y}$ is a $\alpha_2$-homogeneous *F*-spaces, I solve and prove the Hyers-Ulam-Rassias type stability of two forllowing additive-quadratic $g(\lambda)$-functional inequality.

$$\begin{aligned}&\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\&\le\left\| g(\lambda)\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{1}$$

and when I change the role of the function inequality (1), I continue to prove the following function inequality.

$$\begin{aligned}&\left\| 2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\&\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\&\le\left\| g(\lambda)\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{2}$$

$$\mathbf{H}=\left\{h:\mathbb{C}\setminus\{0\}\to\mathbb{C},h(\lambda)=\lambda\right\} \tag{3}$$

where $g\in\mathbf{H}$.

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y)=f(x)+f(y) \tag{4}$$

is called the Cauchy equation.

In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{5}$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The stability of quadratic functional equation was proved by Skof [6] for mappings $f:E_1\to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group.

Recently, the I has studied the additive function inequalities or quadratic function inequalities of mathematicians around the world see [1]-[24], on spaces as complex Banach spaces, non-Archimedan Banach spaces or homogeneous $F$-space let me give two general additive-quadratic functional inequalities and show their solutions exist on $(\alpha_1, \alpha_2)$-homogeneous $F$-space.

In this article, I successfully built quadratic functional inequalities with the number of variables more than 3 on $F$-homogeneous space and I showed their solutions. This is a great step forward in the field of functional equations. Application to solve problems in many spaces with no limit on the number of variables.

The paper is organized as followns: In section preliminarier I remind a basic property such as I only redefine the solution definition of the equation of the additive function and $F^*$-space.

**Section 3**: is devoted to prove the Hyers-Ulam stability of the addive $g(\lambda)$-functional inequalities (1) when when $\mathbf{X}$ is a $\alpha_1$-homogeneous $F$-spaces and that $\mathbf{Y}$ is a $\alpha_2$-homogeneous $F$-spaces.

**Section 4**: is devoted to prove the Hyers-Ulam stability of the addive $g(\lambda)$-functional inequalities (2) when when $\mathbf{X}$ is a $\alpha_1$-homogeneous $F$-spaces and that $\mathbf{Y}$ is a $\alpha_2$-homogeneous $F$-spaces.

**Section 5**: is devoted to prove the Hyers-Ulam stability of the quadratic $g(\lambda)$-functional inequalities (1) when when $\mathbf{X}$ is a $\alpha_1$-homogeneous $F$-spaces and that $\mathbf{Y}$ is a $\alpha_2$-homogeneous $F$-spaces.

**Section 6**: is devoted to prove the Hyers-Ulam stability of the quadratic $g(\lambda)$-functional inequalities (2) when when $\mathbf{X}$ is a $\alpha_1$-homogeneous $F$-spaces and that $\mathbf{Y}$ is a $\alpha_2$-homogeneous $F$-spaces.

## 2. Preliminaries

### 2.1. *F**-Spaces

Let $\mathbf{X}$ be a (complex) linear space. A nonnegative valued function $\|\cdot\|$ is an $F$-norm if it satisfies the following conditions:

1) $\|x\| = 0$ if and only if $x = 0$;
2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all $\lambda$ with $|\lambda| = 1$;
3) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$;
4) $\|\lambda_n x\| \to 0$, $\lambda_n \to 0$;
5) $\|\lambda_n x\| \to 0$, $x_n \to 0$.

Then $(\mathbf{X}, \|\cdot\|)$ is called an $F^*$-space. An $F$-space is a complete $F^*$-space. An $F$-norm is called $\beta$-homgeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in \mathbf{X}$ and for all $t \in \mathbb{C}$ and $(\mathbf{X}, \|\cdot\|)$ is called $\alpha$-homogeneous $F$-space.

### 2.2. Solutions of the Inequalities

The functional equation The functional equation

$$f(x+y)=f(x)+f(y) \tag{6}$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{7}$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping.

## 3. Hyers-Ulam-Rassias Stability Additive $g(\lambda)$-Functional Inequalities (1) in $\alpha$-Homogeneous *F*-Spaces

Now, I first study the solutions of (1). Note that for these inequalities, when $\mathbf{X}$ is a $\alpha_1$-homogeneous *F*-spaces and that $\mathbf{Y}$ is a $\alpha_2$-homogeneous *F*-spaces. Under this setting, I can show that the mapping satisfying (1) is additive. These results are give in the following.

Where: $\alpha_1,\alpha_1 \in \mathbb{R}^+$ and $\alpha_1,\alpha_1 \le 1$.

**Lemma 1.** Let $f:\mathbf{X}\to\mathbf{Y}$ be an odd mapping satilies

$$\begin{aligned}&\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\&\le\left\|\lambda\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}}\end{aligned} \tag{8}$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for $j=1\to n$, if and only if $f:\mathbf{X}\to\mathbf{Y}$ is additive.

*Proof.* Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (8).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (8), we have

$$\|(4k-2)f(0)\|\le|g(\lambda)|^{\alpha_2}\|2kf(0)\|\le 0$$

therefore

So $f(0)=0$.

Next replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,x,\cdots,x)$ in (8), we have

Thus

$$\|f(2kx)-2kf(x)\|\le 0$$

$$f\left(\frac{x}{2k}\right)=\frac{1}{2k}f(x) \tag{9}$$

for all $x\in\boldsymbol{X}$.

From (8) and (9) I infer that

$$\begin{aligned}&\Bigg\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\Bigg\|_{\mathbf{Y}}\\&\le\Bigg\| g(\lambda)\Bigg(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\&-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\Bigg)\Bigg\|_{\mathbf{Y}}\\&=|g(\lambda)|^{\alpha_2}\Bigg\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\Bigg\|_{\mathbf{Y}}\end{aligned}\tag{10}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j=1\to n$, and so

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)=2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)\tag{11}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j=1\to n$.

Next we replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,z,\cdots,z)$ in (11), we have

$$f(kx+kz)+f(kx-kz)=2kf(x)\tag{12}$$

for all $x, z \in \mathbf{X}$.

Now letting $p=kx+kz, q=kx-kz$ when that in (12), we get

$$f(p)+f(q)=2kf\left(\frac{p+q}{2k}\right)=2k\cdot\frac{1}{2k}f(p+q)=f(p+q)\tag{13}$$

for all $p,q \in \mathbf{X}$. So $f$ is an additive mapping. as we expected. The couverse is obviously true. □

**Corollary 1.** *Let* $f:\mathbf{X}\to\mathbf{Y}$ *be an even mapping satilies*

$$\begin{aligned}&f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\\&=g(\lambda)\Bigg(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\&-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\Bigg)\end{aligned}\tag{14}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j=1\to n$, if and only if $f:\mathbf{X}\to\mathbf{Y}$ is additive.

**Note**! The functional equation (14) is called an additive $\lambda$-functional equation.

**Theorem 2.** Assume for $r>\frac{\alpha_2}{\alpha_1}$, $\theta$ be nonngative real number, and Suppose $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|g(\lambda)\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right)\right\|_{\mathbf{Y}}\\&+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|y_j\right\|^r+\sum_{j=1}^{k}\left\|z_j\right\|^r\right)\end{aligned}\tag{15}$$

for all $x_j,y_j,z_j\in X$ for all $j=1\to n$. Then there exists a unique additive mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|f(x)-\phi(x)\right\|\le\frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}-(2k)^{\alpha_2}}\theta\|x\|^r.\tag{16}$$

for all $x\in\mathbf{X}$.

*Proof.* Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (15).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (15), we have

$$\left\|(4k-2)f(0)\right\|\le\left\|2kg(\lambda)f(0)\right\|$$

therefore

$$\left(\left|4k-2\right|^{\alpha_2}-\left|2kg(\lambda)\right|^{\alpha_2}\right)\left\|f(0)\right\|$$

So $f(0)=0$.

Next replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,x,\cdots,x)$ in (15) we have

$$\left\|f(2kx)-2kf(x)\right\|\le\left(2k^{\alpha_1 r+1}+1\right)\theta\|x\|^r\tag{17}$$

for all $x\in\mathbf{X}$. Thus

$$\left\|f(x)-2kf\left(\frac{x}{2k}\right)\right\|\le\frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}}\theta\|x\|^r\tag{18}$$

for all $x\in\mathbf{X}$.

$$\begin{aligned}&\left\|(2k)^l f\left(\frac{x}{(2k)^l}\right)-(2k)^m f\left(\frac{x}{(2k)^m}\right)\right\|\\&\le\sum_{j=1}^{m-1}\left\|(2k)^j f\left(\frac{x}{(2k)^j}\right)-(2k)^{j+1}f\left(\frac{x}{(2k)^{j+1}}\right)\right\|\\&\le\frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}}\theta\sum_{j=1}^{m-1}\frac{(2k)^{\alpha_2 j}}{(2k)^{\alpha_1 rj}}\|x\|^r\end{aligned}\tag{19}$$

for all nonnegative integers $p, l$ with $p > l$ and all $x \in \mathbf{X}$. It follows from (19) that the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $\phi: \mathbf{X} \to \mathbf{Y}$ by

$$\phi(x) := \lim_{n\to\infty} (2k)^n f\left(\frac{x}{(2k)^n}\right)$$

for all $x \in \mathbf{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (19), we get (16).

Form $f: \mathbf{X} \to \mathbf{Y}$ is even, the mapping $\phi: \mathbf{X} \to \mathbf{Y}$ is even.

It follows from (15) that

$$\begin{aligned}
&\left\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-2\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi(z_j)-\sum_{j=1}^{k}\phi(-z_j)\right\|\\
&=\lim_{n\to\infty}(2k)^{\alpha_2 n}\left\|f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)\right.\\
&+f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)\\
&\left.-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\right\|\\
&\le\lim_{n\to\infty}(2k)^{\alpha_2 n}|g(\lambda)|^{\alpha_2}\left\|2kf\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)\right.\\
&+2kf\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)-3\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)\\
&\left.-\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\right\|\\
&+\lim_{n\to\infty}\frac{(2k)^{\alpha_2 n}}{(2k)^{\alpha_1 nr}}\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)\\
&=|g(\lambda)|^{\alpha_2}\left\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}
\end{aligned}\tag{20}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$.

$$\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.$$
$$\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}$$
$$\le\left\| g(\lambda)\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.$$
$$\left.\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right)\right\|_{\mathbf{Y}}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, So by Lemma 1 it follows that the mapping $\phi: \mathbf{X} \to \mathbf{Y}$ is additive. Now we need to prove uniqueness, Suppose $\phi': \mathbf{X} \to \mathbf{Y}$ is also an additive mapping that satisfies (16). Then we have

$$\begin{aligned}\left\|\phi(x)-\phi'(x)\right\| &= (2k)^{\alpha_2 n}\left\|\phi\left(\frac{x}{(2k)^n}\right)-\phi'\left(\frac{x}{(2k)^n}\right)\right\| \\ &\le (2k)^{\alpha_2 n}\left(\left\|\phi\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|+\left\|\phi'\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|\right) \\ &\le \frac{2\cdot(2k)^{\alpha_2 n}\cdot\left(2k^{\alpha_1 r+1}+1\right)}{(2k)^{\alpha_1 nr}\left((2k)^{\alpha_1 r}-(2k)^{\alpha_2}\right)}\theta\|x\|^r\end{aligned} \tag{21}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $\phi(x) = \phi'(x)$ for all $x \in \mathbf{X}$. This proves thus the mapping $\phi: \mathbf{X} \to \mathbf{Y}$ is a unique mapping satisfying (16) as we expected.

**Theorem 3.** Assume for $r < \frac{\alpha_2}{\alpha_1}$, $\theta$ be nonngative real number, and Suppose $f: \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\begin{aligned}&\left\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right) -2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}} \\ &\le\left\| g(\lambda)\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right. \\ &\left.\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right)\right\|_{\mathbf{Y}} \\ &+\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)\end{aligned} \tag{22}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Then there exists a unique additive mapping $\phi: \mathbf{X} \to \mathbf{Y}$ such that

$$\|f(x)-\phi(x)\| \le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_2}-(2k)^{\alpha_1 r}}\theta\|x\|^r. \quad (23)$$

for all $x \in \mathbf{X}$.

*Proof.* Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (22).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (22), we have

$$\|(4k-2)f(0)\| \le \|2kg(\lambda)f(0)\|$$

therefore

$$\left(|4k-2|^{\alpha_2}-|2kg(\lambda)|^{\alpha_2}\right)\|f(0)\|$$

So $f(0)=0$.

Next replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,x,\cdots,x)$ in (22) we have

$$\|f(2kx)-2kf(x)\| \le \left(2k^{\alpha_1 r+1}+1\right)\theta\|x\|^r \quad (24)$$

for all $x\in\mathbf{X}$. Thus

$$\left\|f(x)-\frac{1}{2k}f(2kx)\right\| \le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_2}}\theta\|x\|^r \quad (25)$$

for all $x\in\mathbf{X}$.

$$\begin{aligned}&\left\|\frac{1}{(2k)^l}f\left((2k)^l x\right)-\frac{1}{(2k)^m}f\left((2k)^m x\right)\right\|\\ &\le \sum_{j=1}^{m-1}\left\|\frac{1}{(2k)^j}f\left((2k)^j x\right)-\frac{1}{(2k)^{j+1}}f\left((2k)^{j+1}x\right)\right\| \\ &\le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_2}}\theta\sum_{j=1}^{m-1}\frac{(2k)^{\alpha_1 rj}}{(2k)^{\alpha_2 j}}\|x\|^r\end{aligned} \quad (26)$$

for all nonnegative integers $p,l$ with $p>l$ and all $x\in\mathbf{X}$. It follows from (26) that the sequence $\left\{\frac{1}{(2k)^n}f\left((2k)^n x\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{\frac{1}{(2k)^n}f\left((2k)^n x\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by

$$\phi(x) := \lim_{n\to\infty}\frac{1}{(2k)^n}f\left((2k)^n x\right)$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (26), we get (23).

The rest of the proof is similar to the proof of Theorem 2. □

## 4. Stability Additive $g(\lambda)$-Functional Inequalities (2) in $(\alpha_1, \alpha_2)$-Homogeneous *F*-Spaces

Now, we study the solutions of (2). Note that for these inequalities, when $\mathbf{X}$ is a $\alpha_1$-homogeneous *F*-spaces and that $\mathbf{Y}$ is a $\alpha_2$-homogeneous *F*-spaces. Under this setting, I can show that the mapping satisfying (2) is additive. These results are give in the following.

**Lemma 4.** Let $f:\mathbf{X}\to\mathbf{Y}$ be an odd mapping satilies

$$\begin{aligned}&\Bigg\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\&-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}\\&\le\Bigg\|g(\lambda)\Bigg(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg)\Bigg\|_{\mathbf{Y}}\end{aligned}\tag{27}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j=1\to n$, if and only if $f:\mathbf{X}\to\mathbf{Y}$ is additive.

*Proof.* Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (27).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (27), we have

$$\|2kf(0)\|\le|g(\lambda)|^{\alpha_2}\|(4k-2)f(0)\|$$

So $f(0)=0$.

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(2kx,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (27), we have

Thus

$$\left\|4kf\left(\frac{x}{2k}\right)-2f(x)\right\|\le 0$$

$$f\left(\frac{x}{2k}\right)=\frac{1}{2k}f(x)\tag{28}$$

for all $x\in\mathbf{X}$.

From (27) and (28) we infer that

$$\begin{aligned}&\Bigg\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}\\&\le\Bigg\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\end{aligned}$$

$$-3\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k} f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k} f\left(z_j\right)-\sum_{j=1}^{k} f\left(-z_j\right)\Bigg\|_{\mathbf{Y}}$$
$$\le\left|g(\lambda)\right|^{\alpha_2}\Bigg\| f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k} z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k} z_j\right)$$
$$-2\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k} f\left(z_j\right)-\sum_{j=1}^{k} f\left(-z_j\right)\Bigg\|_{\mathbf{Y}} \tag{29}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j=1\to n$, and so

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k} z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k} z_j\right)=2\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k}\right)$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j=1\to n$, as we expected. The couverse is obviously true. □

**Corollary 2.** Let $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping satilies

$$2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k} z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k} z_j\right)$$
$$-3\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k} f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k} f\left(z_j\right)-\sum_{j=1}^{k} f\left(-z_j\right)$$
$$=g(\lambda)\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k} z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k} z_j\right)\right.$$
$$\left.-2\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k} f\left(z_j\right)-\sum_{j=1}^{k} f\left(-z_j\right)\right) \tag{30}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j=1\to n$, if and only if $f:\mathbf{X}\to\mathbf{Y}$ is additive.

**Note**! The functional equation (30) is called an additive $\lambda$-functional equation.

**Theorem 5.** Assume for $r>\frac{\alpha_2}{\alpha_1}$, $\theta$ be nonngative real number, and Suppose $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that $f(0)=0$ and

$$\Bigg\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k} z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k} z_j\right)$$
$$-3\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k} f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k} f\left(z_j\right)-\sum_{j=1}^{k} f\left(-z_j\right)\Bigg\|_{\mathbf{Y}}$$
$$\le\Bigg\|\lambda\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k} z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k} z_j\right)\right.$$
$$\left.-2\sum_{j=1}^{k} f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k} f\left(z_j\right)-\sum_{j=1}^{k} f\left(-z_j\right)\right)\Bigg\|_{\mathbf{Y}}$$
$$+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|y_j\right\|^r+\sum_{j=1}^{k}\left\|z_j\right\|^r\right) \tag{31}$$

for all $x_j, y_j, z_j \in X$ for all $j=1\to n$. Then there exists a unique additive mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\| f(x)-\phi(x) \right\| \le \frac{(2k)^{\alpha_1 r}}{(2k)^{\alpha_1 r}-(4k)^{\alpha_2}}\theta\|x\|^r. \quad (32)$$

for all $x \in \mathbf{X}$.

*Proof.* Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (38).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (38), we have

$$\left\|2f(0)\right\| \le |\lambda|^{\alpha_2}\left\|(4k-2)f(0)\right\|$$

therefore

$$\left(|4k-2|^{\alpha_2}-|2\lambda|^{\alpha_2}\right)\left\|f(0)\right\| \le 0$$

So $f(0)=0$.

Replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(2kx,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (38) we have

$$\left\|4f\left(\frac{x}{2k}\right)-\frac{1}{k}f(x)\right\|_{\mathbf{Y}} \le (2k)^{\alpha_1 r}\theta\|x\|^r \quad (33)$$

for all $x \in \mathbf{X}$. Thus

$$\left\|4kf\left(\frac{x}{2k}\right)-f(x)\right\| \le (2k)^{\alpha_1 r}k^{\alpha_2}\theta\|x\|^r \quad (34)$$

for all $x \in \mathbf{X}$.

$$\begin{aligned}&\left\|(4k)^l f\left(\frac{x}{(2k)^l}\right)-(4k)^m f\left(\frac{x}{(2k)^m}\right)\right\| \\ &\le \sum_{j=1}^{m-1}\left\|(4k)^j f\left(\frac{x}{(2k)^j}\right)-(4k)^{j+1} f\left(\frac{x}{(2k)^{j+1}}\right)\right\| \\ &\le (2k)^{\alpha_1 r}k^{\alpha_2}\theta\sum_{j=1}^{m-1}\frac{(4k)^{\alpha_2 j}}{(2k)^{\alpha_1 rj}}\|x\|^r\end{aligned} \quad (35)$$

for all nonnegative integers $p,l$ with $p>l$ and all $x \in \mathbf{X}$. It follows from (35) that the sequence $\left\{(4k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{(4k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by

$$\phi(x) := \lim_{n\to\infty}(4k)^n f\left(\frac{x}{(2k)^n}\right)$$

for all $x \in \mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (35), we get (39). Form $f:\mathbf{X}\to\mathbf{Y}$ is even, the mapping

$$\phi:\mathbf{X}\to\mathbf{Y}$$

is even. It follows from (38) that

$$\begin{aligned}
&\left\|2\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-\frac{3}{2k}\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}\phi\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}\phi(z_j)-\frac{1}{2k}\sum_{j=1}^{k}\phi(-z_j)\right\|\\
&=\lim_{n\to\infty}(4k)^{\alpha_2 n}\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^{n+2}}+\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)\right.\\
&+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^{n+2}}-\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{(2k)^{n+1}}\right)\\
&\left.+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{(2k)^{n+1}}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(\frac{z_j}{(2k)^{n}}\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(\frac{-z_j}{(2k)^{n}}\right)\right\|\\
&\le\lim_{n\to\infty}(4k)^{\alpha_2 n}|g(\lambda)|^{\alpha_2}\left\|2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)\right.\\
&+2f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)-2\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)\\
&\left.-\frac{1}{2k}\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\right\|\\
&+\lim_{n\to\infty}\frac{(4k)^{\alpha_2 n}}{(2k)^{\alpha_1 nr}}\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)\\
&=\left\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)-2\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)\right.\\
&\left.-\sum_{j=1}^{k}\phi(z_j)-\sum_{j=1}^{k}\phi(-z_j)\right\|_{\mathbf{Y}}
\end{aligned}\tag{36}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$.

$$\begin{aligned}
&\left\|2\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-\frac{3}{2k}\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}\phi\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}\phi(z_j)-\frac{1}{2k}\sum_{j=1}^{k}\phi(-z_j)\right\|_{\mathbf{Y}}\\
&\le\left\|g(\lambda)\left(\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\right.\\
&\left.\left.-2\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi(z_j)-\sum_{j=1}^{k}\phi(-z_j)\right)\right\|_{\mathbf{Y}}
\end{aligned}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, So by Lemma 4.1 it follows that the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is additive. Now we need to prove uniqueness, Suppose $\phi' : \mathbf{X} \to \mathbf{Y}$ is also a quadratic mapping that satisfies (39). Then we have

$$\begin{aligned}\|\phi(x)-\phi'(x)\| &= (4k)^{\alpha_2 n}\left\|\phi\left(\frac{x}{(2k)^n}\right)-\phi'\left(\frac{x}{(2k)^n}\right)\right\| \\ &\le (4k)^{\alpha_2 n}\left(\left\|\phi\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|+\left\|\phi'\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|\right) \\ &\le \frac{2\cdot(4k)^{\alpha_2 n}\cdot(2k)^{\alpha_1 r}}{(2k)^{\alpha_1 nr}\left((2k)^{\alpha_1 r}-(4k)^{\alpha_2}\right)}\theta\|x\|^r\end{aligned} \tag{37}$$

which tends to zero as $n\to\infty$ for all $x\in X$. So we can conclude that $\phi(x)=\phi'(x)$ for all $x\in\mathbf{X}$. This proves thus the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is a unique mapping satisfying (39) as we expected. □

**Theorem 6.** Assume for $r<\frac{2\alpha_2}{\alpha_1}$, $\theta$ be nonngative real number, $f(0)=0$ and Suppose $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$\begin{aligned}&\Bigg\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right) \\ &-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}} \\ &\le\Bigg\|g(\lambda)\Bigg(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right) \\ &-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg)\Bigg\|_{\mathbf{Y}} \\ &+\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)\end{aligned} \tag{38}$$

for all $x_j,y_j,z_j\in X$ for all $j=1\to n$. Then there exists a unique addtive mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that

$$\|f(x)-\phi(x)\|\le\frac{(2k)^{\alpha_1 r}}{(4k)^{\alpha_2}-(2k)^{\alpha_1 r}}\theta\|x\|^r. \tag{39}$$

for all $x\in\mathbf{X}$.

The proof is similar to theorem 5.

## 5. Hyers-Ulam-Rassias Stability Quadratic $g(\lambda)$-Functional Inequalities (1) in $(\alpha_1,\alpha_2)$-Homogeneous *F*-Spaces

Now, we first study the solutions of (1). Note that for these inequalities, when $\mathbf{X}$ is a $\alpha_1$-homogeneous *F*-spaces and that $\mathbf{Y}$ is a $\alpha_2$-homogeneous *F*-spaces. Under this setting, we can show that the mapping satisfying (1) is quadratic. These results are give in the following.

**Lemma 7.** Let $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping satilies

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|g\left(\lambda\right)\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{40}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, if and only if $f:\mathbf{X}\to\mathbf{Y}$ is quadratic.

*Proof.* Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (40).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (40), we have

$$\left\|(4k-2)f(0)\right\| \le |\lambda|^{\alpha_2}\left\|2kf(0)\right\| \le 0$$

therefore

So $f(0)=0$.

Next replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,x,\cdots,x)$ in (40), we have

Thus

$$\left\|f(2kx)-2kf(x)\right\|\le 0$$

$$f\left(\frac{x}{2k}\right)=\frac{1}{2k}f(x)\tag{41}$$

for all $x\in\mathbf{X}$.

From (40) and (41) we infer that

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|\lambda\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right)\right\|_{\mathbf{Y}}\\&=|\lambda|^{\alpha_2}\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f(\frac{x_j+y_j}{2k})-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}\end{aligned}\tag{42}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, and so

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)=2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+2\sum_{j=1}^{k}f\left(z_j\right) \tag{43}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j=1\to n$.

As we expected. The couverse is obviously true. □

**Corollary 3.** Let $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping satilies

$$\begin{aligned}&f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\\&=g(\lambda)\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\&\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right)\end{aligned} \tag{44}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j=1\to n$, if and only if $f:\mathbf{X}\to\mathbf{Y}$ is quadratic.

**Note**! The functional equation (44) is called an quadratic $g(\lambda)$-functional equation.

**Theorem 8.** Assume for $r>\frac{2\alpha_2}{\alpha_1}$, $\theta$ be nonngative real number, and Suppose $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping such that

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}\\&\le\left\|g(\lambda)\left(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\right.\\&\left.\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right)\right\|_{\mathbf{Y}}\\&+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|y_j\right\|^r+\sum_{j=1}^{k}\left\|z_j\right\|^r\right)\end{aligned} \tag{45}$$

for all $x_j, y_j, z_j \in X$ for all $j=1\to n$. Then there exists a unique quadratic mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that

$$\left\|f(x)-\phi(x)\right\|\le\frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}-(2k)^{\alpha_2}}\theta\left\|x\right\|^r \tag{46}$$

for all $x\in\mathbf{X}$.

*Proof.* Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (45).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (45), we have

$$\|(4k-2)f(0)\| \le \|2kg(\lambda)f(0)\|$$

therefore

$$\left(|4k-2|^{\alpha_2} - |2kg(\lambda)|^{\alpha_2}\right)\|f(0)\|$$

So $f(0)=0$.

Next replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(kx,\cdots,kx,kx,\cdots,kx,x,\cdots,x)$ in (45) we have

$$\|f(2kx)-2kf(x)\| \le \left(2k^{\alpha_1 r+1}+1\right)\theta\|x\|^r \tag{47}$$

for all $x\in\mathbf{X}$. Thus

$$\left\|f(x)-2kf\left(\frac{x}{2k}\right)\right\| \le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}}\theta\|x\|^r \tag{48}$$

for all $x\in\mathbf{X}$.

$$\begin{aligned}&\left\|(2k)^l f\left(\frac{x}{(2k)^l}\right)-(2k)^m f\left(\frac{x}{(2k)^m}\right)\right\|\\ &\le \sum_{j=1}^{m-1}\left\|(2k)^j f\left(\frac{x}{(2k)^j}\right)-(2k)^{j+1} f\left(\frac{x}{(2k)^{j+1}}\right)\right\|\\ &\le \frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_1 r}}\theta\sum_{j=1}^{m-1}\frac{(2k)^{\alpha_2 j}}{(2k)^{\alpha_1 rj}}\|x\|^r\end{aligned} \tag{49}$$

for all nonnegative integers $p,l$ with $p>l$ and all $x\in\mathbf{X}$. It follows from (49) that the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{(2k)^n f\left(\frac{x}{(2k)^n}\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by

$$\phi(x) := \lim_{n\to\infty}(2k)^n f\left(\frac{x}{(2k)^n}\right)$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (49), we get (46).

Form $f:\mathbf{X}\to\mathbf{Y}$ is even, the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ is even.

It follows from (45) that

$$\begin{aligned}&\left\|\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+\phi\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\ &\left.-2\sum_{j=1}^{k}\phi\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}\phi(z_j)-\sum_{j=1}^{k}\phi(-z_j)\right\|\\ &=\lim_{n\to\infty}(2k)^{\alpha_2 n}\left\|f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)\right.\end{aligned}$$

$$+f\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\frac{1}{(2k)^n}\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)$$

$$-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\Bigg\|$$

$$\le\lim_{n\to\infty}(2k)^{\alpha_2 n}|g(\lambda)|^{\alpha_2}\Bigg\|2kf\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)$$

$$+2kf\left(\frac{1}{(2k)^n}\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{(2k)^{n+1}}\sum_{j=1}^{k}z_j\right)-3\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)$$

$$-\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(\frac{1}{(2k)^n}z_j\right)-\sum_{j=1}^{k}f\left(-\frac{1}{(2k)^n}z_j\right)\Bigg\|$$

$$+\lim_{n\to\infty}\frac{(2k)^{\alpha_2 n}}{(2k)^{\alpha_1 nr}}\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right)$$

$$=|g(\lambda)|^{\alpha_2}\Bigg\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right) \tag{50}$$

$$-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$.

$$\Bigg\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)$$

$$-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}$$

$$\le\Bigg\|g(\lambda)\Bigg(2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)$$

$$-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg)\Bigg\|_{\mathbf{Y}}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, So by Lemma 7 it follows that the mapping $\phi: \mathbf{X} \to \mathbf{Y}$ is quadratc. Now we need to prove uniqueness, Suppose $\phi': \mathbf{X} \to \mathbf{Y}$ is also an additive mapping that satisfies (46). Then we have

$$\|\phi(x)-\phi'(x)\|=(2k)^{\alpha_2 n}\left\|\phi\left(\frac{x}{(2k)^n}\right)-\phi'\left(\frac{x}{(2k)^n}\right)\right\|$$

$$\le(2k)^{\alpha_2 n}\left(\left\|\phi\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|+\left\|\phi'\left(\frac{x}{(2k)^n}\right)-f\left(\frac{x}{(2k)^n}\right)\right\|\right) \tag{51}$$

$$\le\frac{2\cdot(2k)^{\alpha_2 n}\cdot\left(2k^{\alpha_1 r+1}+1\right)}{(2k)^{\alpha_1 nr}\left((2k)^{\alpha_1 r}-(2k)^{\alpha_2}\right)}\theta\|x\|^r$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $\phi(x) = \phi'(x)$ for all $x \in \mathbf{X}$. This proves thus the mapping $\phi : \mathbf{X} \to \mathbf{Y}$ is a unique mapping satisfying (46) as we expected.

**Theorem 9.** Assume for $r < \frac{\alpha_2}{\alpha_1}$, $\theta$ be nonngative real number, and Suppose $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that

$$\begin{aligned} &\left\| f\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} + \sum_{j=1}^{k} z_j\right) + f\left(\sum_{j=1}^{k} \frac{x_j + y_j}{2k} - \sum_{j=1}^{k} z_j\right) \right. \\ &\left. -2\sum_{j=1}^{k} f\left(\frac{x_j + y_j}{2k}\right) - \sum_{j=1}^{k} f(z_j) - \sum_{j=1}^{k} f(-z_j) \right\|_{\mathbf{Y}} \\ &\le \left\| g(\lambda)\left( 2kf\left(\sum_{j=1}^{k} \frac{x_j + y_j}{(2k)^2} + \frac{1}{2k}\sum_{j=1}^{k} z_j\right) + 2kf\left(\sum_{j=1}^{k} \frac{x_j + y_j}{(2k)^2} - \frac{1}{2k}\sum_{j=1}^{k} z_j\right) \right. \right. \\ &\left.\left. -3\sum_{j=1}^{k} f\left(\frac{x_j + y_j}{2k}\right) - \sum_{j=1}^{k} f\left(-\frac{x_j + y_j}{2k}\right) - \sum_{j=1}^{k} f(z_j) - \sum_{j=1}^{k} f(-z_j) \right) \right\|_{\mathbf{Y}} \\ &+ \theta\left( \sum_{j=1}^{k} \|x_j\|^r + \sum_{j=1}^{k} \|y_j\|^r + \sum_{j=1}^{k} \|z_j\|^r \right) \end{aligned} \tag{52}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Then there exists a unique quadratic mapping $\phi : \mathbf{X} \to \mathbf{Y}$ such that

$$\|f(x) - \phi(x)\| \le \frac{2k^{\alpha_1 r+1} + 1}{(2k)^{\alpha_2} - (2k)^{\alpha_1 r}} \theta \|x\|^r \tag{53}$$

for all $x \in \mathbf{X}$.

*Proof.* Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (52).

We replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)$ in (52), we have

$$\|(4k-2) f(0)\| \le \|2kg(\lambda) f(0)\|$$

therefore

$$\left(|4k-2|^{\alpha_2} - |2kg(\lambda)|^{\alpha_2}\right)\|f(0)\|$$

So $f(0) = 0$.

Next replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(kx, \cdots, kx, kx, \cdots, kx, x, \cdots, x)$ in (52) we have

$$\|f(2kx) - 2kf(x)\| \le \left(2k^{\alpha_1 r+1} + 1\right)\theta\|x\|^r \tag{54}$$

for all $x \in \mathbf{X}$. Thus

$$\left\| f(x) - \frac{1}{2k} f(2kx) \right\| \le \frac{2k^{\alpha_1 r+1} + 1}{(2k)^{\alpha_2}} \theta \|x\|^r \tag{55}$$

for all $x \in \mathbf{X}$.

$$\begin{aligned}&\left\|\frac{1}{(2k)^{l}}f\left((2k)^{l}x\right)-\frac{1}{(2k)^{m}}f\left((2k)^{m}x\right)\right\|\\&\le\sum_{j=1}^{m-1}\left\|\frac{1}{(2k)^{j}}f\left((2k)^{j}x\right)-\frac{1}{(2k)^{j+1}}f\left((2k)^{j+1}x\right)\right\|\\&\le\frac{2k^{\alpha_1 r+1}+1}{(2k)^{\alpha_2}}\theta\sum_{j=1}^{m-1}\frac{(2k)^{\alpha_1 rj}}{(2k)^{\alpha_2 j}}\|x\|^{r}\end{aligned}\tag{56}$$

for all nonnegative integers $p,l$ with $p>l$ and all $x\in\mathbf{X}$. It follows from (56) that the sequence $\left\{\frac{1}{(2k)^{n}}f\left((2k)^{n}x\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{\frac{1}{(2k)^{n}}f\left((2k)^{n}x\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by

$$\phi(x):=\lim_{n\to\infty}\frac{1}{(2k)^{n}}f\left((2k)^{n}x\right)$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (56), we get (53).

The rest of the proof is similar to the proof of Theorem 5. □

## 6. Stability Quadratic $\lambda$-Functional Inequalities (2) in $(\alpha_1,\alpha_2)$-Homogeneous *F*-Spaces

Now, we study the solutions of (2). Note that for these inequalities, when $\mathbf{X}$ is a $\alpha_1$-homogeneous *F*-spaces and that $\mathbf{Y}$ is a $\alpha_2$-homogeneous *F*-spaces. Under this setting, we can show that the mapping satisfying (2) is quadratic. These results are give in the following.

**Lemma 10.** Let $f:\mathbf{X}\to\mathbf{Y}$ be an even mapping satilies

$$\begin{aligned}&\Bigg\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\&-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg\|_{\mathbf{Y}}\\&\le\Bigg\|g(\lambda)\Bigg(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\\&-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\Bigg)\Bigg\|_{\mathbf{Y}}\end{aligned}\tag{57}$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for $j=1\to n$, if and only if $f:\mathbf{X}\to\mathbf{Y}$ is quadratic.

*Proof.* Assume that $f:\mathbf{X}\to\mathbf{Y}$ satisfies (57).

We replacing $(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k)$ by $(0,\cdots,0,0,\cdots,0,0,\cdots,0)$ in (57), we have

$$\left\|2kf\left(0\right)\right\|\leq\left|g\left(\lambda\right)\right|^{\alpha_2}\left\|\left(4k-2\right)f\left(0\right)\right\|$$

So $f\left(0\right)=0$.

Replacing $\left(x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,\cdots,z_k\right)$ by $\left(2kx,\cdots,0,0,\cdots,0,0,\cdots,0\right)$ in (57), we have

Thus

$$\left\|4kf\left(\frac{x}{2k}\right)-2f\left(x\right)\right\|\leq 0$$

$$f\left(\frac{x}{2k}\right)=\frac{1}{2k}f\left(x\right) \tag{58}$$

for all $x\in\mathbf{X}$.

From (57) and (58) we infer that

$$\begin{aligned}
&\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}\\
&=\left\|2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{\left(2k\right)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{\left(2k\right)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}\\
&\leq\left|g\left(\lambda\right)\right|^{\alpha_2}\left\|f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right\|_{\mathbf{Y}}
\end{aligned} \tag{59}$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for $j=1\rightarrow n$, and so

$$f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)=2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+2\sum_{j=1}^{k}f\left(z_j\right)$$

for all $x_j,y_j,z_j\in\mathbf{X}$ for $j=1\rightarrow n$, as we expected. The couverse is obviously true. □

**Corollary 4.** Let $f:\mathbf{X}\rightarrow\mathbf{Y}$ be an even mapping satilies

$$\begin{aligned}
&2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{\left(2k\right)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{\left(2k\right)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\\
&-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\\
&=g\left(\lambda\right)\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\\
&\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(z_j\right)-\sum_{j=1}^{k}f\left(-z_j\right)\right)
\end{aligned} \tag{60}$$

for all $x_j, y_j, z_j \in \mathbf{X}$ for $j = 1 \to n$, if and only if $f : \mathbf{X} \to \mathbf{Y}$ is quadratic.

**Note**! The functional equation (60) is called a quadratic $g(\lambda)$-functional equation.

**Theorem 11.** Assume for $r > \frac{2\alpha_2}{\alpha_1}$, $\theta$ be nonngative real number, and Suppose $f : \mathbf{X} \to \mathbf{Y}$ be a even mapping such that $f(0) = 0$ and

$$\begin{aligned} &\left\| 2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2kf\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z_j\right)\right.\\ &\left.-3\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}}\\ &\le \left\| g(\lambda)\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\right.\\ &\left.\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}}\\ &+\theta\left(\sum_{j=1}^{k}\|x_j\|^r+\sum_{j=1}^{k}\|y_j\|^r+\sum_{j=1}^{k}\|z_j\|^r\right) \end{aligned} \tag{61}$$

for all $x_j, y_j, z_j \in X$ for all $j = 1 \to n$. Then there exists a unique quadratic mapping $\phi : \mathbf{X} \to \mathbf{Y}$ such that

$$\|f(x)-\phi(x)\| \le \frac{(2k)^{\alpha_1 r}}{(2k)^{\alpha_1 r}-(4k)^{\alpha_2}}\theta\|x\|^r \tag{62}$$

for all $x \in \mathbf{X}$.

*Proof.* Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies (61).

We replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)$ in (61), we have

$$\|2f(0)\| \le |g(\lambda)|^{\alpha_2}\|(4k-2)f(0)\|$$

therefore

$$\left(|4k-2|^{\alpha_2}-|2g(\lambda)|^{\alpha_2}\right)\|f(0)\| \le 0$$

So $f(0) = 0$.

Replacing $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k)$ by $(2kx, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)$ in (61) we have

$$\left\|4f\left(\frac{x}{2k}\right)-\frac{1}{k}f(x)\right\|_{\mathbf{Y}} \le (2k)^{\alpha_1 r}\theta\|x\|^r \tag{63}$$

for all $x \in \mathbf{X}$. Thus

$$\left\|4kf\left(\frac{x}{2k}\right)-f(x)\right\| \le (2k)^{\alpha_1 r}k^{\alpha_2}\theta\|x\|^r \tag{64}$$

for all $x \in \mathbf{X}$.

$$
\begin{aligned}
&\left\|(4k)^{l} f\left(\frac{x}{(2k)^{l}}\right)-(4k)^{m} f\left(\frac{x}{(2k)^{m}}\right)\right\| \\
&\leq \sum_{j=1}^{m-1}\left\|(4k)^{j} f\left(\frac{x}{(2k)^{j}}\right)-(4k)^{j+1} f\left(\frac{x}{(2k)^{j+1}}\right)\right\| \\
&\leq (2k)^{\alpha_1 r} k^{\alpha_2}\theta \sum_{j=1}^{m-1}\frac{(4k)^{\alpha_2 j}}{(2k)^{\alpha_1 rj}}\|x\|^{r}
\end{aligned} \tag{65}
$$

for all nonnegative integers $p,l$ with $p>l$ and all $x\in\mathbf{X}$. It follows from (65) that the sequence $\left\{(4k)^{n} f\left(\frac{x}{(2k)^{n}}\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is complete, the sequence $\left\{(4k)^{n} f\left(\frac{x}{(2k)^{n}}\right)\right\}$ coverges.

So one can define the mapping $\phi:\mathbf{X}\to\mathbf{Y}$ by

$$
\phi(x):=\lim_{n\to\infty}(4k)^{n} f\left(\frac{x}{(2k)^{n}}\right)
$$

for all $x\in\mathbf{X}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (65), we get (62). The rest of the proof is similar to the proof of Theorem 8. □

**Theorem 12.** Assume for $r<\frac{2\alpha_2}{\alpha_1}$, $\theta$ be nonngative real number, $f(0)=0$ and Suppose $f:\mathbf{X}\to\mathbf{Y}$ be a mapping such that

$$
\begin{aligned}
&\left\|2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}+\frac{1}{2k}\sum_{j=1}^{k}z_j\right)+2f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{(2k)^2}-\frac{1}{2k}\sum_{j=1}^{k}z\right)_j\right. \\
&\left.-\frac{3}{2k}\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)+\frac{1}{2k}\sum_{j=1}^{k}f\left(-\frac{x_j+y_j}{2k}\right)-\frac{1}{2k}\sum_{j=1}^{k}f(z_j)-\frac{1}{2k}\sum_{j=1}^{k}f(-z_j)\right\|_{\mathbf{Y}} \\
&\leq\left\|g(\lambda)\left(f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}+\sum_{j=1}^{k}z_j\right)+f\left(\sum_{j=1}^{k}\frac{x_j+y_j}{2k}-\sum_{j=1}^{k}z_j\right)\right.\right. \\
&\left.\left.-2\sum_{j=1}^{k}f\left(\frac{x_j+y_j}{2k}\right)-\sum_{j=1}^{k}f(z_j)-\sum_{j=1}^{k}f(-z_j)\right)\right\|_{\mathbf{Y}} \\
&+\theta\left(\sum_{j=1}^{k}\|x_j\|^{r}+\sum_{j=1}^{k}\|y_j\|^{r}+\sum_{j=1}^{k}\|z_j\|^{r}\right)
\end{aligned} \tag{66}
$$

for all $x_j,y_j,z_j\in X$ for all $j=1\to n$. Then there exists a unique quadratic mapping $\phi:\mathbf{X}\to\mathbf{Y}$ such that

$$
\|f(x)-\phi(x)\|\leq\frac{(2k)^{\alpha_1 r}}{(4k)^{\alpha_2}-(2k)^{\alpha_1 r}}\theta\|x\|^{r}. \tag{67}
$$

for all $x\in\mathbf{X}$.

The proof is similar to theorem 8 and 9.

## 7. Conclusion

In this article, I construct two general functional inequalities with multivariables on homogeneous space and show that their solutions are additive-quadratic maps.

## Conflicts of Interest

The author declares no conflicts of interest.

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