Pseudo-Tetrahedral Complexes

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1 Introduction

A pseudo-triangulation is a cell complex in the plane whose cells are pseudo-triangles, i.e., simple polygons with exactly three convex vertices (so-called corners). Being an interesting and flexible generalization of triangulations, pseudo-triangulations have found their place in computational geometry; see e.g. [8, 11, 7, 1] and references therein.

Unlike triangulations, pseudo-triangulations eluded a meaningful generalization to higher dimensions so far. In this paper, we define pseudo-simplices and pseudo-simplicial complexes in d-space in a way consistent to pseudo-triangulations in the plane. Flip operations in pseudo-complexes are specified, as combinations of flips in pseudo-triangulations [11, 1], and of bistellar flips in simplicial complexes [9, 5, 4]. Our results are based on the concept of maximal locally convex functions on polyhedral domains [1], that allows us to unify several well-known structures, namely pseudo-triangulations, constrained Delaunay triangulations [3, 14], and regular simplicial complexes [2, 5]. Several implications of our results exist, and challenging open questions arise.

2 Polytopes and Corners

We give some notation concerning polytopes in d-space $\mathbb{R}^d$. A connected, bounded, and closed subset P of $\mathbb{R}^d$ is called a *d-polytope* if P is a d-manifold, with piecewise linear boundary $\mathsf{bd}\ P$ that is structured as a $(d-1)$-dimensional cell complex. The components of $\mathsf{bd}\ P$ of dimension j are called the *j-faces* of P. Faces of dimensions $d-1$, 1, and 0, respectively, are also called *facets*, *edges*, and *vertices*. We denote with $\mathsf{vert}\ P$ the set of vertices of P. P is called *simple* if P is homeomorphic to a closed ball in $\mathbb{R}^d$.

A *terminal* of P is a point $x \in P$ such that no line segment $L \subset P$ contains x in its relative interior. All terminals of P belong to $\mathsf{vert}\ P$. A terminal v of P is called a *corner* of P if there exists a hyperplane through v that has all edges of P incident to v on a fixed side. (For $d = 2$, terminals automatically fulfill this requirement and therefore are corners.) All vertices of the convex hull $\mathsf{conv}\ P$ are corners of P. So every d-polytope P has at least $d+1$ corners. In Figure 1, x is a corner, y is a terminal but not a corner, and z is not a terminal.

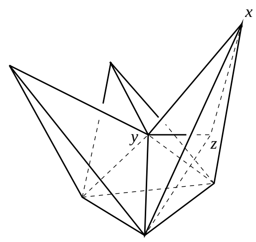

Figure 1: 3-polytope with different vertex types

A boundary-connected d-polytope with exactly $d+1$ corners is termed a *pseudo-simplex*. Every simplex is a pseudo-simplex. For $d = 2$ and $d = 3$, respectively, pseudo-simplices will be also called *pseudo-triangles* and *pseudohedra*. Our definition of a pseudo-triangle is equivalent to the classical definition, see e.g. [11, 1], which requires exactly 3 polygon vertices with an internal angle smaller than π. A *pseudo-complex* is a cell complex in $\mathbb{R}^d$ all whose cells are pseudo-simplices of dimension d. For $d = 2$, pseudo-complexes are pseudo-triangulations.

3 Local Convexity

The theory of maximal locally convex functions is the key to a derivation of pseudo-complexes. For $d = 2$, the relationship between these two concepts has been observed in [1]. Consider a real-valued function f on $\mathbb{R}^d$ whose domain is a simple d-polytope D. Function f is called *locally convex* if f is convex on each line segment $L \subset D$.

Let $\boldsymbol{h}$ be a real-valued vector that assigns a $(d+1)$st coordinate h_i (called *height*) to each vertex $v_i \in \mathsf{vert}\ D$. Our interest is in the maximal locally convex function f^* on D which fulfills $f^*(v_i) \leq h_i$ for each $v_i \in \mathsf{vert}\ D$. The function f^* is unique, because f^* is the pointwise maximum of all locally convex functions which satisfy the constraints in $\boldsymbol{h}$. Moreover, f^* is continuous in the interior $\mathsf{int}\ D$ of D, by its local convexity. For $d \geq 3$, f^* need not be continuous on $\mathsf{bd}\ D$, however.

We can show that f^* is a piecewise linear function on D. In fact, f^* induces a face-to-face cell complex in D, whose cells are maximal connected subset of D where f^* is linear. The continuity of f^*

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on $\mathsf{int}\ D$ implies that the faces of f^* have piecewise linear boundaries, and therefore are j-polytopes, for $0 \leq j \leq d$. As D is a simple d-polytope, the boundary of each cell of f^* is connected: The existence of a hole H in a cell contradicts the convexity of f^* on each line segment $L \subset D$ that crosses the (relative) boundary of H twice.

Each vertex x of f^* either belongs to $\mathsf{vert}\ D$, or x is the intersection of a j-face of f^* with a $(d-j)$-face of D. Accordingly, x will be termed a *primary* or a *secondary* vertex. We extend this terminology to the cells of f^*, by distinguishing whether or not all corners of a cell are primary vertices.

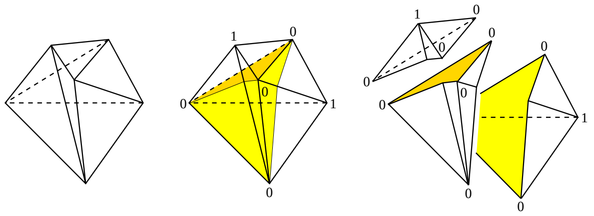

Figure 2: Splitting the Schönhardt polytope with f^*

Figure 2 illustrates a three-dimensional cell complex induced by f^* when D is the Schönhardt polytope [13]. All six vertices of D are corners. Numbers denote vertex heights. The complex consists of three non-tetrahedral primary cells. Note the occurrence of secondary vertices in the relative interior of certain edges of D.

A vertex $v_i \in \mathsf{vert}\ D$ is termed *complete* if $f^*(v_i) = h_i$. A vertex of f^* that is a corner of all its incident cells is called an *allcorner*. For instance, each corner of D is an allcorner of f^*.

Lemma 1 *(a) All terminals of D are complete. (b) A vertex where f^* is discontinuous cannot be a corner of any cell. (c) Allcorners of f^* are characterized by being vertices of D that are complete and where f^* is continuous. (d) No vertex of f^* lies in $\mathsf{int}\ D$, or in the relative interior of a facet of D.*

The polytope D in Figure 1 serves as an example where f^* is discontinuous. As heights for D we choose 1 for vertex y and 0 for the remaining vertices. Then $f^*(p) = 0$ for all $p \in \mathsf{int}\ D$. By Lemma 1(a), we have $f^*(y) = 1$ because y is a terminal. Thus f^* is discontinuous at y.

Maximal locally convex functions constitute a natural generalization of convex hulls. Let H be the point set in $\mathbb{R}^{d+1}$ that results from lifting $\mathsf{vert}\ D$ by its height vector $\boldsymbol{h}$. Denote by low_H the (convex) function whose graph is the lower convex hull of H, i.e., the part of $\mathsf{bd}\ \mathsf{conv}\ H$ visible from $-\infty$ on the $(d+1)$st coordinate axis.

Theorem 2 *If the domain D is convex then we have $f^* = \mathsf{low}_H$, for every height vector $\boldsymbol{h}$.*

4 Pseudo-Complexes

For a given simple d-polytope D and a height vector $\boldsymbol{h}$ for $\mathsf{vert}\ D$, let $\mathcal{PC}(D, \boldsymbol{h})$ denote the polytopal cell complex induced by f^* in D. We call $\boldsymbol{h}$ *generic* (for D) if there exists some $\epsilon > 0$ such that $\mathcal{PC}(D, \boldsymbol{h}_\epsilon) = \mathcal{PC}(D, \boldsymbol{h})$ holds for all possible ϵ-perturbations $\boldsymbol{h}_\epsilon$ of $\boldsymbol{h}$. To ease the exposition, this property of $\boldsymbol{h}$ will be implicitly assumed henceforth.

We call $\boldsymbol{h}$ *convex* if $\mathsf{low}_H(v_i) = h_i$ holds for each $v_i \in \mathsf{vert}\ D$. In particular, $\boldsymbol{h}$ is called *parabolic* if $h_i = v_i^2$ for each i. If $\boldsymbol{h}$ is convex then $\mathcal{PC}(D, \boldsymbol{h})$ shows several nice properties.

Theorem 3 *Let $\boldsymbol{h}$ be a convex height vector. Then all cells of $\mathcal{PC}(D, \boldsymbol{h})$ are primary cells, pseudo-simplices, and simple d-polytopes. All primary vertices of $\mathcal{PC}(D, \boldsymbol{h})$ are allcorners. Moreover, f^* is continuous on the entire domain D.*

By Theorem 3, locally convex functions generate pseudo-complexes if the height vector $\boldsymbol{h}$ is convex. In fact, this is the case for arbitrary $\boldsymbol{h}$; see Section 5. If D (and with it, $\boldsymbol{h}$) is convex then all cells are simplices, and $\mathcal{PC}(D, \boldsymbol{h})$ is a regular simplicial complex [2, 5] in D, by Theorem 2. If, in addition, $\boldsymbol{h}$ is parabolic then the well-known Delaunay simplicial complex [6] for D is obtained. When $\boldsymbol{h}$ is convex but D is not, then $\mathcal{PC}(D, \boldsymbol{h})$ need not be simplicial; see Figure 2. The case $d = 2$ has been treated in [1]. $\mathcal{PC}(D, \boldsymbol{h})$ then is a constrained regular pseudo-triangulation of the simple polygon D. In particular, if $\boldsymbol{h}$ is parabolic, then the constrained Delaunay triangulation [3] of D is obtained, which is the Delaunay triangulation [6] provided D is a convex polygon.

5 Bistellar Pseudoflips

We now investigate the properties of $\mathcal{PC}(D, \boldsymbol{h})$ for arbitrary height vectors $\boldsymbol{h}$, by defining flip operations in pseudo-complexes that result from controlled changes in $\boldsymbol{h}$. These operations generalize both the d-dimensional Lawson flip [9, 4] and the flips in 2-dimensional pseudo-triangulations [11, 1]. From now on, let $n = |\mathsf{vert}\ D|$.

Moving Heights Let $\boldsymbol{h}_0$ and $\boldsymbol{h}_1$ be two (generic) height vectors for $\mathsf{vert}\ D$. Assume that $\boldsymbol{h}_0$ is convex, and that $\boldsymbol{h}_0 > \boldsymbol{h}_1$ (elementwise). We continuously deform $\mathcal{PC}(D, \boldsymbol{h}_0)$ into $\mathcal{PC}(D, \boldsymbol{h}_1)$ and study the changes in the structure of cells.

To this end, let $\boldsymbol{h}_\lambda = \lambda \boldsymbol{h}_1 + (1-\lambda)\boldsymbol{h}_0$, for λ increasing from 0 to 1. By Theorem 3, in $\mathcal{PC}(D, \boldsymbol{h}_0)$ all primary vertices are complete and are allcorners, and all cells are primary and are pseudo-simplices. $\mathcal{PC}(D, \boldsymbol{h}_\lambda)$ changes its shape exactly at values λ where $\boldsymbol{h}_\lambda$ is not generic. Fix such a value λ. Consider a cell U of $\mathcal{PC}(D, \boldsymbol{h}_\lambda)$ which is not a cell of $\mathcal{PC}(D, \boldsymbol{h}_{\lambda-\varepsilon})$, for sufficiently small $\varepsilon > 0$. Denote with $\mathcal{PC}_{\lambda-\varepsilon}$ the restriction of $\mathcal{PC}(D, \boldsymbol{h}_{\lambda-\varepsilon})$ to U. The crucial observation is that, for $\lambda - \varepsilon$, f^* on $\mathsf{int}\, U$ is determined by its values at the allcorners of $\mathcal{PC}_{\lambda-\varepsilon}$. This follows from Lemma 1(b). Therefore $\mathcal{PC}_{\lambda-\varepsilon}$ has exactly $d+2$ allcorners (apart from special cases which can be avoided by perturbing $\boldsymbol{h}_0$ slightly). In particular, U has at most $d+2$ corners.

In $\mathcal{PC}(D, \boldsymbol{h}_{\lambda+\varepsilon})$, the polytope U is restructured into a cell complex $\mathcal{PC}_{\lambda+\varepsilon}$. The replacement of $\mathcal{PC}_{\lambda-\varepsilon}$ by $\mathcal{PC}_{\lambda+\varepsilon}$ is termed a *pseudoflip*.

Anatomy of Pseudoflips To study the structure of pseudoflips, let us consider any complex $PC(U, \boldsymbol{h})$ with exactly $d+2$ allcorners $v_1, \ldots, v_{d+2}$. By Lemma 1(c), full height is assumed at and only at $v_1, \ldots, v_{d+2}$. W.l.o.g., let $\boldsymbol{h}$ contain entries ∞ for all other vertices of U. Let $\boldsymbol{h}^-$ be the vector obtained from $\boldsymbol{h}$ by changing the signs of finite entries. Then, for any generic choice of heights $h_1, \ldots, h_{d+2}$ for $v_1, \ldots, v_{d+2}$, one of the complexes $PC(U, \boldsymbol{h})$ or $\mathcal{PC}(U, \boldsymbol{h}^-)$ has to arise, because the relative position of the $d+2$ points $\binom{v_i}{h_i}$ in $\mathbb{R}^{d+1}$ already determines f^*.

As a consequence, each pseudoflip can be simulated by replacing $PC(U, \boldsymbol{h})$ by $\mathcal{PC}(U, \boldsymbol{h}^-)$. Moreover, as $\mathcal{PC}(U, \boldsymbol{h}^-)$ has at most $d+2$ allcorners, the cells of $\mathcal{PC}(U, \boldsymbol{h}^-)$ are pseudo-simplices, provided the same holds for $PC(U, \boldsymbol{h})$. (If $\mathcal{PC}(U, \boldsymbol{h}^-)$ has $d+1$ allcorners then its only cell is the pseudo-simplex U.) Recalling that the original complex $\mathcal{PC}(D, \boldsymbol{h}_0)$ was a pseudo-complex, we get an inductive argument showing that the final complex $\mathcal{PC}(D, \boldsymbol{h}_1)$ is a pseudo-complex.

We face two different types of pseudoflips. An *exchanging* pseudoflip transforms $PC(U, \boldsymbol{h})$ into a complex with the same number of allcorners, whereas a *removing* pseudoflip transforms $PC(U, \boldsymbol{h})$ into a single cell. (The inverse of a removing pseudoflip is also considered a valid pseudoflip; we call it an *inserting* pseudoflip.) $PC(U, \boldsymbol{h})$ contains primary cells and, in general, also secondary cells, because secondary vertices may arise in the relative interior of faces of U. Unfortunately, neither the number of primary cells nor the number of secondary cells is bounded by a function of d. Already for $d = 3$ there are examples where $\Theta(k)$ primary cells and $\Theta(k^2)$ secondary cells occur, for $k = |\mathsf{vert}\, U|$. An upper bound for primary cells in this case is $O(k^2)$, see Theorem 4.

For general d and arbitrary complexes $\mathcal{PC}(D, \boldsymbol{h})$, the number of primary cells is bounded by Theorem 2 and the observation that this number is maximal if D is convex. The number of secondary cells can be shown to be finite but remains unclear. Concerning the total number of pseudoflips, we can show that each $(d+2)$-tuple of vertices of D gives rise to at most one pseudoflip when heights are moved as above. We conclude:

Theorem 4 *$\mathcal{PC}(D, \boldsymbol{h})$ is a pseudo-complex for arbitrary (generic) $\boldsymbol{h}$. The number of primary cells of $\mathcal{PC}(D, \boldsymbol{h})$ is $O(n^{\lceil d/2 \rceil})$. Given two height vectors $\boldsymbol{h}$ and $\boldsymbol{h}'$, the distance between $\mathcal{PC}(D, \boldsymbol{h})$ and $\mathcal{PC}(D, \boldsymbol{h}')$ by pseudoflips is $O(n^{d+2})$.*

A challenging open question is whether pseudoflip sequences between simplicial complexes do exists such that cell sizes are bounded by $O(d)$. The complexity of pseudoflips does not depend on n in this case.

Examples We illustrate some pseudoflips for $d = 3$. The 3-polytope U where a flip takes place has at most 5 corners; they are labeled by numbers in the following figures. The pseudoflips are viewed best when imagining that the height of vertex 4 is lowered.

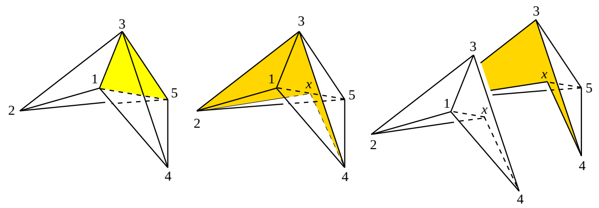

Figure 3: Exchanging pseudoflip

Figure 3 shows an exchanging pseudoflip. Before the flip, U contains the two tetrahedral cells 1235 and 1345. They are are adjacent in the triangular facet 135. (The tetrahedron 1245 avoids the interior of U and yields no cell.) After the flip, which destroys the facet 135 and creates the pseudo-triangular facet $234x$, two pseudohedra arise as cells. Their corners are $1, 2, 3, 4$ and $2, 3, 4, 5$, respectively. The secondary vertex x arises as a noncorner of both cells. All involved cells are primary cells.

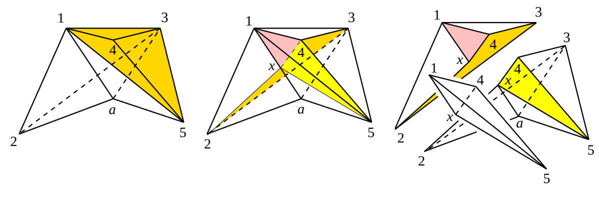

Figure 4: Pseudoflip generating a secondary cell

The exchanging pseudoflip in Figure 4 is more complicated. Again, U contains only two cells before the flip, the tetrahedron 1345 and the pseudohedron with corners $1, 2, 3, 5$ and the noncorner a. Both cells are primary cells. Their common triangular facet 135 is destroyed in the flip. After the flip, two new primary cells are present, namely, the pseudohedra with corners $1, 2, 3, 4$ and $2, 3, 4, 5$, respectively. In addition, the tetrahedron $145x$ arises as a secondary cell. Its corner x is the secondary vertex where the pseudo-triangular facet with corners $2, 3, 4$ intersects the edge $1a$.

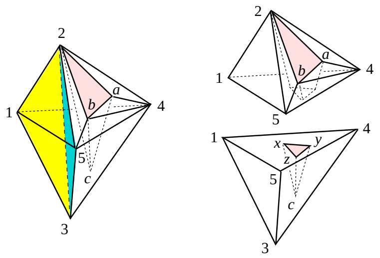

Figure 5: Pseudoflip that creates a tunnel

Figure 5 depicts an exchanging pseudoflip that creates a non-simple cell. Three primary cells are present before the flip: The tetrahedra 1234 and 1235, and the pseudohedron with corners $2, 3, 4, 5$ and non-corners a, b, c. These cells are pairwise adjacent in triangular facets which are destroyed in the flip. A single facet F with corners $1, 4, 5$ and the secondary vertices x, y, z as noncorners is created. As xyz is a hole, F is not a valid pseudo-triangle, but rather a polygonal region with three corners. Two primary cells are adjacent in F. The cell with corners $1, 2, 4, 5$ contains a tunnel, defined by the edges $2x$, ay, and bz.

6 Extensions

Several extensions of our results exist. For fixed D, consider the class of pseudo-complexes

$$\mathcal{R}(D) = \{\mathcal{PC} \mid \exists\, \boldsymbol{h} \text{ with } \mathcal{PC} = \mathcal{PC}(D, \boldsymbol{h})\}.$$

The existence of a convex n-polytope can be established that represents all the members of $\mathcal{R}(D)$. This generalizes the polytope constructions in [10] (the associahedron) and in [2] (the secondary polytope), which concern the regular simplicial complexes for convex domains D, as well as the polytope in [1], for constrained regular pseudo-triangulations of simple polygons D. In particular, the subclass

$$\mathcal{M}(D) = \{\mathcal{PC} \mid \exists\, \boldsymbol{h} \text{ convex with } \mathcal{PC} = \mathcal{PC}(D, -\boldsymbol{h})\}$$

constitutes a generalization to $\mathbb{R}^d$ of minimum (or pointed) pseudo-triangulations [11].

A pseudohedron need not be tetrahedrizable. Still, pseudo-complexes for convex height vectors provide a way of decomposing a given nonconvex polytope in 3-space into $O(n^2)$ tetrahedra, when secondary vertices are used as Steiner points.

Pseudo-complexes give rise to several burning questions. A question of major interest in motion planning applications [8] is whether visible space between polyhedral objects can be tiled and maintained with pseudo-simplices and pseudoflips, respectively. A related important problem is whether any two simplicial complexes in d-space can be transformed into each other by pseudoflips. For Lawson flips [9], the answer is negative for $d \geq 6$, and unknown for $3 \leq d \leq 5$; see [12].

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