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# Economic Growth: The Log Symmetry of Collaborative Learning Populations

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## Abstract

Human beings are not only intelligent, but also comprise a social species capable of extensive cooperation. Moreover, they collaborate. Individual human efforts segregate into incongruous silos of multimodal distributions of expression and low performance. A social psychological, economic and mathematical review indicates that collaboration can integrate mental facilities into a congruent unimodal normal distribution of expression and high performance not otherwise possible. The ontology of symmetry is remarkable and compelling. The learning outcomes are natural for economic growth and human development.

## Keywords

Economic Growth, Collaboration, Innovation, Normal Distribution, Cognitive Reintegration, Learning

## 1. Introduction

The terms collaboration and cooperation are sometimes confused. So are economic growth and economic development. In this research, we are interested in collaboration and economic growth and development based on epistemological, metaphysical, and axiological insights (Randrup, Druckemiller, & Briggs, 2016), so for clarity of purpose we begin with the following definitions.

*Definition.* Normal refers to a distribution containing a most frequently occurring typical value at its peak (central mean) and atypical values with lower and lower frequency as they occur further and further away from the mean.

*Definition.* Cooperation is a plan and execution thereof by participants, each with their own personal self-interest and economic gain in mind yet yielding unintended mutual benefits.

*Definition.* Collaboration is the plan and execution thereof by participants for their intentional mutual benefit of shared goals, objectives, and rewards.

*Definition.* Economic growth is the improvement in per capita real gross domestic product adjusted for purchasing power parity.

*Significance.* The path to extraordinary economic growth is collaboration → capitalization → gross domestic product. Total market capitalization is Pareto distributed (~lognormal) and wealth is a function of total market capitalization. Wealth is unlimited but the rich must necessarily get richer in order to raise the human condition and to uplift the poor.

Whereas cooperation can result in the unintended consequences of mutual benefit, collaboration is planned intentionally by participants to provide for their mutual benefit (Tomasello, 2001, 2009; Tomasello et al., 2005, 2012). Capitalism, democracy, and rule of law are fundamental elements of economic collaboration in pursuit of extraordinary economic growth. Economic cooperation (or going along to get along) is an activity that people perform with self-interest and personal gain in mind. Rand (1961, 1990) speaks to the morality of objectivist self-interest. But there is no shared benefit required. Collaboration is a phenomenon associated with and only with human beings and humanity. Adam Smith (1776, 2010) described how the butcher, the brewer, and the baker divide their labor, each with their own self-interest in mind. And how the pursuit of their own advantage leads to action that is most advantageous to society. At first this has the appearance of collaboration of sorts or at least cooperation. But it is clear that self-interest cannot be collaboration since the outcome for society is not intentional. The field of economics speaks to cooperation for the satisfaction of self-interest, but little consideration is given to collaboration. Economic cooperation implied by the division of labor and trade will create the efficient deployment of capital and concomitant ordinary economic growth. But this occurs in an environment of a depreciating production component of endogenous capital stock. Ridley (2018, 2020a) showed that 21% of per capita real gross domestic product (GDP) adjusted for purchasing power parity (GDPppp) is required for reinvestment in growth and to cover maintenance, depreciation and obsolescence. To add new growth, capital must be constructed from exogenous human capital ideas of imagination and creativity. While inventors and scientists are at work, there is no commercial product to trade and no clearly identifiable self-interests or beneficiaries. There is no guarantee of how traders might perceive the value of any potential product. All that inventors can envisage is the application, utility, and benefit to their community. We attribute the willingness of the human being to invent something new when the profits and direction of profits are unknown, to human propensity for altruistic collaboration over and above cooperation. This paper contributes what might be considered the study of the advancement from economic cooperation to economic collaboration for the purpose of extraordinary economic growth. Whereas John Heywood (1497-1580) might have said "many hands make light work" with regards to cooperation, Maxwell (2002) might have said "teamwork makes the dream work" with respect to collaboration.

To paraphrase Charles Darwin (1809-1882) "It is not the strongest of the species that survive, nor the most intelligent, but the one most responsive to change." Human beings are known to be the most sociable species on earth. Not only do they take care of their children they care of each other and work together to accomplish remarkable feats not possible by individuals acting separately. While obviously intelligent primates, this propensity for sociability might account for their becoming the dominant species. Singular efforts lacking collaboration can all amount to naught. Or the minimum at best. Human beings can apply force. It is obvious from what we know about physics that two or more forces aligned in the same direction can move more mass than either one of the forces alone. Therefore, human beings can collaborate to move more mass than they otherwise would.

The purpose of this paper is to explore the way in which individual, disparate facilities of the human brain can consciously combine physical and mental effort to solve problems that integrate human ideas of imagination and creativity. Such entrepreneurial activities are known to create massive wealth (Ridley, 2020a, 2020b; de Silva, Ridley, & Green, 2020; Llaugel & Ridley, 2018; Ngnepieba et al., 2018). Ridley, Ngnepieba and de Silva (2021) show how theoretical and empirical bimodal distributions of two subpopulations of university calculus test scores can combine into a unimodal normal distribution when collaborative learning is implemented. In addition to higher test score averages, normally distributed test scores are obtained. Both of these outcomes are assumed to be natural and desired for human development. In this research, we extend the scope to multimodal distributions of several subpopulations. We are interested in the implications of collaboration and normal or otherwise symmetrical outcomes for learning, business, management, industry, and ultimately extraordinary economic growth.

There are many examples of single individuals working alone, and groups of people working together and simultaneously. The difference between invention and innovation is that innovation is the process of making a breakthrough invention available, affordable, and reliable (see also Schumpeter (1911, 1928, 1954)) who initiated the theory of economic growth based on entrepreneurship). Invention is an individual effort and outcome that is very important and useful. But the really difficult part is innovation (Ridley, 2020). Ridley (2020) argues that we give too much credit to inventors and not enough to innovators who refine and improve an invention to make it valuable to users. The power of trial and error cannot be overemphasized. But, also not well appreciated is the amount of collaboration that is required to take it to the next level.

The novelty of this research is to show the inevitability of the Pareto 80/20 rule as it applies to the distribution of capital. It demonstrates the importance of collaboration (over and above cooperation) for extraordinary economic growth. It derives the mathematical mechanism for how combining from different probability distributions of capital can occur. And why low-income people must collaborate with high-income innovators to create labor saving and leisure devices that raise the standard of living of the least among us.

The remainder of the paper is organized as follows. Section 2 is a review of relevant literature. Section 3 is a brief discussion on aptitude. Section 4 is a review of psychology as it applies to the harnessing of human skills. Section 5 is a review of socialization as it applies to collaboration and cooperation. Section 6 is a discussion on how human facilities of the mind work in collaboration and cooperation for economic growth and development. Section 7 is a mathematical proof that combining several multimodal subpopulations can integrate them into a unimodal normal or otherwise symmetrical distribution. Section 8 summarizes conclusions and suggestions for future research.

## 2. Literature Review

*Collaboration*

Human collaboration has come a long way from the prehistoric times to the present. At the dawn of the human evolution, collaboration was essential to the survival of individuals and tribes. At present, collaboration among organizations, industries and countries is needed for the advancement of the economic development of mankind. The ability of human beings to collaborate is fundamental to the development of humanity (Gilbert, 1989; Tuomela, 2007). A renewed interest in the ontogeny of human collaboration has recently become the center of empirical research of developmental psychologists (Brownell, Ramani, & Zerwas, 2006; Tomasello et al., 2005; Warneken, Chen, & Tomasello, 2006; Warneken & Tomasello 2007; Callaghan et al., 2011; Sterelny, 2011, 2012).

Game theories have been used to model and explain human evolution (Smith, 1982a; Smead, 2012), evolution of learning (Simpson, 1953; Roth & Erev, 1995; Samuelson, 1998; Skyrms, 2004; Smead, 2014) and the dynamics of collaboration (Skyrms, 2004; Tomasello, 2009). Human collaborative activities can be traced back to foraging when it was used in Stag Hunt-type situations which required coordinated collaborative actions from each participant for the benefit of all (Alvard, 2012; Tomasello et al., 2012). As evidenced from comparative experimental data, human beings use cognitive and motivational proximate mechanisms for cooperating in these types of situations (Sterelny, 2011). With time, human beings learned to overcome the main collaborative issues: sharing of spoils, free ride, foregoing individual gains, and to pursue "Stag Hunt collaboration as an evolutionarily stable subsistence strategy" (Tomasello et al., 2012). A few mechanisms helped build cooperation of population members in the Stag Hunt-type situations: Group selection (Binmore, 2005; Huttegger & Smead, 2011), dynamic network formation (Skyrms & Pemantle, 2000; Skyrms, 2004), signaling and interaction (Zollman, 2005), and other mechanisms.

The ontogeny of human collaboration is evidenced through a depth of empirical research on a child's ability to collaborate that "emerges in an ontogenetically regimented manner despite variation in environmental input" (Sober, 1994a, 1994b; Sober, 1998; Ariew, 1996, 1999, 2007; Callaghan et al., 2011). While research studies have proven collaboration to be a type of behavior, developmental psychologists, anthropologists and neuroscientists are seeking evidence of whether human cognitive systems affect collaborative activities. Scientists have found that the capacity to collaborate is connected to cognitive systems that encode and decode human language (Kiang, 1984; Kellis et al., 2010; Kubanek et al., 2013; Leonard et al., 2015; Huth et al., 2016; Khalighinejad et al., 2017), shared knowledge (Searle, 1995; Wyman, Rakoczy, & Tomasello, 2013), the need to engage in joint activities in pursuit of shared outcomes, social obligation, fairness, and predicting decisions about human interactions (Grafenhain et al., 2009; Bruning & Schraw, 2011; Hollmann et al., 2011; Hamann et al., 2011, 2012; Warneken et al., 2011).

Recent findings on a child's collaborative activities provide evidence that collaboration emerges as children grow between 12 and 24 months (Tomasello et al., 2005; Warneken, Chen, & Tomasello, 2006; Brownell, Ramani, & Zerwas, 2006; Warneken & Tomasello, 2007; Carpenter, 2009; Brownell, 2011). Children first learn to collaborate on new activities with adults (Warneken et al., 2006; Callaghan et al., 2011) and closer to their second birthday, they are skillfully collaborating with their peers (Brownell, Ramani, & Zerwas, 2006; Brownell, 2011). Importantly, the development of social cognitive activities start at about their third birthday, which is revealed through children's understanding and development of joint commitments to attain joint goals (Grafenhain et al., 2009; Hamann et al., 2012) and evidence of equal sharing (Hamann et al., 2011).

One may argue that children from different cultures and populations may have developmental differences in the capacity to collaborate. Warneken, Chen and Tomasello (2006) conducted an experiment to evaluate the ability of 18-month-old and 24-month-old European children to solve two collaborative tasks and to play two collaborative games. Research findings demonstrated that members of both age groups were able to engage in the collaborative tasks and games, and to reengage with the adult experimenter. Further, study results also revealed that children in the 24month-old group were more skilled in both collaborative tasks and games, and more frequently reengaged the adult experimenter when she stopped participating in the activity. Later, Callaghan et al. (2011) replicated the study to examine whether there was a significant variation between Canadian, Peruvian and Indian populations. Callaghan et al. (2011) conducted an experiment with samples of 57 Canadian, Peruvian, and Indian children between 17 and 28 months and did not find significant differences between the sample populations. Both studies provide evidence that even at an early age, a child understands the nature of collaboration, which suggests the ontogeny of collaboration is an innate human ability.

*A dilemma*

While the benefit of collaboration is clear, there is the potential dilemma of intra institutional collaboration and inter institutional competition. Collaboration within a company is in part, for the purpose of competing against other companies. For example, consider the development a new product. This may seem good from the point of view of the company. But, to what extent could between company collaboration result in an even better product for society? For another example, consider the two alternatives of nation states versus tribalism. Tribes expend appreciable collaborative effort in waging war against other tribes. Collaboration might occur effectively within a tribe of likeminded people, but it is limited in what it can achieve in terms of a national totality and diversity of ideas, technology and natural resources. The tribes might be better off uniting into a nation state.

*Human capital*

Human capital is the sole source of wealth. In the below section on economic growth, we will show how capital is distributed lognormally. This might be rooted in human psychology. Logarithms and log-normal distributions occur in many phenomena involving human activity and perception (Goldstein, 2009; Mathews, 2000). The relationship between the time individuals take to choose an alternative and the number of choices available is logarithmic according to Hick's law (Welford, 1968). Fitts's law predicts that the time required to rapidly move to a target area is a logarithmic function of the distance to and the size of the target (Fitts, 1954). The Weber-Fechner law and Stevens's power law in psychophysics propose a logarithmic relationship between stimulus and sensation (Banerjee, 1994; Nadel, 2005). Clementi and Gallegati (2005), and Wataru (2002) found evidence of Pareto (shape similar to lognormal) incomes of individuals.

## 3. Normal Aptitude

We recognize that human beings are a highly intelligent species with many skills. But failure to combine these skills can be unproductive. Some populations comprise people who try to do everything and nothing particularly well. A strategy that can slow progress while reducing quality. But some populations have found ways to leverage their efforts by raising their skills through specialization then by combining them. Ridley, Ngnepieba and de Silva (2021) comment on the College Board (2019) report on scholastic aptitude test (SAT). SAT test scores (both writing and mathematics) are remarkably normally distributed. It is as if the symmetry is a promise of aptitude. On the other hand, the College Board (2011) report on AP test scores, with the only exception of English, is all multimodal. These disparate subpopulations matriculate into universities where their multimodalities are plain to see. University test scores are typically not normally distributed. Quite to the contrary, they are multimodal silos of thought enclaves. When people are compartmentalized, the compartments are referred as silos. Instead of fulfilling the promise of scholastic aptitude, the university experience is one of frustration often ending in failure. Ridley, Ngnepieba and de Silva (2021) reported an experiment in which they implemented active learning teaching methodology (as opposed to didactic lecture) such that professor supervised student-student collaboration resulted in unimodal normal test scores and higher score averages.

## 4. Humanity and Psychology

### 4.1. Child Psychology and Play

Play is essential for the development of a child. As part of physical, cognitive, emotional, and social development, play is an ontogenetic adaptation that serves a function in child development (Oppenheim, 1981; Lambert & Johnson, 2011). Play is considered to be a quintessential developmental construct. While ethologists (Martin & Caro, 1985) and child developmental psychologists (Rubin, Fein, & Vandenberg, 1983) refer to play as behavior that lacks functionality or where the processes are more important than the results, play has been an indispensable part of childhood that takes a substantial portion of children's time and energy (Hinde, 1989). In the ethological literature, play has been divided into three main types: Object play, physical activity play (rhythmic stereotypes: exercise play, rough-and-tumble play), and social or pretend play (Fagen, 1981; McCune-Nicholich & Fenson, 1984; Martin & Caro, 1985), each one providing experiences and opportunities for children to develop respective skills and abilities (Piaget, 1962; Pellegrini & Bjorklund, 1997; Pellegrini & Smith, 1998; Smith, 2005).

Most child development theories posit that there are no immediate benefits of play in childhood because they are assumed to be deferred until past childhood (Groos, 1898, 1901; Vygotsky, 1978). Children's play has been viewed as the period of preparation for adulthood through learning and development of skills important for adults. Play functionality has been seen in building a collection of skills and when the skills were acquired, the play would stop (Bruner, 1972; Bateson, 1976; Smith, 1982a, 1982b). Play scenarios have provided opportunities for children to explore new roles and different activities, develop social skills, master existing skills and develop cognitive and physical abilities seen in adults: Collaboration, empathy, self-efficacy, social signaling, leadership traits and skills, decision making skills, negotiation skills, conflict resolution skills, bone and muscle development, and other skills (Martin & Caro, 1985; Bjorklund & Green, 1992; Pellegrini & Smith, 1998). Some of these skills will stand the child in good stead later as an adult collaborator.

Geary and Bjorklund (2000) posit that evolutionary developmental psychologists apply evolutionary thinking to examine human development. Through theoretical developments and practical evaluation, evolutionary developmental psychology helps us understand why and how the human development evolves throughout the course of ontogeny. Bjorklund and Pellegrini (2000) argue that an evolutionary perspective helps better understand "human ontogeny in contemporary society" while a developmental perspective helps comprehend evolutionary psychology.

### 4.2. Adult Psychology and Work

While play is essential for the development of a child, work serves the same or similar functionality for an adult from young adulthood in the second decade of life well into the third decade of life. The transition to adulthood has presented young people with complex intertwined paths of work and family alongside a range of institutional transitions, normative and developmental tasks, and demands (Evans & Heinz, 1994; Fleeson & Cantor, 1995; European Group for Integrated Social Research, 2001; Cohen et al., 2003; Arnett, 2004; Shulman & Nurmi, 2010).

Young adults mainly focus on work and personal life goals (Nurmi & Salmela-Aro, 2002; Roisman et al., 2004) which naturally progress to the setting of goals such as starting a family and having children (Salmela-Aro, Aunola, & Nurmi, 2007). In a recent five-wave longitudinal study, Upadyaya and Salmela-Aro (2017) investigated the cross-lagged associations between study or work engagement and life satisfaction in young adults. As part of the longitudinal Finnish Educational Transitions research, the principal investigators were observing 821 participants from age 17 through age 25 (from high school to higher education or work). It was found that gender and educational tracks did not have statistically significant effects on the developmental dynamics of life satisfaction and study or work engagement. Importantly, as demonstrated by the developmental dynamics model, psychological and emotional well-being and inner life satisfaction in young adults predicted their successful engagement in study or work both during the high school period of life and after the transition to higher education or work. There was also a significantly positive correlation between study or work engagement of young adults, especially males, and life satisfaction during their third decade of life. These results suggested that general wellbeing directly affects study and work variables and promotes positive personal development in adults (Upadyaya & Salmela-Aro, 2017; Symonds et al., 2019).

Employees with positive core self-evaluations, as part of general wellbeing, have higher levels of job satisfaction. Those satisfied with their jobs actively contribute to shared team and organizational goals through higher levels of job involvement, organizational commitment, employee engagement, and organizational citizenship behaviors (Podsakoff et al., 2000)—the very behaviors built on the fundamental collaborative skills.

## 5. Humanity and Collaboration (Socialization)

### 5.1. Child Collaboration

Human beings constantly engage in complex collaborative activities. By the first birthday, children demonstrate early attempts to collaborate and interact with peers and adults culturally: The skills that will be mastered throughout childhood (Tomasello et al., 2005). Based on shared goals and joint actions, collective (cooperative) and collaborative activities progress in complexity as the child grows, institutionalizes, and crystallizes into generational beliefs, values and norms.

Human beings socialize from a young age through individual and team sports, social interactions, and structured early childhood Montessori-type education that promote the development of social skills. From track and field, tennis and swimming, that promote individual achievement with a team component, to cricket, football, hockey, baseball, and basketball, team games that build various cooperative and collaborative skills. Human beings constantly socialize across continents, cultures and through history. In the context of team sports, researchers have examined in-group and out-group dynamics, shared identities, cultural affinity, building social networks, friendship, competition, collaboration, socialization, and other social variables and behaviors that emerge in early childhood and evolve through the life span (Wagg, 2005; Appadurai, 2015; Mercado & Bernthal, 2016; Thomas, 2020; Marqués-Sánchez et al., 2021).

Tomasello et al. (2005) argued that collaboration is a skill derived from socialization. We propose that collaboration is a human talent that is deployed through socialization. When adolescent children participate in track and field, they cooperate to pursue their personal goals. When adolescent children participate in team sports, they collaborate to plan and execute the pursuit of team (shared, or joint) goals. Therefore, team sports socialize them to collaborate later in the workplace.

### 5.2. Adult Collaboration

As discussed earlier, collaboration capacity undergoes transformation from the childhood through juvenile period and into adulthood. Team processes in personal and professional lives are a vivid demonstration of collaborative activities. Collaborative behaviors in adults are seen in intra-organizational and inter-organizational joint activities at all levels of power, across industries, across disciplines and professions, locally, regionally, and globally. The theory of collaborative advantage emerged in academia in the 1980s, aimed to explain a rapid increase in collaborations within countries and around the world (Bird & Osland, 2006; Ansell & Gash, 2008; Beamish & Lupton, 2009; Vangen & Huxham, 2013). There was a boom of collaborative activities: Collaborations came in different types and sizes, from "dyads to international networks" from the private to the "almost every aspect of the public and not-for-profit sectors" (Vangen & Huxham, 2013). Social changes and a new economic order created an environment where collaborative advantage was derived from the "integrated process of collaboration between policymakers, business and society" (Johnsen & Ennals, 2011).

In our contemporary world, business operations and social settings are comprised of a multitude of teams: Problem-solving, self-managed, cross-functional, virtual, and other types. These adult teams have familiar and similar, if not the same, processes and composition variables already learned in childhood in the framework of contextual factors such as cooperation, rewards, trust, leadership, and resource allocation. Johnsen and Ennals (2011) argue that collaboration is a soft skill like networking, social learning, and partnership building: They have a common denominator that relates "to the social, interactional and relational side of social and economic activity". Collective foraging gave human beings an "apprentice learning model" of collaboration (Tomasello, 2009; Sterelny, 2011, 2012) in which a wealth of knowledge and experiences was passed from elder members of a group or tribe to the younger members. Globalization gave human beings a teamwork model of collaboration which generates positive synergy through coordinated efforts.

## 6. Economic Growth through Collaboration

Economic development is the process by which economies advance. Ridley (2021) explains the process by which the overall health, well-being, and academic level of the general population improves through collaboration. Consideration is given to both economic and social conditions. Aggregate living standard (material wellbeing) is more narrowly focused and is measured by real per capita GDP adjusted for purchasing power parity (GDPppp). GDP is the sum of all economic activity in a nation over a specified period. It is the value of all the products and services that an economy produces. Economic growth is the process by which low living standards become high living standards by expanding the size of the economy, albeit without particular regard to wealth distribution. But wealth does not just exist and cannot therefore be distributed. Any governmental injunction to distribute wealth is meaningless. It has no positive impact on economic growth and is not required for it to occur. The human being is naturally impecunious, so wealth has to be created. The way that wealth is held across the population is a worthwhile consideration for development but there will be no development and no wealth for anybody if there is no GDP. Therefore, in this research we consider the primary component of economic development to be economic growth. We are interested to know the role of collaboration in economic growth. In the process of this investigation, it is observed that certain statistically distributional features of wealth appear to be inevitable. And while wealth might be unlimited, and the masses of the population that are the least wealthy amongst us can have a comfortable living standard, there will always be what is described as the few wealthy at the top. Acceptance of this counter intuitive ontological disposition might go a long way to reducing angst and social disorder. The most talented and least talented can collaborate to design and build labor saving devices that are safe and that elevate the least amongst us. This is not only inevitable it is required for economic growth.

### 6.1. Innovation as a Proxy for Collaboration

There are no published data for collaboration by country. The closest data that can be used as a proxy for collaboration is innovation. The use of innovation data is supported by noted commercial practices in which collaboration and innovation are used interchangeably. The following are anecdotal and serve only as examples.

"Using a highly collaborative approach to drive innovation is exactly what Procter & Gamble used in 2000 when A.G. Lafley took the helm as CEO. Lafley took a huge leap forward and made a strategic bet by turning to a collaborative approach for driving innovation. This concept came to be known as 'Connect & Develop' and his goal was, 'Half of our new products would come from our own labs, and half would come through them.' Lafley's bet paid off and spurred a huge wave of innovations that came from collaborations with 'outsiders.' A good example was the Swiffer, a highly successful brand of cleaning supplies. Looking for opportunities to expand the line, the P & G team had worked on a handheld dusting tool, but without much success. On a trip to Japan, the R & D leader for Home Care found the answer in the cubicle of a P & G employee: a sleek, user-friendly handheld duster that was better than the products P & G was testing. Its curly fiber captured dust, dirt, and hair far better than anything P & G had come up with, but it was owned by Japan's UniCharm. The problem turned into a collaborative opportunity when P & G bought the rights to UniCharm's duster outside of Japan. A win for P & G and a win for UniCharm. The Swiffer Duster was an instant success: in the first four months, it cleaned up $100 million in sales."

In a second example, "Hastings and Meyer (2020) tell the Netflix secret to success. How to maximize innovation, risk taking and collaboration. Netflix cofounder Reed Hastings reveals for the first time the unorthodox culture behind one of the world's most innovative, imaginative, and successful companies. They tap into the creativity of employees at all levels." See also the Ridley (2017) micro intrapreneurship proposal for tapping into creativity of low-level employees.

### 6.2. Innovation Index and Gross Domestic Product

We are interested in how economic growth is related to collaboration. Since there are no available data for collaboration by country, we use innovation as the closest available data. We obtained the year 2014 GDPppp reported by the IMF (http://www.imf.org/external/data.htm) and the global innovation index (GII) from the world intellectual property organization (WIPO) (Indicator Rankings & Analysis/Global Innovation Index). GDPppp measures standard of living. The GII comprises an innovation input sub-index and an innovation output sub-index. The innovation input sub-index comprises institutions, human capital and research, infrastructure, market sophistication and business sophistication. The innovation output sub-index comprises knowledge and technology outputs and creative outputs.

These data are for 79 countries for which all data are available. The total population size in these countries represents almost all people in the world for whom all listed data are reported. The remaining countries have populations less than one million and/or do not provide all data. The data are listed in Table 1 and plotted in Figure 1. From the graph we see that GDPppp is highly positively correlated with innovation. As innovation declines so does GDPppp. The innovation values are relative in rank but have no meaning as an absolute measure. There are no innovation values below 24.4 (see Bangladesh in Table 1). The least squares fit to this line is GDPppp = −30,627.3 + 1335.6 GII. At an innovation rank of 22.9, the GDPppp estimate is approximately zero. If innovation could theoretically go below 22.9 and down to zero, GDPppp would be negative and wealth that is subject to depreciation and obsolescence would decline. The actual correlation coefficient is 0.85. GII is a reflection of human ideas of imagination and creativity and is therefore an exogenous variable, expected to yield unbiased estimates of the regression coefficients.

### 6.3. Innovation Index and Market Capitalization

Innovation is an abstract terminology the automatic operationalization of which is not obvious. We are interested in the pathway by which innovation leads to GDPppp. To uncover that, we consider the relationship between innovation and

**Table 1.** Real per capita GDPppp by country (2014), Global innovation index (GII) and Market Capitalization.

| Country | Real per capita GDPppp | Global Innovation Index (GII) | Market Capitalization Per capita | LN Market Capitalization | Country | Real per capita GDPppp | Global Innovation Index (GII) | Market Capitalization Per capita | LN Market Capitalization |
|---|---|---|---|---|---|---|---|---|---|
| Argentina | 22,302 | 35.1 | 580 | 6.364 | Latvia | 23,793 | 44.8 | 566 | 6.338 |
| Armenia | 8164 | 36.1 | 44 | 3.786 | Lebanon | 18,052 | 33.6 | 1751 | 7.468 |
| Australia | 46,550 | 55 | 53,584 | 10.889 | Lithuania | 27,259 | 41 | 1372 | 7.224 |
| Austria | 46,640 | 53.4 | 12,189 | 9.408 | Macedonia | 13,398 | 25.5 | 270 | 5.599 |
| Bangladesh | 3391 | 24.4 | 166 | 5.110 | Malawi | 1112 | 27.6 | 45 | 3.801 |
| Belgium | 43,139 | 51.7 | 26,540 | 10.186 | Malaysia | 25,145 | 45.6 | 15,431 | 9.644 |
| Bolivia | 6224 | 27.8 | 405 | 6.003 | Mauritius | 18,689 | 40.9 | 5686 | 8.646 |
| Botswana | 17,050 | 30.9 | 2142 | 7.670 | Mexico | 17,950 | 36 | 4294 | 8.365 |
| Brazil | 16,155 | 36.3 | 5979 | 8.696 | Mongolia | 11,919 | 37.5 | 421 | 6.043 |
| Bulgaria | 17,926 | 40.7 | 920 | 6.824 | Morocco | 7813 | 32.2 | 1574 | 7.362 |
| Canada | 44,967 | 56.1 | 56,026 | 10.933 | Namibia | 10,656 | 28.5 | 561 | 6.329 |
| Chile | 23,057 | 40.6 | 17,223 | 9.754 | Netherlands | 47,960 | 60.6 | 38,314 | 10.553 |
| China | 13,224 | 46.6 | 2689 | 7.897 | Nigeria | 6054 | 27.8 | 302 | 5.709 |

**Continued**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Colombia | 13,480 | 35.5 | 5400 | 8.594 | Norway | 67,166 | 55.6 | 48,514 | 10.790 |
| Cote d'Ivoire | 3101 | 27 | 357 | 5.879 | Oman | 43,847 | 33.9 | 6930 | 8.844 |
| Croatia | 20,947 | 40.7 | 5095 | 8.536 | Panama | 19,546 | 38.3 | 3288 | 8.098 |
| Denmark | 44,625 | 57.5 | 39,398 | 10.581 | Peru | 11,860 | 34.7 | 3259 | 8.089 |
| Dominican Republic | 14,014 | 32.3 | 14 | 2.632 | Philippines | 6974 | 29.9 | 2568 | 7.851 |
| Egypt | 10,918 | 30 | 654 | 6.483 | Poland | 25,247 | 40.6 | 4618 | 8.438 |
| El Salvador | 8060 | 29.1 | 1648 | 7.407 | Portugal | 27,069 | 45.6 | 6315 | 8.751 |
| Estonia | 27,880 | 51.5 | 1778 | 7.483 | Romania | 19,744 | 38.1 | 799 | 6.684 |
| Finland | 40,661 | 60.7 | 28,851 | 10.270 | Russia | 24,449 | 39.1 | 5970 | 8.694 |
| France | 40,538 | 52.2 | 28,263 | 10.249 | Saudi Arabia | 52,311 | 41.6 | 11,578 | 9.357 |
| Germany | 46,216 | 56 | 18,246 | 9.812 | Serbia | 13,378 | 35.9 | 1047 | 6.954 |
| Ghana | 4137 | 30.3 | 114 | 4.735 | Singapore | 83,066 | 59.2 | 74,820 | 11.223 |
| Greece | 25,954 | 38.9 | 4137 | 8.328 | Slovakia | 28,279 | 41.9 | 850 | 6.745 |
| Hungary | 25,019 | 44.6 | 2108 | 7.653 | Slovenia | 29,867 | 47.2 | 3128 | 8.048 |
| India | 5808 | 33.7 | 983 | 6.891 | South Africa | 13,094 | 38.2 | 11,142 | 9.318 |
| Indonesia | 10,651 | 31.8 | 1534 | 7.335 | Spain | 33,835 | 49.3 | 21,435 | 9.973 |
| Iran | 17,443 | 26.1 | 1782 | 7.485 | Sweden | 46,219 | 62.3 | 56,900 | 10.949 |
| Ireland | 51,284 | 56.7 | 23,518 | 10.066 | Switzerland | 58,149 | 64.8 | 129,905 | 11.775 |
| Israel | 33,136 | 55.5 | 17,538 | 9.772 | Thailand | 15,579 | 39.3 | 5870 | 8.678 |
| Italy | 35,131 | 45.7 | 7918 | 8.977 | Trinidad and Tobago | 32,170 | 31.6 | 11,236 | 9.327 |
| Jamaica | 8610 | 32.4 | 2347 | 7.761 | Turkey | 19,698 | 38.2 | 3921 | 8.274 |
| Japan | 37,519 | 52.4 | 29,028 | 10.276 | Uganda | 1939 | 31.1 | 209 | 5.344 |
| Jordan | 11,971 | 36.2 | 2829 | 7.948 | Ukraine | 8681 | 36.3 | 484 | 6.182 |
| Kazakstan | 24,108 | 32.8 | 1332 | 7.195 | United Kingdom | 39,826 | 62.4 | 46,384 | 10.745 |
| Kenya | 3099 | 31.9 | 313 | 5.745 | United States | 54,370 | 60.1 | 57,812 | 10.965 |
| Korea, South | 34,355 | 55.3 | 22,903 | 10.039 | Vietnam | 5656 | 34.9 | 355 | 5.872 |
| Kyrgyzstan | 3262 | 27.8 | 27 | 3.313 | | | | | |
| | | | | | | | | Mean | 8.025 |
| | | | | | | | | Standard deviation | 2.007 |
| | | | | | | | | Skewness($S$) | −0.397 |
| | | | | | | | | Kurtosis ($K$) | 2.798 |
| | | | | | | | | Jarque Bera (JB) | 2.208 |

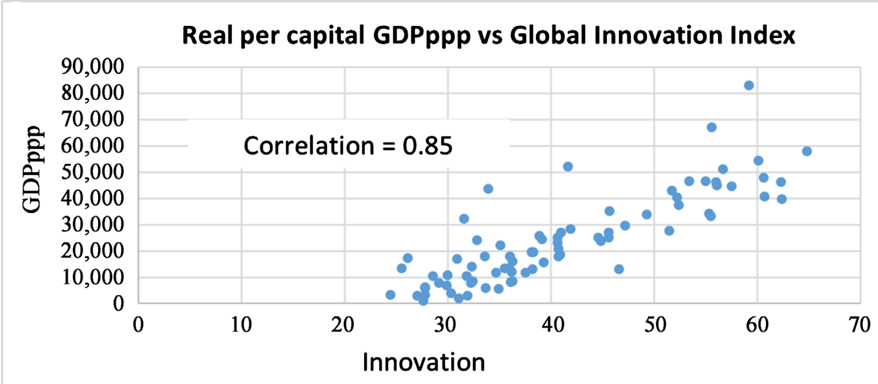


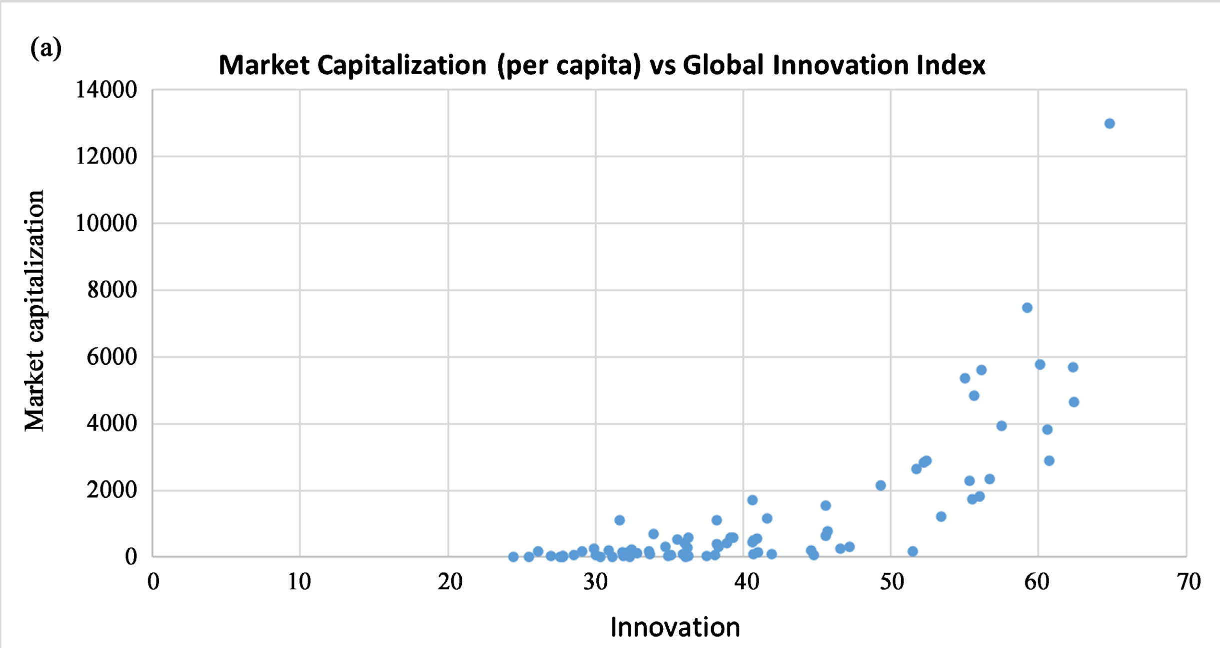

**Figure 1.** Real per capita GDPppp vs Global innovation index (GII).

capitalization. We define capitalization as total market capitalization. It measures capitalism, the degree to which capital in a country is organized for profitable investment. It includes exogenous ex nihilo human ideas of imagination and creativity, and endogenous capital stock of knowledge, computers and software, recordings, patents, skills, etc., and fixed capital of plant and machinery. Exogenous capital that begins in the human mind is converted into endogenous capital stock. We expect that collective learning with knowledge acquisition and accumulation will outlive its creators (Christian, 2012, 2014). Nevertheless, capital stock is subject to depreciation and obsolescence. Therefore, the only source of wealth is human capital ideas of imagination and creativity. Capitalization (US$ mundi) was obtained from http://www.indexmundi.com/facts/indicators/CM.MKT.LCAP.CD/rankings and listed in Table 1. From Figure 2(a), we see that as innovation increases, capital explodes exponentially. Taking Napierian logarithms (Figure 2(b)) results in a linear relationship and a high correlation of 0.8. The least squares fit to this line is LN Market Cap = −14.8 + 6.2 LN GII and Market Cap = $3.74 \times 10^{-7}$ $\text{GII}^{6.2}$.

### 6.4. Market Capitalization and Gross Domestic Product

Next, we consider the relationship between capitalization and GDPppp. From Figure 3(a), we see that as capitalization increases, GDPppp increases with diminishing returns. Taking Napierian logarithms (Figure 3(b)) results in a linear relationship and a high positive correlation of 0.82. The least squares fit to this line is LN GDPppp = 6.9 + 0.35 LN Market Cap, and GDPppp = 992 Market $\text{Cap}^{0.35}$. In addition to exogenous human ideas, Market Cap contains endogenous capital stock and therefore cannot be expected to yield unbiased estimates of the regression coefficients. Still, our objective is not coefficient precision, but the efficiency of Market Cap to account for GDPppp.

So, we see that innovation is positively correlated with total capital and capital is deployed for the generation of gross domestic product. That is, the path to economic growth is Innovation → Capital → GDPppp or Collaboration → Capital → GDPppp. We know from Ridley (2020a, 2020b) that this is facilitated by



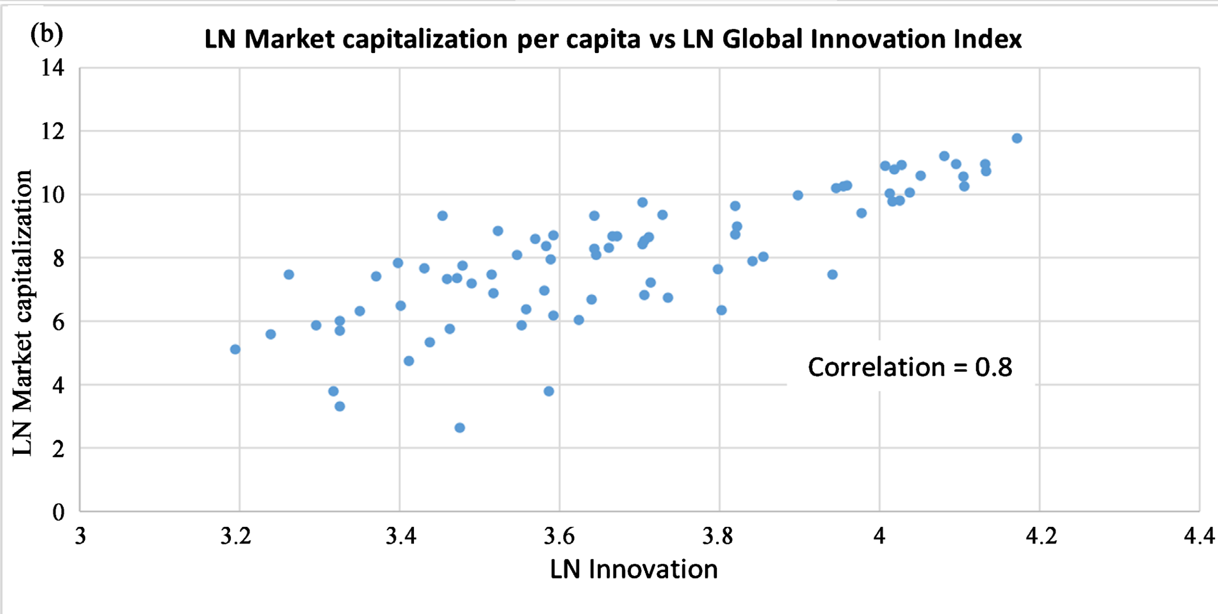




**Figure 2.** (a) Market capitalization vs. innovation; (b) LN Market capitalization vs. LN Innovation.

the catalysts democracy and rule of law. See also the Acemoglu et al. (2019) suggestion that democracy does cause growth. Rule of law is the reverse of corruption that is responsible for capital insecurity. Rule of law provides the stability that attracts capital and democracy provides additional pathways for its optimal deployment (Ridley, 2020a, 2020b, Ridley & Nelson, 2022). The number of pathways increases exponentially. For $n$ persons the number of pathways increases according to the square of $n$, calculated from $n(n-1) = n^2 - n$. For a mere 10 persons there are as many as $10^2 - 10 = 90$ pathways. A genius who does not collaborate must function alone with the whole world resting on his shoulders.

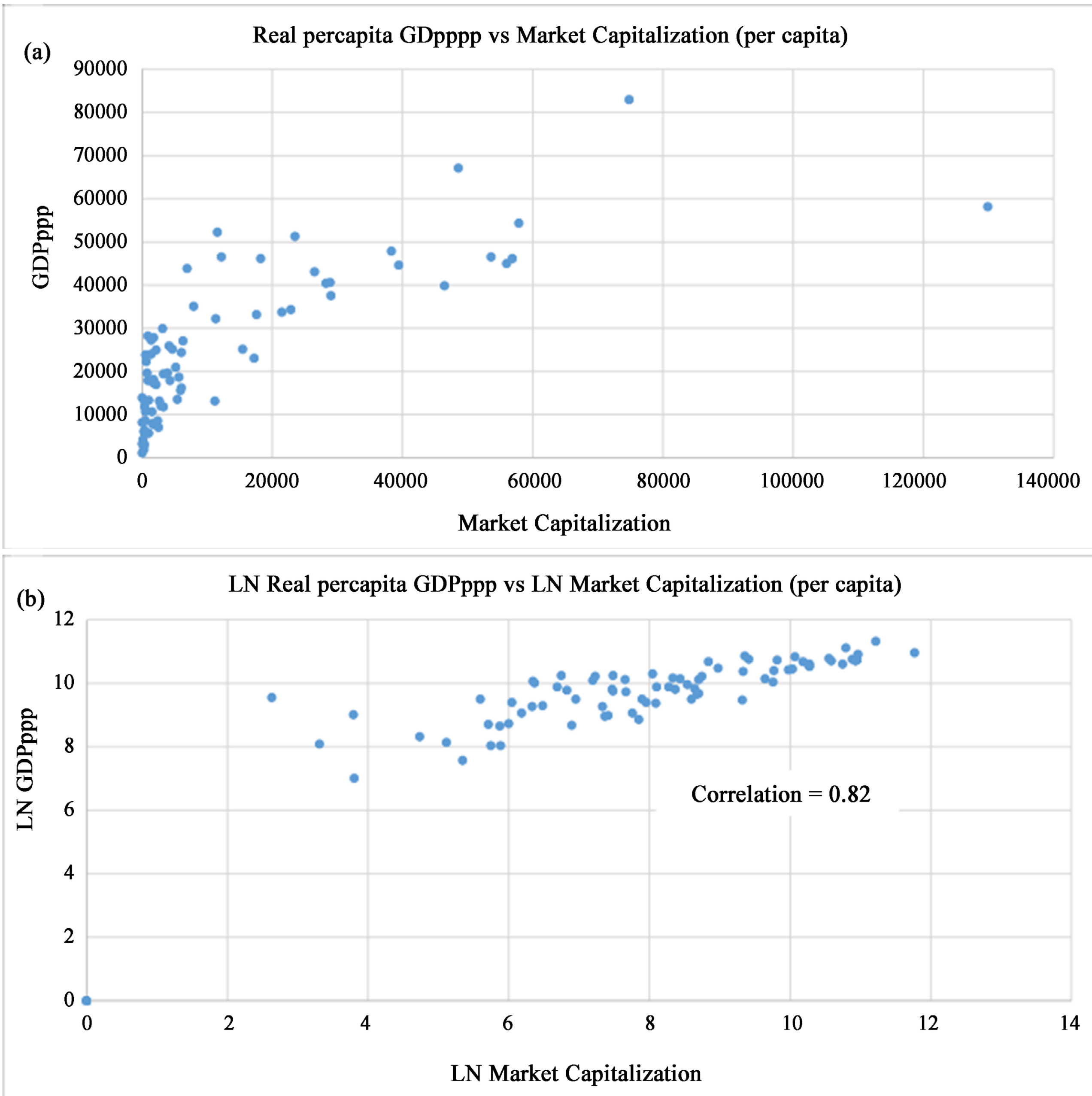



Figure 3. (a) Market capitalization vs. innovation; (b) LN Market capitalization vs. LN Innovation

This may be too much even for a genius. Collaboration as measured here by the innovation index is the way that business operationalizes capital for economic growth.

So, let us look at how capital is statistically distributed across countries. Figure 4(a) shows that the distribution of capital is exponential (approximately lognormal). This is consistent with the Pareto (1906) principle that a minority (~20 percent) possess the majority (~80 percent) of the available capital. The precise number obtained from these data is 75%. If the remaining unreported countries of the population were included, it is possible that the number could

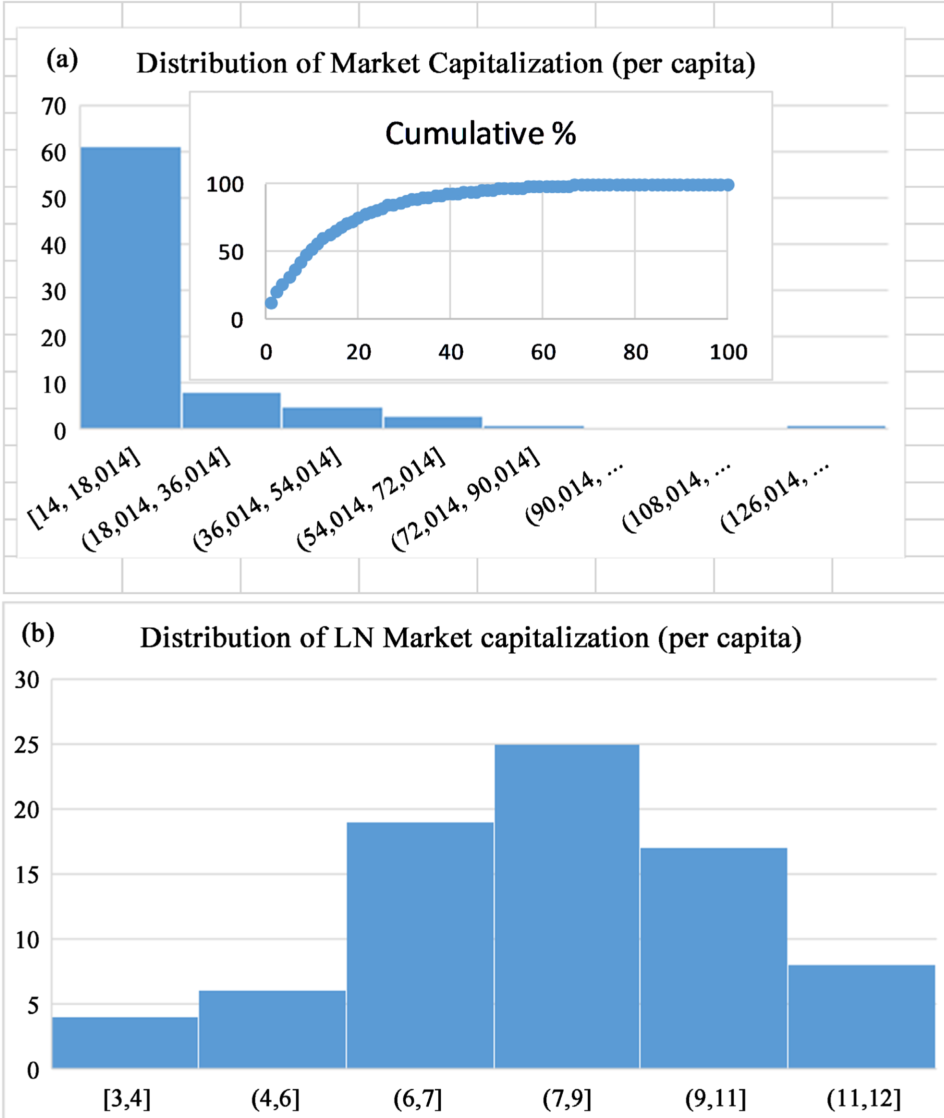



**Figure 4.** (a) Global distribution of Market cap; (b) Global distribution of LN Market cap.

be 80%. As this capital is deployed to generate GDPppp that contributes to wealth, the Pareto principle applies also to wealth. There is no requirement that the 20% always comprise the same people. Taking Napierian logarithms, Figure 4(b) shows a symmetric pattern associated with the normal distribution. That is, the log symmetry that is the thesis of this paper.

To test the hypothesis that LN Market Capitalization is normally distributed, consider the hypotheses:

$H_0$: LN Market Capitalization is normally distributed.

$H_1$: LN Market Capitalization is not normally distributed.

From Table 1, the LN Market capitalization statistics are: Average = 8.025, Standard deviation = 2.007, Skewness ($S$) = −0.397, Kurtosis ($K$) = 2.798, and the Jarque Bera (1980, 1987) test statistic $JB = (n/6)(S^2 + (1/4)(K - 3)^2) = (79/6)(-0.397^2 + (1/4)(2.798 - 3)^2) = 2.208$. The theoretical JB statistic follows a Chi square distribution. With a 5% level of significance and 2 degrees of freedom, Chi square = 5.99. Since JB = 2.208 < 5.99, we fail to reject $H_0$ and accept that LN Market capitalization is normally distributed.

The Pareto principle is pervasive in human activity. And the normal distribution is just as pervasive. There is no ignoring or getting around these two facts. If it were just personal income that is Pareto distributed one might argue the possibility of ill begotten gains along pathways of dishonest collaboration. But we are seeing the distribution in capitalization and capital is innately human. The starting point of collaboration. The conceptual desire for equality is also pervasive but it does not occur anywhere in practice. The implication of this is that to raise the wealth of the least among us, the rich must become even richer. It is imperative that there be free market economies. Freedom to organize capital maximally. Ridley (2020a, 2020b) showed that GDPppp can be explained almost entirely (90%) by the policy variables; capitalism, democracy and rule of law, when combined with the physical variables; natural resources and geographic latitude. Of this 90%, capitalization constitutes 60%. Natural resources and geography contribute only 6% and 4% respectively and are not adjustable. So, capitalization is crucial to economic growth and development. Without it very little economic growth can occur, and the members of the economy will remain in their natural state as peasants.

All economics begin at the microeconomic level. One person at a time, then in subgroups of human beings. We are investigating the mechanics of collaboration, and all collaboration occurs at the microscopic level. If capital were exponentially distributed (**Figure 4(a)**) within small economic enclaves, we would be interested to know how such multimodal exponentials can combine via arithmetic or geometric averages into a single unimodal exponential distribution. In that case the sum of exponential distributions is Gamma distributed. When the means are equal it becomes Erlang distributed. There are a wide range of means and shape factors for which these approximate an exponential or lognormal distribution (see the Appendix). In either case their logarithm is symmetric, resembling a normal distribution. The products of exponentials are lognormal.

At the time when human ideas of imagination and creativity emerge, they might be normally distributed across individual human beings. There is evidence for this in normal SAT scores, multimodal AP test scores and the empirical results from Ridley, Ngnepieba and de Silva (2021). As collaboration occurs, capitalization grows. As capital is reinvested the ensemble will grow exponentially from the microscopic to the mesoscopic to the macroscopic, and from normally distributed to exponentially distributed. Ridley (2020a: p. 12) contains the seminal presentation of a CDR econometric model for the a priori computation of world average endogenous growth. The estimate reported there is 1.8%. We mention in passing an interesting observation that this equates to $\frac{1}{4}e^2$, where $e$ is Napier's constant (Euler's number) and the base for the natural logarithm. If LN market capital is a normally distributed random variable denoted by $X$ (**Figure 4(b)**) then market capital will be exponentially distributed: $e^X$ (**Figure 4(a)**). This natural phenomenon is observed throughout economies and elsewhere in the physical world. If capital were normally distributed (**Figure 4(b)**), we would be interested in the arithmetic and geometric averages of normal distributions. The below section on mathematical integration and symmetry explains ways in which any number of multimodal distributions can combine to produce a unimodal distribution. The way to accomplish this in practice is through collaboration.

### 6.5. No Uniformity

Inequality has always been a subject of inquiry in the fields of economics and sociology. There is a large body of literature about the role that human capital plays in the intergenerational transmission of inequality in income and wealth (e.g. Becker & Tomes, 1979; Becker & Tomes, 1986; Becker, 1993; Mayer & Lopoo, 2005; Solon, 1999; Sun, 2020). Some researchers (e.g. Bourguignon, 1981; Chatterjee, 1994; Caselli & Ventura, 2000; Li, Xie, & Zou 2000) argue that if the market is perfect and there are no personal differences and no continuous random shocks, the inequality in income and wealth will eventually disappear over time. See also Galor and Tsiddon (1997) on the distribution of human capital and economic growth. But assumptions of uniformity and personnel equality are unrealistic. And if there were no random shocks then there would be no innovations and no economic growth. Subject to depreciation and obsolescence all capital would converge to zero. The reality is that capital is Pareto distributed.

Social-minded equalitarians may abhor the notion of the rich getting richer in order to improve the lot of the masses and that this must result from deliberate collaboration. But they can take solace in the fact that none of this occurs at the expense of the poor. Instead, it alleviates poverty. The Pareto 20% does not always have to be same people. Mobility is permitted and does in fact occur. Before the invention of capitalism (about the turn of the 19th century and the industrial revolution), apart from feudal lords, beneficiaries of the 17th century Amsterdam stock exchange, the Dutch and English East India Companies, and certain skilled artisans, all people were impecunious. The wealth created from capitalization occurs via entrepreneurship. Entrepreneurs create novel products and services that improve the lives of all. They take risks, invest, and dedicate many waking hours to creating a better world. This is especially true in the case of labor-saving devices. Entrepreneurs give up their leisure time to create leisure time for others. Then there are the never-ending strategies and tactics to reduce cost so as to make innovations affordable for the common man. If anywhere, somebody produces products at a lower price with the same quality or produces better quality for the same price, the total economic pie must increase for all to benefit. Furthermore, if wealth comes from ideas, then since ideas are unlimited, wealth is unlimited. The marginal benefits are greatest to those with the least who might otherwise toil the most. The marginal benefits are least to those with the most. The rich can only consume but so much. They can only live in one house at a time, sleep in one bed at a time, drive one car at a time, eat one full complement of meals per day, etc. Instead of focusing on equality of income, equalitarians might consider equality of consumption. Entrepreneurs are our creator's gift to mankind and entrepreneurship is an act of giving. Entrepreneurship has amassed enormous wealth for the capitalist, but it has also generated philanthropy by virtue of companies that create and manage entities that are dedicated to raising the human condition. Said philanthropy being greater than that of any poor society or any government. Anything done to obstruct the entrepreneur risks our return to mass poverty.

## 7. Mathematical Integration and Symmetry

The above empirical observations are most compelling. Let us now look at a theoretical construction that will help us complete our understanding. Ridley, Ngnepieba and de Silva (2021) show how theoretical distributions of two subpopulations of normal distributions can combine into unimodal normal distributions. Then, they show how an active learning teaching method can combine subpopulations of university calculus test scores to produce unimodal normal distributions. In this research we extend the analysis into multimodal distributions of several subpopulations. This is more like what we would expect to find in a national economy.

Consider $N$ subpopulations of human beings $X_1, X_2, \cdots, X_N$, where the subscripts represent population number, and their subpopulation size is M. The subpopulations are classified according to a particular one of $N$ different human skills that is possessed by the people in their subpopulation. The skill is uniquely characteristic to them. The level of each skill is distributed normally over the subpopulation to which it belongs. Let the concatenation of the distributions be denoted by $X = X_1 \mathbin{+\!\!\!+} X_2 \mathbin{+\!\!\!+} \cdots \mathbin{+\!\!\!+} X_N$. The multimodal distribution is depicted in **Figure 5**. When individual people in one subpopulation work alone, independently of each other, there is no strengthening of the force from applying their related skill. If the members of the subpopulation work cooperatively, the combined force could be as much as $M$ times. If the skills in all subpopulations are applied cooperatively, the combined force could be as much as $N \times M$.

There are many mathematical formulations that perform in parallel with the real world that they are intended to represent. They are applied to science and engineering to gain insight, and to design devices and systems. Consider the mathematical modeling of the multimodal distributions as a collection of normal

**Figure 5.** Concatenation $X = X_1 \mathbin{+\!\!\!+} X_2 \mathbin{+\!\!\!+} \cdots \mathbin{+\!\!\!+} X_N$.

subpopulations. Consider two ways in which the subpopulations can combine to reflect collaboration between the members of the subpopulations. One way is the arithmetic average of forces applied through collaboration. The arithmetic average $X = (X_1 + X_2 + \cdots + X_N)/N$ is depicted in the 10,000 random number computer simulation in **Figure 6**. It is well known from the central limit theorem that the sum of normal distributions is normally distributed (see also de Moivre, 1738 and the Appendix below). Furthermore, even if the component distributions are not normal the sum approaches a normal distribution for large sample size. Another way of combining is the geometric average of forces applied through collaboration. The geometric average $X = (X_1 X_2 \cdots X_N)^{1/N}$ is depicted in the computer simulation in **Figure 7**. The geometric average involves the

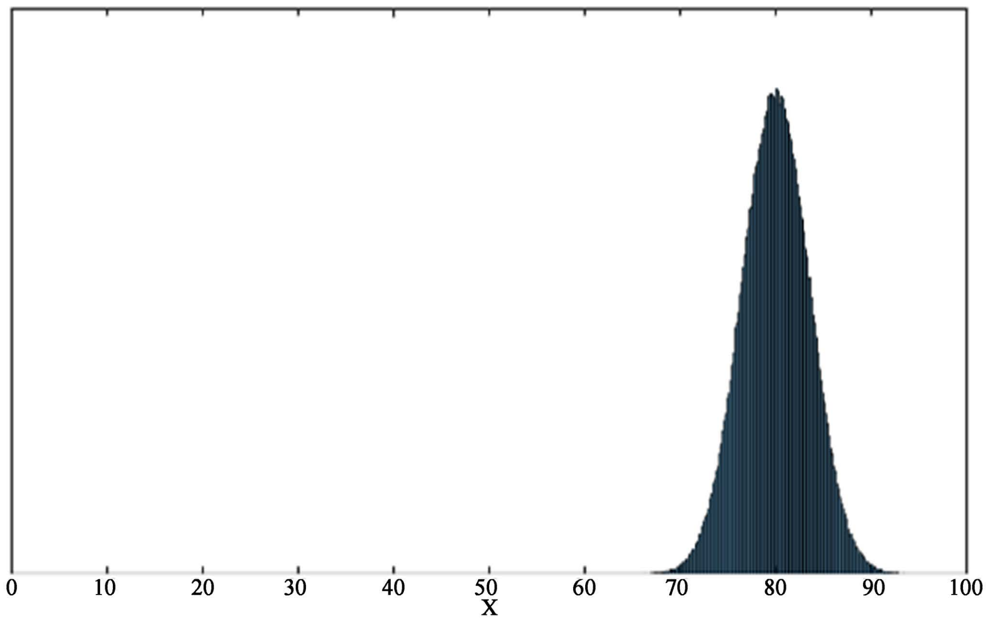



**Figure 6.** Arithmetic average $X = (X_1 + X_2 + \cdots + X_N)/N$.

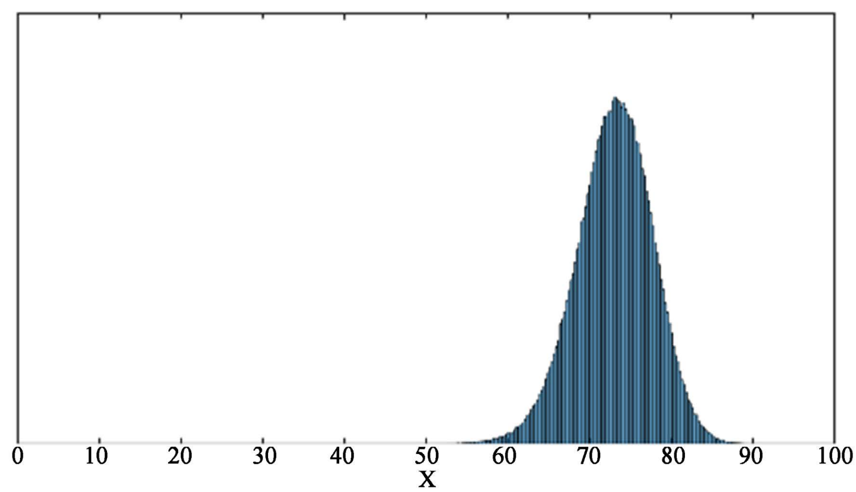



**Figure 7.** Geometric average $X = (X_1 X_2 \cdots X_N)^{1/N}$.

products of several normal distributions. A distribution of normal product is given in the Appendix below. It is shown there that the normal product is symmetrical. And as the product of inverse coefficients of determination is greater than one ($\prod_{i=1}^{N} \frac{\mu_i}{\sigma_i} > 1$), the distribution is normal. The depictions in **Figure 6** and **Figure 7** are computer simulations. Clearly, they appear normally distributed. In terms of *X*, the total force from either method of combining all subpopulations is *NX*.

## 8. Conclusion

We know from Ridley (2020a, 2020b) that intangible total market capitalization, democracy and rule of law together with natural resources and geographic latitude explain almost 90% of GDPppp. The source of wealth is human capital ideas of imagination and creativity. Natural resources and geography have a small effect on GDPppp. Government spending, country size, location, culture and population physical characteristics have negligible effect on GDPppp. We also know from Ridley, Ngnepieba and de Silva (2021) that learning that takes place in a collaborative mode produces university course grades that are higher and that are symmetric normally distributed. They argue that normally distributed grades are the signal that learning is optimized, ceteris paribus. The purpose of this research is to extend the investigation into the relationship between collaboration and economic growth.

There are no published data for collaboration, so innovation is used as a proxy for collaboration. The two are often used interchangeably. Still, innovation by itself is not observable. However, this study shows that capitalization is positively correlated with innovation and capitalization is observable. Capitalization is the measure of how much a country is able to organize capital. This is distinctly different from capital that is disorganized regardless of the amount of capital. We show that GDPppp is positively correlated with capitalization and therefore collaboration. We show that capitalization is Pareto distributed and the logarithm of capitalization is symmetric normally distributed. That is, the log symmetry that is the thesis of this paper. Annual GDPppp as used here is one year's contribution to wealth. Therefore, the strategy for wealth creation and higher standard of living is the organization of capital through collaboration. The Pareto distribution of capitalization implies that to raise the living standard of the least among us, the rich must become even richer. Implicit in human capital is the notion of human intelligence. However, it is the organization of said capital through collaboration that is responsible for extraordinary economic growth. Because human ideas are unlimited, wealth is unlimited. Future research might include ways to teach acceptance of the reality of the income imbalance dilemma, and then identify mechanisms for increasing global collaboration for the advancement of economic growth with the objective being to maximize total wealth and the amount available to the least among us. This paper focuses on how human beings collaborate for the benefit of economic growth. A suggestion for future research is on collaboration for the benefit of the economic development of people.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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## Appendix

**The arithmetic exponential average**

If $X_1$ and $X_2$ are independent exponential random variables with rate parameters $\mu_1$ and $\mu_2$ respectively, then the probability density of $Z = X_1 + X_2$ is given by (the shape will be the same as for the average $0.5(X_1 + X_2)$)

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(z - x_1) dx_1$$

$$f_Z(z) = \int_0^z \mu_1 e^{-\mu_1 x_1} \mu_2 e^{-\mu_2(z - x_1)} dx_1$$

$$f_Z(z) = \mu_1 \mu_2 e^{-\mu_2 z} \int_0^z e^{(\mu_2 - \mu_1)x_1} dx_1$$

$$f_Z(z) = \begin{cases} \dfrac{\mu_1\mu_2}{\mu_2 - \mu_1}\left(e^{-\mu_1 z} - e^{-\mu_2 z}\right) & \text{if } \mu_1 \neq \mu_2 \\ \mu^2 z e^{\mu z} & \text{if } \mu_1 = \mu_2 = \mu \end{cases}$$

This is a Gamma distribution (**Figure 8**). When $\mu_1 = \mu_2$ it becomes an Erlang 2 distribution which can be extended to $N$ sums and an Erlang $N$ distribution. **Figure 9** shows a computer simulation for the arithmetic average of exponentials. These distributions are similar to the lognormal in shape. Whether exponential or lognormal, when the logarithm is taken the result is the symmetric distribution resembling the normal distribution (see **Figure 4(a)**, **Figure 4(b)**).

**The geometric exponential average**

If $X_1$ and $X_2$ are independent exponential random variables with rate parameters $\mu_1$ and $\mu_2$ respectively, then the probability density of $Z = X_1 X_2$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(z/x_1) \frac{1}{|X_1|} dx_1$$

$$f_Z(z) = \int_0^z \mu_1 e^{-\mu_1 x_1} \mu_2 e^{-\mu_2(z/x_1)} \frac{1}{|X_1|} dx_1$$

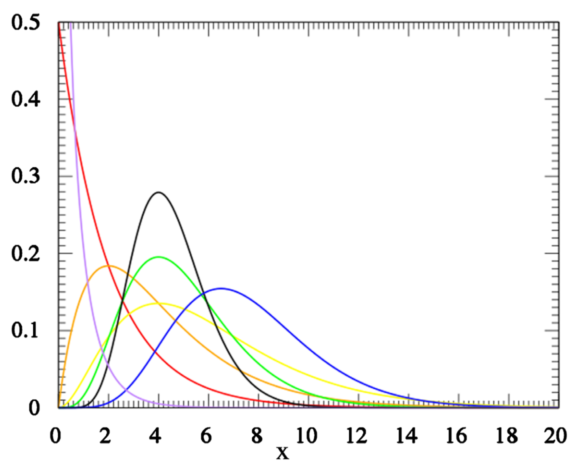


**Figure 8.** Gamma distribution of the arithmetic average of two exponentials for various shape and scale parameters.

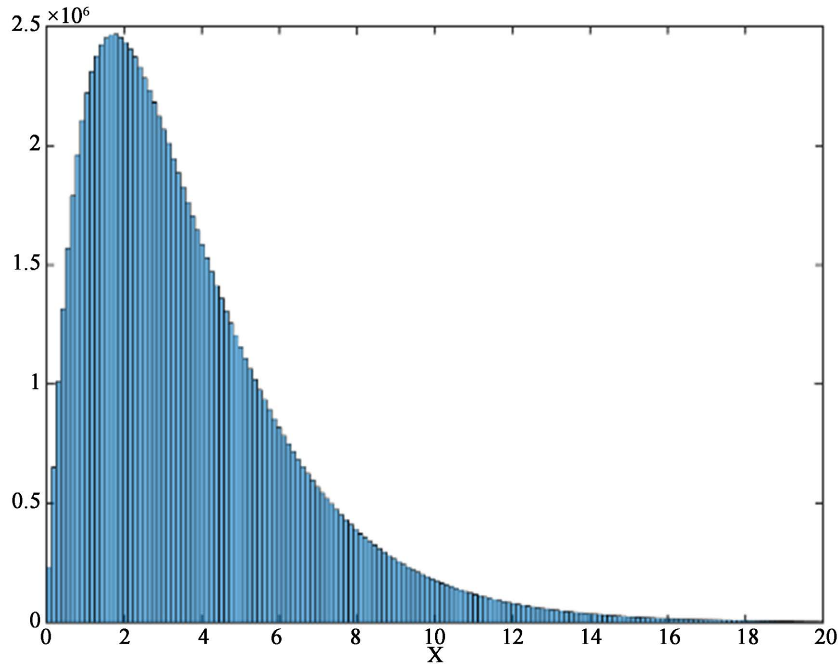


**Figure 9.** Computer simulation of the arithmetic average of two exponentials with rate parameters 1/2.5 and 1/5.

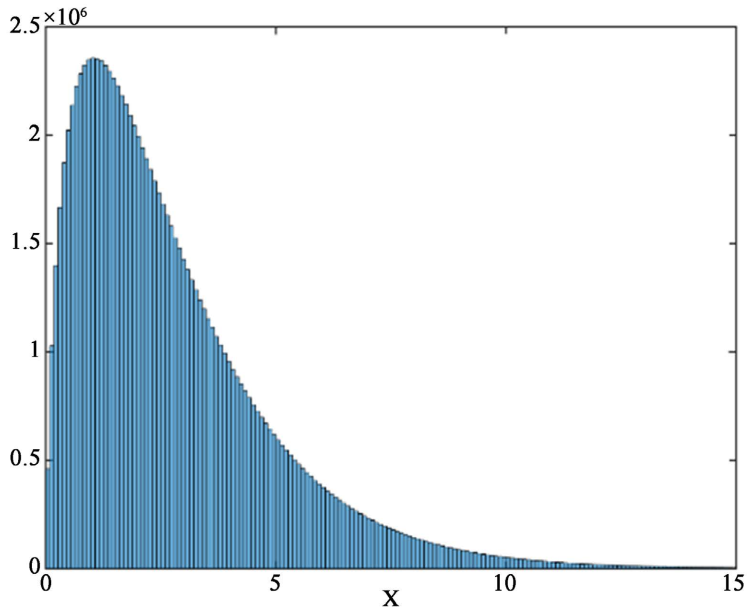


**Figure 10.** Computer simulation of the geometric average of two exponentials with rate parameters 1/2.5 and 1/5.

$$f_Z\left(z\right) = \mu_1\mu_2\int_0^z e^{-\mu_1 x_1} e^{-\mu_2 (z/x_1)} \frac{1}{|X_1|} dx_1 .$$

A computer simulation for the geometric average of two exponentials with rate parameters 1/2.5 and 1/5 is given in **Figure 10**. It appears lognormal and is similar to an exponential.

**The arithmetic normal average**

If $X_i, i = 1, 2, \cdots, N$ are independent normally distributed $\sim \mathbb{N}\left(\mu_i, \sigma_{x_i}^2\right)$, then it is well known that the population arithmetic average $\sum_1^N X_i / N$ is normally distributed $\sim \mathbb{N}\left(\sum_1^N X_i / N, \sum_1^N \sigma_{x_i}^2 / N^2\right)$.

**The normal product**

We know from Weisstein (2020) that if $X_1$ and $X_2$ are normally distributed, with zero means $\mu_1$ and $\mu_2$, and variances $\sigma_{X_1}^2$ and $\sigma_{X_2}^2$ respectively, the product $X_1X_2$ follows a normal product distribution with density

$$P_{X_1,X_2}(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-x_1^2 / \mathrm{e}^{2\sigma_{x_1}^2}}}{\sigma_{x_1}\sqrt{2\pi}} \frac{\mathrm{e}^{-x_2^2 / \mathrm{e}^{2\sigma_{x_2}^2}}}{\sigma_{x_2}\sqrt{2\pi}} \delta(x_1x_2 - u)\mathrm{d}x_1\mathrm{d}x_2 .$$

This distribution is symmetric around zero, unbounded at $X_1X_2 = 0$. Craig (1936) obtained a moment generating function for the product. Aroian (1947) showed that under certain circumstances this distribution is normal. Seijas-Macías and Oliveira (2012) and Ware and Lad (2003) showed that as the inverse coefficient of variation $\mu/\sigma$ increases, the distribution of the product of two independent normal variables tends towards a normal distribution. And $\mu/\sigma > 1$ is sufficient to approximate normality.

Now consider the generalization of this product to $N$ distributions

$$P_{X_1,X_2,\cdots,X_N}(u) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-x_1^2 / \mathrm{e}^{2\sigma_{x_1}^2}}}{\sigma_{x_1}\sqrt{2\pi}} \frac{\mathrm{e}^{-x_2^2 / \mathrm{e}^{2\sigma_{x_2}^2}}}{\sigma_{x_2}\sqrt{2\pi}} \cdots \frac{\mathrm{e}^{-x_N^2 / \mathrm{e}^{2\sigma_{x_N}^2}}}{\sigma_{x_N}\sqrt{2\pi}} \delta(x_1x_2\cdots x_N - u)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_N .$$

$$P_{X_1,X_2,\cdots,X_N}(u) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{i=1}^N \frac{\mathrm{e}^{-x_i^2 / \mathrm{e}^{2\sigma_{x_i}^2}}}{\sigma_{x_i}\sqrt{2\pi}} \delta(x_i - u)\mathrm{d}x_i .$$

where there are $N$ normal distributions of contributions from $M$ skills, and $\prod_{i=1}^N \frac{\mu_i}{\sigma_i} > 1$.

**The geometric normal average**

If $\sqrt[N]{X_i}, i = 1, 2, \cdots, N$ are independent normally distributed, then $\left(X_iX_j\right)^{1/N}, i \neq j$ follows this normal multiple product distribution with $X_i$ replaced by $\sqrt[N]{X_i}$. To investigate the normality of distribution in $\sqrt[N]{X}$ when $X \sim \mathbb{N}\left(\mu, \sigma^2\right)$, consider the probability density function (PDF) of $X$,

$$f_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-(t-\mu)^2}{2\sigma^2}} .$$

Next, consider $Y = g(X)$, where $g$ is a continuous one-to-one function with inverse $g^{-1}$.

The cumulative distribution function (CDF) of $X$ is

$$P(X < a) = \int_{-\infty}^{a} f_X(t)\mathrm{d}t = F_X(a).$$

The CDF of $Y$ is

$$F_Y(a) = P(Y < a) = P(g(X) < a) = P(X < g^{-1}(a)),$$

$$F_Y(a) = F_X(g^{-1}(a)).$$

Differentiating the CDF to obtain the PDF,

$$f_Y(a) = (F_Y(a))' = \left[F_X(g^{-1}(a))\right]',$$

$$f_Y(a) = F_X'(g^{-1}(a))\left[g^{-1}(a)\right]',$$

$$f_Y(a) = f_X(g^{-1}(a))\left[g^{-1}(a)\right]'.$$

In particular, if $g(X) = \sqrt[N]{X}$, $g^{-1}(X) = X^N$, $X > 0$, then, if $X \sim \mathbb{N}(\mu, \sigma^2)$, the PDF of $Y = \sqrt[N]{X}$ may be obtained using $g^{-1}(a) = a^N$ and $\left[g^{-1}(a)\right]' = Na^{N-1}$. Therefore,

$$f_Y(a) = \frac{Na^{N-1}}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-(a^N - \mu)^2}{2\sigma^2}}, \quad a \geq 0.$$

For certain conditions, this distribution of $\sqrt[N]{X}$ is symmetric and asymptotically normal. This distribution was simulated for a wide range of $\mu/\sigma$ values likely to occur in actual data, and the results were all normal. We see that both arithmetic and geometric combinations transform the bimodal distribution to a unimodal normal distribution.