Quality Triangulations Made Smaller

Alper Üngör*

Abstract

We study alternative types of Steiner points (to circumcenters) for computing quality guaranteed Delaunay triangulations in three dimensions. We show through experiments that their effective use results in smaller (in the number of tetrahedra) triangulations than the output of the traditional circumcenter refinement methods.

1 Introduction

We consider the following optimization problem: *Compute the smallest size triangulation of a given domain such that all the simplices in the triangulation are of good quality.* Quality constraint is motivated by the numerical methods used in many engineering applications. A simplex is said to be good if its radius-edge ratio (circumradius over shortest edge length) is bounded from above. Under the quality constraint, our objective is to make the triangulation size as small as possible for their efficient use in the applications. There has been quite a few solutions for this problem [1, 2, 6, 8, 9]. Earliest algorithms that provide both size optimality (within a constant factor) and quality guarantee used balanced quadtrees to generate first a nicely spread point set and then the Delaunay triangulation of these points [1]. Subsequently, Delaunay refinement techniques are developed based on an incremental point insertion strategy and provide the same theoretical guarantees [8]. Delaunay refinement has become much more popular than the quadtree-based algorithms mostly due to its superior performance in generating smaller triangulations. Due to its importance in a wide range of applications, this problem is frequently revisited and several versions of the Delaunay refinement is suggested [2, 6, 8, 9].

Delaunay refinement method involves first computing an initial Delaunay triangulation of the input domain, and then iteratively adding points called *Steiner points* to improve the quality of the triangulation. Traditionally, circumcenters of bad simplices are used as Steiner points [8, 9]. We recently introduced a new type of Steiner points, called *off-centers*, as an alternative to circumcenters and propose a new variant of the Delaunay refinement algorithm in two dimensions [11]. We showed that the off-center insertion algorithm generates size-optimal quality-guaranteed triangulations. Moreover, experimental study indicates that our refinement algorithm with off-centers inserts fewer Steiner points than the circumcenter insertion algorithms and results in smaller triangulations. This implies substantial reduction not only in triangulation time, but also in the running time of the subsequent application algorithms. In this extended abstract, we present recent research progress on off-center based Delaunay refinement. We extend the off-center definition to three dimensions and present preliminary experimental results.

*Dept. of Computer & Information Science & Engineering, University of Florida, `ungor@cise.ufl.edu`

2 Quality Triangulations in 2D

Replacing the circumcenters with off-centers enabled us to make progress both on theoretical and practical fronts. In theory, using off-centers, we first improved the earlier parallel complexity results [10], then designed the first time-optimal Delanuay refinement algorithm [5]. In practice, off-center insertion algorithm results in significant reduction in the output size (see Figure 1). It is now used in the popular Delaunay refinement software `Triangle`[1].

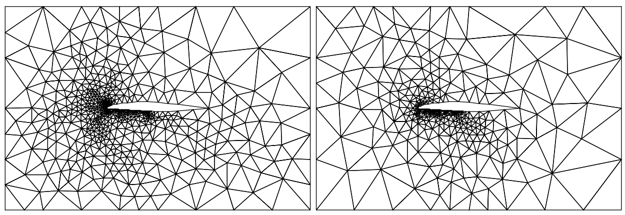

Figure 1: Airfoil mesh. Smallest angle in both output triangulations is 31°. Circumcenter insertion introduces 624 Steiner points resulting a mesh with 1222 triangles (left). Off-center insertion introduces only 359 Steiner points resulting a mesh with 699 triangles (right).

Off-center, c, of a bad triangle pqr is defined as the closest point to the circumcenter of pqr on the bisector of the shortest edge, say pq, such that pqc is (barely) a good triangle [11]. In our experiments we observed that, a perturbation from this theoretical definition gives the best results. We control the amount of perturbation by a parameter called α_1, which rescales the distance between the off-center and the shortest edge. While $\alpha_1 = 1$ means that there is no perturbation,

[1]Available at `http://www-2.cs.cmu.edu/~quake/triangle.html`

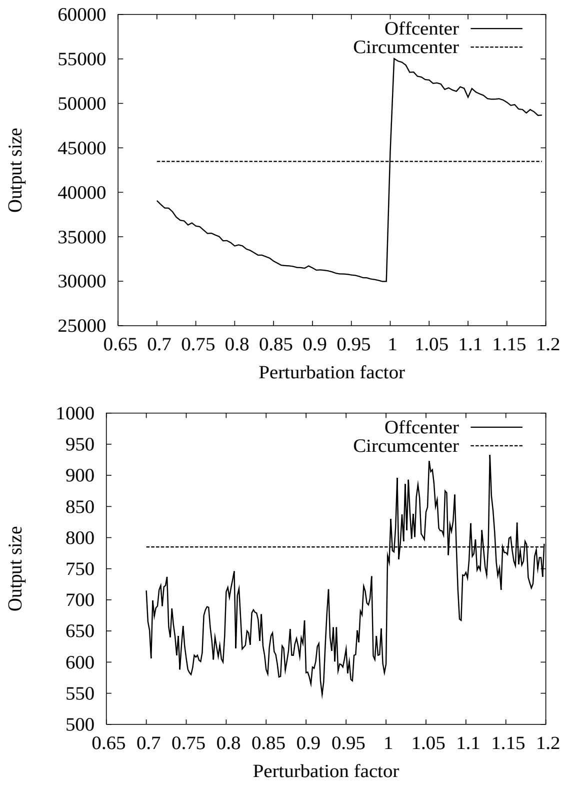

Figure 2: Impact of α_1 on the output size for random point (top) and airfoil (bottom) data sets.

$\alpha_1 < 1$ perturb the off-center on the bisector towards the shortest edge, and $\alpha_1 > 1$ move it away.

While the best choice for α_1 varies as we change the radius-edge ratio threshold and the data set, there is a clear pattern in the performance behavior. There is a sudden large shift in the output size from small to large as α_1 becomes larger than 1 (Figure 2). Best performance is usually observed when α_1 is in the interval (0.95, 1). Note that with a perturbation we not only make sure that the new triangle formed by the shortest edge points and the off-center is of good quality but also potentially fix more bad triangles at the same iteration.

3 Quality Triangulations in 3D

An extension of the circumcenter insertion algorithm to three dimensions is given by Shewchuk [9]. We briefly review this algorithm below and refer to [3, 9] for details.

3.1 Delaunay Refinement with Circumcenters

In three dimensions, a collection Ω of vertices, segments, and facets is called a *piecewise linear complex* (PLC) if (i) all lower dimensional elements on the boundary of an element in Ω also belong to Ω, and (ii) if any two elements intersect, then their intersection is a lower dimensional element in Ω [7]. We first compute the Delaunay triangulation of the set of vertices of the input PLC Ω. Then, we add new points (i) to recover the edges and facets that are not conformed by the Delaunay triangulation and (ii) to improve the quality of the triangulation. A point is said to *encroach* upon a simplex if it is inside the smallest sphere that contains the simplex. A tetrahedron is considered *bad* if its radius-edge ratio is larger than a pre-specified constant $\beta \geq 2$. We maintain the Delaunay triangulation as we add new points using the following rules.

1. If a segment is encroached upon, we add its midpoint.
2. If a facet is encroached upon, we add its circumcenter unless Rule 1 applies.
3. If a tetrahedron is of bad quality, we add its circumcenter unless Rule 1 or 2 applies.

3.2 Delaunay Refinement with Off-centers

Here, we describe two new types of off-centers as Steiner points for three dimensional refinement.

3.2.1 Off-center on triangle bisector

Let pqr be the face of $pqrs$ with the smallest circumradius. Let a be the circumcenter of the triangle pqr, and c be the circumcenter of the tetrahedron $pqrs$. We call the ray that starts from a and goes through c, the bisector of the triangle pqr. We define the Type I *off-center* to be the circumcenter of $pqrs$ if the radius-edge-ratio of $pqrc$ is smaller than or equal to β. Otherwise, the Type I *off-center* is the point on the bisector of pqr, which makes the radius-edge ratio of the triangle based on p, q, r and the off-center itself exactly β (shown as b in Figure 3 (a)).

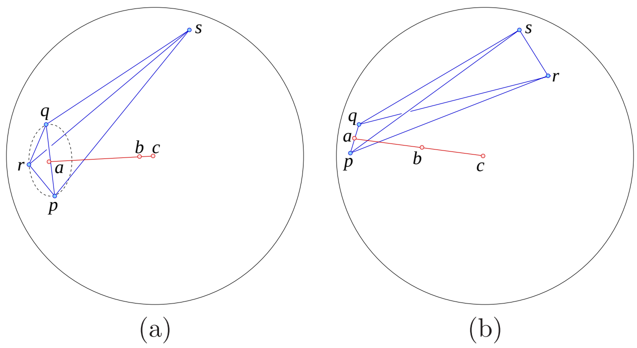

Figure 3: Off-center on triangle and edge bisectors.

Let pq be the shortest edge of pqr and t be the circumradius of pqr. We compute the location of b by rescaling the length of the vector $c - a$ to $|ab|$:

$$|ab| = \alpha_2 \sqrt{\frac{2\beta^2|pq|^2 - t^2 + 2\sqrt{\beta^2|pq|^2\left(\beta^2|pq|^2 - t^2\right)}}{\|c-a\|^2}},$$

where $\alpha_2 \leq 1$ is the perturbation factor, similar to the one described in Section 2 for two dimensional off-center insertion. The choice of $\alpha_2 = 1$ means that the tetrahedron $pqrb$ is just good. Our experiments show that a good choice for α_2 is 0.9.

Note that we use this type of off-centers only if $\beta^2|pq|^2 > t^2$. Otherwise, the radius-edge ratio β cannot be satisfied with the location of b.

3.2.2 Off-center on edge bisector

The line that goes through the midpoint of an edge of a tetrahedron and its circumcenter is called the *bisector* of the edge. Given a bad tetrahedron $pqrs$, suppose that its shortest edge is pq. Let c denote the circumcenter of $pqrs$. We define the Type II *off-center* to be the circumcenter of $pqrs$ if the radius-edge-ratio of pqc is smaller than or equal to β. Otherwise, the Type II *off-center* is the point on the bisector (and inside the circumsphere), which makes the radius-edge ratio of the triangle based on p, q and the off-center itself exactly β (shown as b in Figure 3 (b)). We compute the length of ab as follows:

$$|ab| = \alpha_3 \left(\beta + \sqrt{\beta^2 - 1/4}\right) |pq|,$$

where α_3 is the perturbation factor.

When $\alpha_3 \leq 1$, diametral sphere of pqb has radius $\beta|pq|$, hence tetrahedra formed by p, q, b, and a fourth point x can be a good tetrahedron. As the value of α_3 approaches to 1, the chances of $pqbx$ being a good tetrahedron converges to 0. Experimentally, we found that a good choice for α_3 is 0.6. Note that the factor multiplying $|pq|$ above can be precomputed.

3.2.3 Algorithm

The structure of the Delaunay refinement algorithm as presented in Section 3.1, remains the same. We just replace the type of Steiner points used. The two types of off-centers give us the opportunity to explore several versions of the algorithm. We can use a single (either) type of off-center, or both. We give a comparison of these three approaches in the next section. When facets on the boundary are to be split, we use the two-dimensional off-center insertion algorithm.

3.2.4 Experiments

We implemented the Delaunay refinement with off-centers by replacing the circumcenter procedure in the `Pyramid` software. Computing off-centers and circumcenters are very similar and take roughly the same time. Hence, savings in the number of Steiner points reflects the amount of savings in triangulation time.

It is known that the insertion order of the Steiner points has an impact on the output mesh size. In this study, for fairness of comparison, we use the same ordering strategy (larger radius-edge ratio first) for both the circumcenter and the off-center insertion schemes. We shall note that there is room for further improvement by using a more appropriate ordering strategy for the off-center insertion method.

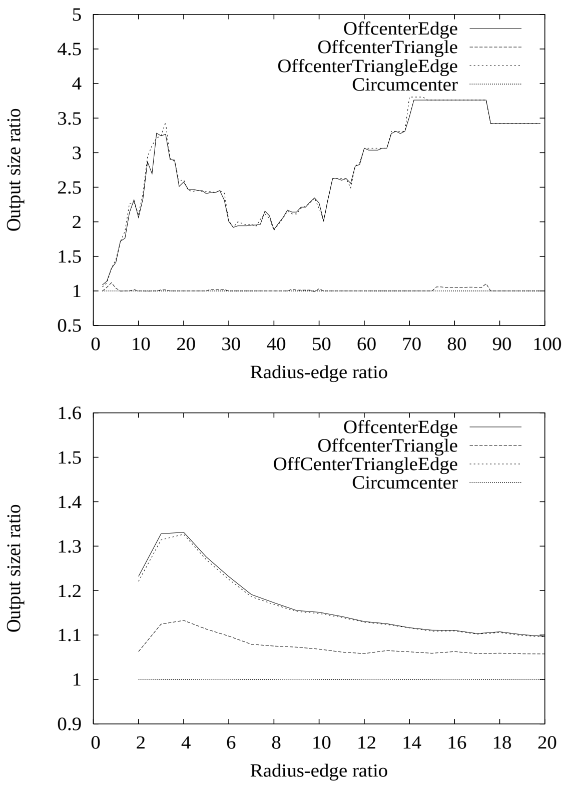

Figure 4: Output size ratio with respect to radius-edge ratio constraint for the tiny feature in the middle of a box (top) and ten thousand points on an ellipsoid (bottom) data sets..

Figure 4 presents a summary of our experiments on two data sets. First data set consists of a tiny feature (two vertices within a distance of 10^{-4}) located at the center of a unit box. Second data set consists of 10,000 points randomly located on the surface of an ellipsoid, which is contained inside a bounding box. We report the ratio of the output size M_c/M_o, where M_c and M_o are the number of elements generated by the circumcenter and the off-center insertion methods, respectively. We ran experiments on various data sets. In most cases, the difference in the output is visible (see Figure 5). We summarize our observations as follows:

- We get significant size improvements with the use of off-centers, especially when there is grading in the mesh (due to relatively small input features with respect to the domain size).

- Use of both type of off-centers or the use of Type II off-centers alone outperforms the use of Type I off-centers alone, which in turn outperforms the use of circumcenters.

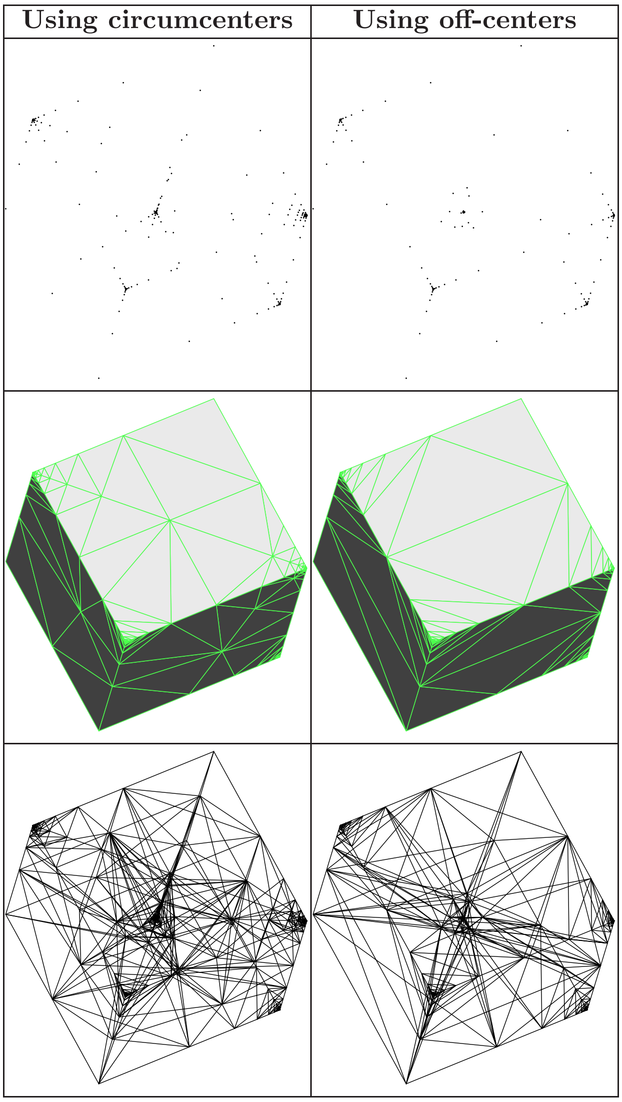

Figure 5: Input consists of 20 points and 6 facets. Largest radius-edge ratio in both triangulations is 15. The `Pyramid` software inserted 785 circumcenters resulting 1343 edges and 392 tetrahedra (left). Our algorithm inserted only 322 off-centers resulting 797 edges and 219 tetrahedra (right).

- Output size ratio M_c/M_o varies largely (more so than in two dimensions) as we change data sets.
- Performance behavior with respect to radius-edge ratio constraint (Figure 4) is somewhat different than that pattern in two dimensions [11], where we got the best size improvements for the smallest radius-edge ratio values.

4 Discussions

Our experimental study of the off-center insertion algorithm in three dimensions is by no means complete. Here, we described two types of off-centers as Steiner points and present how effective off-center insertion can be for computing small size quality-guaranteed triangulations. We should note that, off-center insertion do not always output smaller triangulations than the output of circumcenter insertion, especially when the perturbation factors α_1, and α_2 are not carefully chosen. We believe that it is worth to explore a perturbation strategy based on the local point distribution. In fact, our goal is to combine the off-center insertion algorithm with the perturbation based sliver removal approach presented in [4] to compute small size sliver-free triangulations in three dimensions.

Acknowledgements

I am thankful to Jonathan Shewchuk for using the off-center insertion algorithm in the `Triangle` software and for providing the `Pyramid` software. I also gratefully acknowledge the preliminary implementations Ethan Eade has done while taking my course at Duke University.

References

[1] M. Bern, D. Eppstein, and J. Gilbert. Provably good mesh generation. *J. Comp. System Sciences* 48:384–409, 1994.

[2] L.P. Chew. Guaranteed-quality triangular meshes. TR-89-983, Cornell Univ., 1989.

[3] H. Edelsbrunner. *Geometry and Topology for Mesh Generation*. Cambridge Univ. Press, 2001.

[4] H. Edelsbrunner, X. Li, G.L. Miller, A. Stathopoulos, D. Talmor, S.-H. Teng, A. Üngör, and N. Walkington. Smoothing and cleaning up slivers. *Proc. 32nd ACM Symp. on Theory of Computing*, 273–277, 2000.

[5] S. Har-Peled and A. Üngör. A time-optimal Delaunay refinement algorithm in two dimensions. *Submitted. Available at* `arXiv:cs.CG/0501007`.

[6] G.L. Miller. A time efficient Delaunay refinement algorithm. *Proc. ACM-SIAM Symp. on Disc. Algorithms*, 400–409, 2004.

[7] G.L. Miller, D. Talmor, S.-H. Teng, N. Walkington, and H. Wang. Control volume meshes using sphere packing: Generation, refinement, and coarsening. *Proc. 5th Int. Meshing Roundtable*, 47–61, 1996.

[8] J. Ruppert. A new and simple algorithm for quality 2-dimensional mesh generation. *Proc. 4th ACM-SIAM Symp. on Disc. Algorithms*, 83–92, 1993.

[9] J.R. Shewchuk. *Delaunay Refinement Mesh Generation*. Ph.D. thesis, Carnegie Mellon University, 1997.

[10] D.A. Spielman, S.-H. Teng, and A. Üngör. Parallel Delaunay refinement with off-centers. *Proc. EUROPAR*, LNCS 3149, 812–819, 2004.

[11] A. Üngör. Off-centers: A new type of Steiner points for computing size-optimal quality-guaranteed Delaunay triangulations. *Proc. LATIN*, LNCS 2976, 152–161, 2004.