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Pricing Bermudan Option with Variable Transaction Costs under the Information-Based Model

Matabel Odin[1]*, Jane Akinyi Aduda[2], Cyprian Ondieki Omari[3]

[1]Department of Mathematics, Pan African University Institute for Basic Sciences, Technology and Innovation (PAUSTI), Nairobi, Kenya
[2]Department of Statistics and Actuarial Sciences, Jomo Kenyatta University of Agriculture and Technology (JKUAT), Nairobi, Kenya
[3]Department of Statistics and Actuarial Sciences, Dedan Kimathi University of Technology (DeKUT), Nyeri, Kenya
Email: *matabel.odiaga@jkuat.ac.ke

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Abstract

The Bermudan option pricing problem with variable transaction costs is considered for a risky asset whose price process is derived under the information-based model. The price is formulated as the value function of an optimal stopping problem, which is the value function of a stochastic control problem given by a non-linear second order partial differential equation. The theory of viscosity solutions is applied to solve the stochastic control problem such that the value function is also the solution of the corresponding Bellman equation. Under some regularity assumptions, the existence and uniqueness of the solution of the pricing equation are derived by the application of the Perron method and Banach Fixed Point theorem.

Keywords

Bermudan Option, Information-Based Model, Variable Costs, Bellman Equation, Viscosity Solutions

1. Introduction

The problem of pricing and hedging in the context of arbitrage-free option pricing in complete markets is well developed in the literature. However, the possibility of a perfect hedge is restricted to the complete market, to certain models and restrictive assumptions. The seminal work of [1] [2], provided the first theoretical framework for pricing options in a market with no imperfections like transaction costs, and under strict assumptions such as constant volatility, and normality of asset returns. Over the past five decades, several attempts have been made to modify the Black-Scholes-Merton (BSM) framework to ensure that theoretical option pricing models give prices that are close to true market positions. Several extensions have been developed to incorporate jump processes, dividend payments on the underlying, transaction costs of trading, stochastic volatility and arbitrage possibilities.

[3] proposed a new framework for pricing options based on the emergence of information also referred to as the "Information-based model" (IBM) or the "Brody-Hughston-Macrina" (BHM) framework or more generally "X-factor Theory". The primary idea under the IBM is that market filtration is modelled such that it is generated by a set of processes that carry information about the future cash flows generated by financial securities. Such cash flows can be seen as random variables or represented as a function of one or more random variables referred to as "market factors" or "X-factors". These X factors are linked to the information processes that generate market filtration so that the value of the X-factor is known at a given fixed time by the associated information process. The conventional examples of information processes studied in literature are based on the Brownian, Gamma, Variance Gamma and Lévy Random Bridges.

Unlike standard option pricing models where the market filtration is chosen implicitly by specifying the law of the price process, the IBM constructs the market filtration explicitly thereby allowing price movements to show more structure. The approach also incorporates both deterministic and stochastic volatility when valuing contracts. The IBM has been majorly applied to value European-style contracts such as credit risky bonds under the assumption of market completeness with no transaction costs, see [3] [4] [5]. However, some extensions, for example in [6] [7] examine information inefficiency under the IBM that reflects an incomplete market setting. No attempt has been made so far to account for transaction costs of trading under the IBM. [4] also suggested that the information-based framework could be used to value contracts with pre-maturity default times. Therefore, the approach can be applied to price Bermudan-style options which are exercisable at pre-determined times before the date of expiration.

The valuation of the Bermudan-style option is generally a discrete-time optimal stopping problem, which in most cases, cannot be solved explicitly. Many different approaches for numerically solving optimal stopping problems, and in particular Bermudan pricing problems have been studied in the literature, for example in [8] [9] [10]. In discrete time and for a finite time horizon, the problem can easily be solved using dynamic programming such that the optimal stopping problem is transformed into a stochastic optimal control problem represented by the Bellman equation. This means that the solution of the Bellman equation is the value function representing the price of the Bermudan option. In the literature, an important approach for approximating the solution of the Bellman equation is by applying the theory of viscosity solutions introduced by [11]. See [12] [13] [14] [15] [16], for different forms of the Bellman equation solved using the notion of viscosity solutions. The common practice entails showing that the Bellman equation exists and proving that it admits a unique solution given some general conditions.

This article extends the IBM by incorporating transaction costs of trading in valuing Bermudan options which correspond to an incomplete market setting that replicates the real market position. We derive the Bellman equation arising in pricing Bermudan options with variable costs for assets driven by Lévy Random Bridge market information process. This is the first attempt to incorporate transaction costs in deriving the Bermudan pricing equation and its associated Bellman equation under the information-based framework. Further, we prove under general conditions on the value function, the existence and uniqueness of viscosity solutions of the derived Bellman equation.

The rest of the paper is organized as follows. Section 2 presents the modelling framework under the Information-based model. Section 3 involves the derivation of the Bermudan option pricing equation under variable transaction costs for an asset process evolving according to IBM. Section 4 presents the Bellman equation arising in the stochastic optimal control problem. Finally, Section 5 presents mathematical results of the existence and uniqueness of the solution to the Bellman equation.

2. Modelling Framework under the Information-Based Model

Consider a financial market consisting of three assets: a risk-free asset *B*, a risky asset *S* and an option *V*. Given the time horizon $[0,T], T \in \mathbb{N}$ and the probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ such that Ω is a non-empty set representing all possible future states of the financial market, $\mathcal{A}$ is a σ-algebra representing all outcomes of a random process, and $\mathbb{Q}$ is the risk-neutral probability measure such that $\mathbb{Q}: \mathcal{A} \to [0,1]$.

For every $t \in [0,T]$, let $\mathcal{F}_t$ be a sub-sigma algebra of $\mathcal{A}$ such that $\mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \mathcal{A}$ is a filtration satisfying the condition of right continuity. The risky asset $S \in \mathcal{F}_t$ is assumed to generate cashflows X_t such that the sequence $S_t = \{X_1, X_2, \cdots, X_T\}$ of random variables can be modeled as measurable mappings $S_t: \Omega \to \mathbb{R}$. X_t is assumed to be integrable and has a priori continuous distribution ν.

2.1. Asset Price Dynamics under the Information-Based Model

IBM assumes the existence of a market information process ξ_t such that

$$\mathcal{F}_t = \sigma(\xi_t) \tag{1}$$

where ξ_t reveals all information about the price of S_t to market participants. The class of processes termed the Lévy Random Bridges (LRBs) is used as the market information process. LRBs are a generalization of the Brownian and Gamma bridges which are Lévy processes conditioned to have a specified terminal marginal density. The generalization of the gamma bridge process is useful in modelling an accumulation process. In contrast, the generalization of the Brownian bridge process which is adopted in this study, allows one to model the emergence of both actual and noisy information. According to [4], LRBs are a good choice because of their Markov property, dynamic consistency and ability to incorporate any form of distribution for the terminal value.

In the original IBM, the LRB market information process generalising the Brownian bridge is given by

$$\xi_t = \lambda t X_T + \beta_t,\ \ \xi_T = X_T \tag{2}$$

where $\lambda t X_T$ denotes the actual information relating to the value of X_T with λ denoting the constant rate of information flow to market participants, and $\beta_t \sim N\left(0, \frac{t(T-t)}{T}\right)$ representing the Brownian bridge process that consists of market noise or rumours relating to the value of X_T. The Brownian bridge process β_t is assumed to be independent of X_T, and therefore represents pure noise.

According to [17], market information includes but is not limited to the number of news releases about the volume of the risky asset traded, investor opinion and volatility of the price. Such information is expected to grow with time reducing uncertainty about the value of the terminal cash flow. Therefore, as opposed to a constant rate of information flow, this study assumes that information arrives steadily to market participants and any inconsistencies or market noise is modelled by the Brownian bridge process. The information flow rate increases linearly leading to greater information about the value of X_T such that λ in the LRB market information process is replaced by $\lambda_i = \frac{t}{T}$, for $i = 0, \Delta t, 2\Delta t, \cdots, T$. This means that at time 0, market participants do not have any access to information about the value of terminal cash flow X_T, and the rate of information flow grows linearly up to time T when the value X_T is known by all market participants. The information process is given as

$$\xi_t = \lambda_i t X_T + \beta_t,\ \ \xi_T = X_T \tag{3}$$

The risk-neutral price of S_t is defined by

$$S_t = \mathrm{e}^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[X_T \mid \xi_t\right] \tag{4}$$

where r is the risk-free rate of interest. Expanding Equation (4) yields

$$S_t = \mathrm{e}^{-r(T-t)} \frac{\int_0^\infty x \exp\left(\frac{T}{T-t}\left(\lambda_i x \xi_t - \frac{1}{2}\lambda_i^2 x^2 t\right)\right)\nu \mathrm{d}x}{\int_0^\infty \exp\left(\frac{T}{T-t}\left(\lambda_i x \xi_t - \frac{1}{2}\lambda_i^2 x^2 t\right)\right)\nu \mathrm{d}x} \tag{5}$$

The stochastic differential Equation (SDE) for S_t is given as follows as shown in [8].

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t \mathrm{d}W_t,\ \ S_0 = s \tag{6}$$

where μ is the mean return on the risky asset, W_t is an $\mathcal{F}_t$-Brownian motion defined as

$$W_t = \xi_t - \int_0^t \frac{1}{T-s}\left(\lambda_i t X_T - \xi_s\right)\mathrm{d}s \tag{7}$$

and σ_t is the volatility process defined as

$$\sigma_t = \mathrm{e}^{-r(T-t)}\frac{\lambda_i T}{T-t} Var\left(X_T \mid \xi_t\right) \tag{8}$$

2.2. Bermudan Option Pricing under the Information Based Model

Consider a Bermudan call option with maturity at time *T*. The option can be exercised $P \geq 1$ discrete times that are equally spaced. This means that $t_1 = \Delta t$, $t_2 = 2\Delta t, \cdots, t_P = P\Delta t$, with $\Delta t = \frac{T}{P}$ and $t_1 < t_2 < \cdots < t_P$. For a strike price *K*, and $t_p \in [0,T]; p = 1,2,\cdots,P$, the payoff process is given as

$$\Phi\left(S_{t_p}\right) = \max\left(S_{t_p} - K, 0\right) \tag{9}$$

where S_{t_p} evolves according to the process in Equation (6). The valuation of Bermudan options is an optimal stopping problem which involves finding the optimal stopping time for exercising the option. Let the stopping time τ be defined as $\left\{\tau = t_p\right\} \in \xi_{t_p}$. The risk-neutral time t_p-price of the Bermudan call option is defined by the value function

$$V\left(S, t_p\right) = \mathrm{e}^{-r\left(\tau - t_p\right)}\mathbb{E}^{\mathbb{Q}}\left[\Phi\left(S_\tau\right) \mid \xi_{t_p}\right]$$

The optimal stopping problem can be formulated as follows

$$\max_{\tau \in [0,T]} V\left(S, t_p\right) \tag{10}$$

The value of τ that realizes the maximum is the optimal stopping time defined by

$$\hat{\tau} = \left\{\min \tau \geq 0; V\left(S,\tau\right) = \Phi\left(S_\tau\right)\right\} \tag{11}$$

The optimal stopping problem in Equation (10) is transformed to a stochastic control problem such that $V(S,\tau)$ satisfies the discrete Bellman equation given by

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{L}V\left(S,\tau\right) = 0 \\ V\left(S,\tau\right) = \Phi\left(S_\tau\right) \end{cases} \tag{12}$$

where $\mathcal{L}$ is the partial differential operator of the IBM partial differential Equation (PDE) with variable transaction costs. This is derived in Section 3 via a hedging argument.

3. Pricing Equation under Variable Costs

The assumption of no-arbitrage still holds in the presence of variable transaction costs such that any portfolio constructed based on the three assets: *S*, *B* and *V*, is riskless and grows according to the risk-free rate of interest. The portfolio must also be self-financing. Consider a self-financing portfolio Π consisting of one Bermudan call option in a long position at the price V and Π_S shares of the underlying asset with variable transaction costs of trading denoted by C. It follows that the value of the portfolio at time t is $\Pi = V + \Pi_S S - C$. The self-financing assumption implies that

$$\mathrm{d}\Pi = \mathrm{d}V + \Pi_s \mathrm{d}S - \Delta C \tag{13}$$

In Equation (13), the risky asset satisfies the IBM with the dynamics $\mathrm{d}S = \mu S \mathrm{d}t + \sigma_t \mathrm{d}W$ and the option V, by the Itô's lemma, evolves according to the process $\mathrm{d}V = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 V}{\partial S^2 \sigma_t^2} \right) \mathrm{d}t + \frac{\partial V}{\partial S} \sigma_t \mathrm{d}W$. What remains is to find ΔC. The notations $S = S_t$, $W = W_t$, $V(S_t, t) = V$ and $\Pi(t) = \Pi$ are used for convenience.

The derivation of ΔC follows the ideas of [18] [19] but with slight modifications. In [18], the cost of transaction is assumed to be a fixed percentage of the trading volume such that $C = \frac{S_{ask} - S_{bid}}{S}$ where $S = \frac{S_{ask} + S_{bid}}{2}$ and $S_{ask} \geq S_{bid}$. The value of the portfolio changes over the time interval $[t, t + \mathrm{d}t]$ by either selling $\Delta\Pi_s < 0$ or purchasing $\Delta\Pi_s > 0$ shares of the underlying. This means that trading in $|\Delta\Pi_s|$ shares of the underlying yields an additional transaction cost ΔC for the option holder given by

$$\Delta C = \frac{S}{2} C |\Delta\Pi_s| \tag{14}$$

where ΔC is approximated by its expected value that is, $\Delta C \approx E[\Delta C]$.

In the modification of [19], the cost of transaction is assumed to be a non-increasing function of the change in the number of shares traded $|\Delta\Pi_s|$ such that $C = C(\Delta\Pi_S)$ and

$$\Delta C = \frac{S}{2} C(\Delta\Pi_S) |\Delta\Pi_s| \tag{15}$$

In the current approach, variable transaction costs are introduced for each small-time interval as a non-increasing function of the LRB market information process ξ_t and change in the number of shares traded $|\Delta\Pi_s|$ such that $C = C(\Delta\Pi_S, \xi_t)$. The transaction costs are assumed to decrease as the market information process driving the price process increases. This is motivated by the fact that more market information about the prices reduces the bid-ask spreads and uncertainty about the price, hence lowering the transaction costs for market participants [17]. Similar to [19], the transaction costs are expected to be discounted for investors trading in a higher number of shares of the underlying. The information process is assumed to be independent of the number of shares invested. It follows that

$$\Delta C = \frac{S}{2} C(\Delta\Pi_S, \xi_t) |\Delta\Pi_S| \tag{16}$$

Using $\Delta C \approx E[\Delta C]$,

$$\Delta C = \frac{S}{2} E\left\{C(\Delta\Pi_S, \xi_t)|\Delta\Pi_S|\right\} \tag{17}$$

$$= \frac{S}{2} E\left[|\Delta\Pi_S|\right] \cdot E\left[C(\Delta\Pi_S, \xi_t)\right] \tag{18}$$

The first expectation in Equation (18), $E\left[|\Delta\Pi_S|\right]$ is obtained by applying the delta hedging strategy and Itô's lemma such that $\Pi_S = -\frac{\partial V}{\partial S} \Rightarrow \frac{\partial \Pi_S}{\partial S} = -\frac{\partial^2 V}{\partial S^2}$. Thus,

$$\Delta\Pi_S = -\frac{\partial^2 V}{\partial S^2} \mathrm{d}S = -\frac{\partial^2 V}{\partial S^2}(\mu S \mathrm{d}t - \sigma_t \mathrm{d}W) \tag{19}$$

$$= -\frac{\partial^2 V}{\partial S^2} \sigma_t \mathrm{d}W \quad \text{since} \quad \mathrm{d}t \approx 0 \tag{20}$$

Taking expectation in Equation (20), it follows that

$$E\left[|\Delta\Pi_S|\right] = \left|\frac{\partial^2 V}{\partial S^2}\right| \sigma_t E[\mathrm{d}W] = \left|\frac{\partial^2 V}{\partial S^2}\right| \sigma_t \sqrt{\frac{2}{\pi}} \sqrt{\mathrm{d}t} \tag{21}$$

The second expectation in Equation (18), $E\left[C(\Delta\Pi_S, \xi_t)\right]$, can be derived depending on the nature of the transaction cost function $C(|\Delta\Pi_s|, \xi_t)$ resulting in different forms of the pricing equation.

Briefly, we present two examples of realistic non-increasing transaction cost functions studied in the literature namely; the non-increasing exponential transaction cost function and non-increasing Linear Transaction Cost Function. We define the fixed percentages, C_0 as the constant cost of trading, C_1 as the reduced cost per amount of share traded, and $C_2 > 0$ as the reduced cost per unit time as $t \to T$.

3.1. The Non-Increasing Exponential Transaction Cost Function

Consider the following non-increasing exponential transaction cost function as introduced by [19].

$$C(|\Delta\Pi_s|, \xi_t) = C_0 \mathrm{e}^{-C_1 \xi_t - C_2 |\Delta\Pi_S|} \tag{22}$$

The expected value is given as

$$E\left[C(\Delta\Pi_S, \xi_t)\right] = E\left[C_0 \mathrm{e}^{-C_1 \xi_t - C_2 |\Delta\Pi_S|}\right] = C_0 E\left(\mathrm{e}^{-C_1 \xi_t}\right) E\left(\mathrm{e}^{-C_2 |\Delta\Pi_S|}\right) \tag{23}$$

Since $\xi_t \sim N\left(\lambda_i t X_T, \frac{t(T-t)}{T}\right)$, it follows that the expectation containing ξ_t in Equation (23) yields

$$E\left(\mathrm{e}^{-C_1 \xi_t}\right) = \mathrm{e}^{-C_1\left(\lambda_i t X_T + \frac{t(T-t)}{2T}\right)} \tag{24}$$

Using $\Delta\Pi_S = -\frac{\partial^2 V}{\partial S^2} \sigma_t \mathrm{d}W \approx -\frac{\partial^2 V}{\partial S^2} \sigma_t \phi \sqrt{\mathrm{d}t}$ where $\phi \sim N(0,1)$, it follows that the expectation containing $|\Delta\Pi_S|$ in Equation (23) yields

$$E\left(\mathrm{e}^{-C_2|\Delta\Pi_S|}\right)=\mathrm{e}^{-C_2\frac{\partial^2V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}}E\left[\mathrm{e}^{-\phi}\right]=\mathrm{e}^{-C_2\frac{\partial^2V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}}\cdot\mathrm{e}^{-\frac{1}{2}} \tag{25}$$

Combining Equations (24) and (25)

$$E\left[C\left(\Delta\Pi_S,\xi_t\right)\right]=C_0\mathrm{e}^{-C_1\left(\lambda_i tX_T+\frac{t(T-t)}{2T}\right)}\mathrm{e}^{-C_2\frac{\partial^2V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}}\cdot\mathrm{e}^{-\frac{1}{2}} \tag{26}$$

$$=C_0\exp\left(-\frac{1}{2}-C_1\left(\lambda_i tX_T+\frac{t(T-t)}{2T}\right)-C_2\frac{\partial^2V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}\right) \tag{27}$$

Substituting Equations (21) and (27) into Equation (18) yields

$$\Delta C=\frac{S}{2}C_0\exp\left(-\frac{1}{2}-C_1\left(\lambda_i tX_T+\frac{t(T-t)}{2T}\right)-C_2\frac{\partial^2V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}\right)\left|\frac{\partial^2V}{\partial S^2}\right|\sigma_t\sqrt{\frac{2}{\pi}}\sqrt{\mathrm{d}t} \tag{28}$$

Replacing the dynamics of $\mathrm{d}S$ and $\mathrm{d}V$ as well as the value of ΔC obtained in Equation (28) into Equation (13) gives

$$\begin{aligned}\mathrm{d}\Pi=&\left(\frac{\partial V}{\partial t}+\frac{\partial V}{\partial S}\mu S+\frac{1}{2}\frac{\partial^2V}{\partial S^2}\sigma_t^2\right)\mathrm{d}t+\frac{\partial V}{\partial S}\sigma_t\mathrm{d}W+\Pi_S\left(\mu S\mathrm{d}t+\sigma_t\mathrm{d}W\right)\\&-\frac{S}{2}C_0\exp\left(-\frac{1}{2}-C_1\left(\lambda_i tX_T+\frac{t(T-t)}{2T}\right)-C_2\frac{\partial^2V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}\right)\left|\frac{\partial^2V}{\partial S^2}\right|\sigma_t\sqrt{\frac{2}{\pi}}\sqrt{\mathrm{d}t}\end{aligned} \tag{29}$$

The portfolio is expected to be risk-less, this means that the terms containing the Brownian motion $\mathrm{d}W$ are zero such that

$$\begin{aligned}\mathrm{d}\Pi=&\left(\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2V}{\partial S^2}\sigma_t^2-\frac{S}{2}C_0\exp\left(-\frac{1}{2}-C_1\left(\lambda_i tX_T+\frac{t(T-t)}{2T}\right)\right.\right.\\&\left.\left.-C_2\frac{\partial^2V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}\right)\left|\frac{\partial^2V}{\partial S^2}\right|\sigma_t\sqrt{\frac{2}{\pi\mathrm{d}t}}\right)\mathrm{d}t\end{aligned} \tag{30}$$

Furthermore, under the no-arbitrage assumption, the portfolio must earn a risk-free rate. This means that $\mathrm{d}\Pi=r\Pi\mathrm{d}t$ so that,

$$\begin{aligned}&\left(\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2V}{\partial S^2}\sigma_t^2-\frac{S}{2}C_0\exp\left(-\frac{1}{2}-C_1\left(\lambda_i tX_T+\frac{t(T-t)}{2T}\right)\right.\right.\\&\left.\left.-C_2\frac{\partial^2V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}\right)\left|\frac{\partial^2V}{\partial S^2}\right|\sigma_t\sqrt{\frac{2}{\pi\mathrm{d}t}}\right)\mathrm{d}t=r\left(V-\frac{\partial V}{\partial S}S\right)\mathrm{d}t\end{aligned} \tag{31}$$

Dropping the $\mathrm{d}t$ term on both sides of Equation (31) yields the PDE

$$\begin{aligned}&\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2V}{\partial S^2}\sigma_t^2\left[1-\frac{SC_0}{\sigma_t\sqrt{\mathrm{d}t}}\exp\left(-\frac{1}{2}-C_1\left(\lambda_i tX_T+\frac{t(T-t)}{2T}\right)\right.\right.\\&\left.\left.-C_2\frac{\partial^2V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}\right)\sqrt{\frac{2}{\pi}}sign\left(\frac{\partial^2V}{\partial S^2}\right)\right]+rS\frac{\partial V}{\partial S}-rV=0\end{aligned} \tag{32}$$

3.2. The Non-Increasing Linear Transaction Cost Function

Consider the following linear non-increasing transaction cost function similar to the one introduced by [20]

$$C\left(\left|\Delta\Pi_s\right|,\xi_t\right)=C_0-C_1\left|\Delta\Pi_s\right|-C_2\xi_t \tag{33}$$

$$\Delta C=\frac{S}{2}\left\{E\left[C_0\left|\Delta\Pi_S\right|-C_1\left|\Delta\Pi_s\right|^2-C_2\xi_t\left|\Delta\Pi_S\right|\right]\right\} \tag{34}$$

$$=\frac{S}{2}\sigma_t\left(C_0\left|\frac{\partial^2V}{\partial S^2}\right|\sqrt{\frac{2}{\pi \mathrm{d}t}}-C_1\left(\frac{\partial^2V}{\partial S^2}\right)^2\sigma_t-C_2\lambda_i tX_T\left|\frac{\partial^2V}{\partial S^2}\right|\sqrt{\frac{2}{\pi \mathrm{d}t}}\right)\mathrm{d}t \tag{35}$$

The change in the value of the portfolio is

$$\begin{aligned}\mathrm{d}\Pi=&\left(\frac{\partial V}{\partial t}+\frac{\partial V}{\partial S}\mu S+\frac{1}{2}\frac{\partial^2V}{\partial S^2}\sigma_t^2\right)\mathrm{d}t+\frac{\partial V}{\partial S}\sigma_t\mathrm{d}W+\Pi_S\left(\mu S\mathrm{d}t+\sigma_t\mathrm{d}W\right)\\&-\frac{S}{2}\sigma_t\left(C_0\left|\frac{\partial^2V}{\partial S^2}\right|\sqrt{\frac{2}{\pi \mathrm{d}t}}-C_1\left(\frac{\partial^2V}{\partial S^2}\right)^2\sigma_t-C_2\lambda_i tX_T\left|\frac{\partial^2V}{\partial S^2}\right|\sqrt{\frac{2}{\pi \mathrm{d}t}}\right)\mathrm{d}t\end{aligned} \tag{36}$$

The resulting PDE is

$$\begin{aligned}&\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2V}{\partial S^2}\sigma_t^2\left[1-\frac{SC_0}{\sigma_t}\sqrt{\frac{2}{\pi}}sign\left(\frac{\partial^2V}{\partial S^2}\right)-\frac{SC_1}{\sigma_t}\frac{\partial^2V}{\partial S^2}\right.\\&\left.-\frac{SC_2}{\sigma_t}\lambda_i tX_T sign\left(\frac{\partial^2V}{\partial S^2}\right)\right]+rS\frac{\partial V}{\partial S}-rV=0\end{aligned} \tag{37}$$

4. The Stochastic Optimal Control Problem

The price of the Bermudan call option is given by Equation (32) and Equation (37) for the non-increasing exponential transaction cost function and the non-increasing linear transaction cost function respectively. These equations are second-order non-linear PDEs whose solution cannot be defined analytically but can be obtained at least in some adequate weak sense using the theory of viscosity solutions. The definition of viscosity solutions can be found in [11]. Let $\Omega\in\mathbb{R}$, the discrete Dirichlet problem for the Bellman Equation (12) can be rewritten as

$$\begin{cases}\frac{\partial V}{\partial t}+B\left(\nabla^2V,\nabla V,S,V\right)=0 & \text{in } \Omega\\ V\left(S,\tau\right)=\Phi\left(S\right) & \text{in } \partial\Omega\end{cases} \tag{38}$$

In this case, $V\left(S,\tau\right)=\Phi\left(S\right)$ is the stopping or exercise region, and $\frac{\partial V}{\partial t}+B\left(\nabla^2V,\nabla V,S,V\right)=0$ is the continuation region where B is the discrete-time Bellman equation defined by

$$B=\max\mathcal{L}V\left(S,\tau\right) \tag{39}$$

where $\mathcal{L}$ is the differential operator for the IBM-PDE with variable transaction costs given by

$$\mathcal{L}=\frac{\partial}{\partial S}rS+\frac{1}{2}\frac{\partial^2}{\partial S^2}\hat{\sigma}_t^2-rV \tag{40}$$

For the non-increasing exponential transaction cost function of the form in

Equation (22)

$$\hat{\sigma}_t^2 = \sigma_t^2\left[1 - \frac{SC_0}{\sigma_t\sqrt{\mathrm{d}t}}\exp\left(-\frac{1}{2} - C_1\left(\lambda_i t X_T + \frac{t(T-t)}{2T}\right)\right.\right.$$
$$\left.\left. - C_2\frac{\partial^2 V}{\partial S^2}\sigma_t\sqrt{\mathrm{d}t}\right)\sqrt{\frac{2}{\pi}}sign\left(\frac{\partial^2 V}{\partial S^2}\right)\right] \tag{41}$$

For the linearly non-increasing function of the form in Equation (33)

$$\hat{\sigma}_t^2 = \sigma_t^2\left[1 - \frac{SC_0}{\sigma_t}\sqrt{\frac{2}{\pi}}sign\left(\frac{\partial^2 V}{\partial S^2}\right) - \frac{SC_1}{\sigma_t}\frac{\partial^2 V}{\partial S^2} - \frac{SC_2}{\sigma_t}\lambda_i t X_T sign\left(\frac{\partial^2 V}{\partial S^2}\right)\right] \tag{42}$$

Assuming C_0, C_1, C_2 are small enough, $\hat{\sigma}_t > 0$ for both cost functions.

5. Existence and Uniqueness of the Solution to the Non-Linear IBM PDE under Variable Costs

It suffices to show that the solution to the Dirichlet problem in Equation (38) exists and is unique which corresponds to the optimal price of the Bermudan option. The following regularity assumptions on Equation (38) are introduced.

1) $\Phi(S)$ is continuous and bounded in $\mathbb{R}$, $\Rightarrow V(S,\tau)$ is also continuous.

2) $V(S,\tau)$ does not have to be everywhere differentiable w.r.t. *S*.

3) $V(S,\tau)$ is strictly non-decreasing and grows quadratically is *S*.

***Proposition* 1:** *Under assumptions i-iii, there exists a unique solution V of the Bellman equation.*

Existence Proof via Perron's Method

Let $S' \in \Omega$ be a limit point and define the upper semicontinuous (USC) and lower semicontinuous (LSC) envelopes of the function $V(S,\tau)$ by respectively, $\bar{V}(S,\tau) = \lim_{S\to S'}\sup V(S,\tau)$ and $\underline{V}(S,\tau) = \lim_{S\to S'}\inf V(S,\tau)$. By comparison principle, $V(S,\tau)$ satisfies $\underline{V}(S,\tau) \le V(S,\tau) \le \bar{V}(S,\tau)$. Under Perron's method, the idea is to show that given any stopping time *τ*, $\bar{V}(S,\tau)$ is a viscosity sub-solution and $\underline{V}(S,\tau)$ is a viscosity super solution, hence $V(S,\tau)$ is a viscosity solution of the Bellman equation [21]. Lemmas 1 and 2 are fundamental to the proof of existence and lemma 3 proves the uniqueness of the solution.

Lemma 1*: *V*(*S*, *τ*) *is a viscosity sub-solution of the Bellman equation

Proof: By definition [11], this requires showing that $\bar{V}(S,\tau)$ satisfies the viscosity sub-solution property such that, if $\bar{V}(S,\tau) - \Phi(S)$ attains a local maximum at a limit point $S' \in \Omega$, then

$$\frac{\partial V}{\partial t} + B\left(\nabla^2 V, \nabla V, S, V\right) \le 0 \tag{43}$$

Let $\mathcal{C}^2$ denote the class of twice differentiable continuous functions. It is clear that $\Phi(S) \in \mathcal{C}^2$ and suppose $\bar{V}(S,\tau) - \Phi(S)$ has a local maximum at S', such that

$$\max\left\{\bar{V}(S,\tau) - \Phi(S)\right\} = \bar{V}(S',\tau) - \Phi(S') = 0 \tag{44}$$

By assumption iii, define another function

$$\Psi(S) = \Phi(S) + \bar{V}(S',\tau) - \Phi(S') + |S - S'|^4 \tag{45}$$

It follows that $V(S,\tau) - \Psi(S)$ attains its strict maximum at S' such that

$$\begin{aligned} \bar{V}(S,\tau) - \Psi(S) + |S - S'|^4 &= \bar{V}(S,\tau) - \Phi(S) \le \bar{V}(S',\tau) - \Phi(S') \\ &= \bar{V}(S',\tau) - \Psi(S') = 0 \end{aligned} \tag{46}$$

This yields

$$\bar{V}(S,\tau) - \Psi(S) \le -|S - S'|^4 \tag{47}$$

Since Ω is a complete metric space, all sequences S_n are Cauchy sequences such that $S_n \to S'$ as $n \to \infty$. This implies that $V(S_n,\tau) \to V(S',\tau)$ and for each *n*, there exists V_n such that $V_n(S_n,\tau) \ge V(S_n,\tau) - \frac{1}{n}$. It follows from Equation (47) that

$$|S_n - S'|^4 \le \Psi(S_n) - \bar{V}(S_n,\tau) \tag{48}$$

$$\le \Psi(S_n) - V_n(S_n,\tau) \tag{49}$$

$$\le \Psi(S_n) - V(S_n,\tau) + \frac{1}{n} \tag{50}$$

$$\le 0 \tag{51}$$

Since $S_n \to S'$, implies $\Psi(S_n) \to \Psi(S')$ and $V(S_n,\tau) \to V(S',\tau)$ thus,

$$B\left(\nabla^2 V(S',\tau), \nabla V(S',\tau), S', V(S',\tau)\right) \le 0 \tag{52}$$

This implies that $V(S,\tau)$ is a viscosity sub-solution.

Lemma 2**:** ***V*****(*****S*****,** ***τ*****)** ***is a viscosity super-solution of the Bellman equation***

Proof: Similarly, the super-solution property amounts to show that if $\underline{V}(S,\tau) - \Phi(S)$ attains a local minimum at S', then

$$\frac{\partial V}{\partial t} + B\left(\nabla^2 V, \nabla V, S, V\right) \ge 0 \tag{53}$$

The proof is by contradiction. Suppose $\underline{V}(S,\tau)$ is not a super-solution, then there exists $\Phi(S) \in \mathcal{C}^2$ and a limit point $S' \in \Omega$, such that

$$\min\left\{\underline{V}(S,\tau) - \Phi(S)\right\} = \underline{V}(S',\tau) - \Phi(S') = 0 \tag{54}$$

Suppose $\omega(S,\tau)$ is a viscosity super-solution of the Bellman equation, then by comparison principle $V(S,\tau) \le \omega(S,\tau)$. If $\Phi(S') = \omega(S',\tau)$, then $\omega(S,\tau) - \Phi(S)$ has a local minimum at S' which is a contradiction since $\omega(S,\tau)$ is a super-solution. It remains to show that $\Phi(S') < \omega(S',\tau)$. Using Equation (45), there exists $\epsilon > 0$ such that $\Psi(S) \le \underline{V}(S,\tau)$ and $\Psi(S) + \epsilon \le \omega(S,\tau)$ implies

$$B\left(\Delta^2 \Psi(S), \Delta\Psi(S), S, \bar{V}(S,\tau)\right) \le -\epsilon \tag{55}$$

Define another function $g(S,\tau) = \max\left\{\Psi(S) + \epsilon, \underline{V}(S,\tau)\right\}$. It can be seen that $\omega(S,\tau) \ge g(S,\tau) \Rightarrow \omega(S,\tau) > \underline{V}(S,\tau)$. By definition, there exists $S_n \to S'$ as $n \to \infty$ such that $\omega(S_n,\tau) \to \omega(S',\tau)$ and $\underline{V}(S_n,\tau) \to \underline{V}(S',\tau)$. This completes the proof.

Lemma 3*: *The value function V(S, τ) is the unique viscosity solution of the Bellman equation

Proof: Given any stopping time τ it has been shown that the solution of the value function exists. The application of the Banach Fixed Point Theorem [22] proves the uniqueness of this solution.

Define another function $f:\Omega \to \Omega$ such that $f\left(V\left(S,\tau\right)\right)=V\left(S,\tau\right)$ where $V\left(S,\tau\right)$ is any fixed point of *f*. Since $V\left(S,\tau\right)$ is continuous and non-decreasing on the interval $\left[0,T\right]$, the application of the Intermediate Value Theorem implies that there exists a value *K* in between $V\left(S_0,0\right)$ and $V\left(S_T,T\right)$ such that

$$V\left(S_0,0\right)<K<V\left(S_T,T\right) \tag{56}$$

Dividing both sides of the inequality in Equation (56) by $V\left(S_T,T\right)$ yields

$$\frac{V\left(S_0,T\right)}{V\left(S_T,T\right)}<K<1 \tag{57}$$

Such that $\frac{V\left(S_0,T\right)}{V\left(S_T,T\right)}\geq 0$. This implies that there exists a value $K\in\left[0,1\right)$ which is the Lipschitz constant of the function *f*. It follows from Mean Value Theorem that the derivative of *f* exists on the interval $\left[0,T\right]$ and that there exists a value $V\left(S,\tau\right)$ in between $V\left(S_0,0\right)$ and $V\left(S_T,T\right)$ such that

$$\begin{aligned}&\left|f\left(V\left(S_T,T\right)\right)-f\left(V\left(S_0,0\right)\right)\right|\\&=\left|\frac{\partial f\left(V\left(S_\tau,\tau\right)\right)}{\partial S}\right|\left|V\left(S_T,T\right)-V\left(S_0,0\right)\right|\leq K\left|V\left(S_T,T\right)-V\left(S_0,0\right)\right|\end{aligned} \tag{58}$$

Therefore, *f* is Lipschitz with respect to *S* and is a contraction mapping since $K<1$. The Lipschitz continuity property also guarantees the existence of the solution of $V\left(S,\tau\right)$. By Banach Fixed Point Theorem, *f* admits a unique fixed point which we can write as $V\left(S,\hat{\tau}\right)$ where $\hat{\tau}$ is the optimal exercise time for the Bermudan option.

6. Conclusion

This paper extends IBM by allowing for variable transaction costs in determining the price of a Bermudan option. Rather than a closed form formula, the price of the option is derived as a non-linear second order PDE with modified volatility due to the transaction costs of trading. Mathematical results showing the existence and uniqueness of the solution of the PDE are presented by the application of stochastic optimal control and the theory of viscosity solutions. Therefore, it can be deduced that the price of a Bermudan option for an asset driven by the LRB market information process is the unique viscosity solution of its corresponding Bellman equation. Further research can be focused on identifying an effective numerical approximation scheme that can compute the Bermudan option prices under the information-based asset pricing model with variable transaction costs.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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