Minimizing Local Minima in Terrains with Higher-Order Delaunay Triangulations

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Abstract

We show that triangulating a set of points with elevations such that the number of local minima of the resulting terrain is minimized is NP-hard for degenerate point sets. The same result applies when there are no degeneracies for higher-order Delaunay triangulations. Two heuristics are presented to minimize the number of local minima for higher-order Delaunay triangulations, and they are compared experimentally.

1 Introduction

A fundamental geometric structure in computational geometry is the triangulation. It is a partitioning of a point set or region of the plane into triangles. A triangulation of a point set P partitions the convex hull of P into triangles whose vertices are exactly the points of P. There are many different ways in which one can define the quality of a triangulation of a set of points. A criterion that is always important for triangulations is the nice shape of the triangles. This can be formalized in several ways [2, 3]. In this paper, nice shape is formalized by *higher-order Delaunay triangulations* [4]. They provide a class of triangulations that are all reasonably well-shaped, depending on a parameter k.

Definition 1 *A triangle in a point set P is* order-k *if its circumcircle contains at most k points of P. A triangulation of a set P of points is an* order-k Delaunay triangulation *if any triangle of the triangulation is order-k.*

So a Delaunay triangulation is an order-0 Delaunay triangulation. For any positive integer k, there can be many different order-k Delaunay triangulations. We also define the *useful order* of an edge as the lowest order of a triangulation that includes this edge.

When using triangulations for terrain modeling, one should realize that terrains are formed by natural processes. This implies that there are linear depressions (valleys) formed by water flow, and very few local minima occur [7]. Local minima can be caused by erroneous triangulation: an edge may stretch from one side of a valley to the opposite side. Such an edge is an artificial dam, and upstream from the dam in the valley, a local minimum appears. It is generally an artifact of the triangulation. Therefore, minimizing local minima is an optimization criterion for terrain modeling.

This paper discusses triangulations of a point set P of which elevations are given. The objective is to triangulate P with an order-k Delaunay triangulation, such that there are as few local minima as possible. In Section 2 we show that over all possible triangulations, minimizing local minima is NP-hard. This result relies heavily on degenerate point sets. For order-k Delaunay triangulations, NP-hardness can also be shown for non-degenerate point sets, for $k = \Omega(n^{\epsilon})$ and $k = O(n^{1/4})$. (For $k = 1$, an $O(n \log n)$ time algorithm that minimizes local minima was given in [4].) Then we discuss two heuristics for minimizing local minima. In Sections 3 and 4 we present the flip and hull heuristics and their efficiency. The latter was introduced before in [4]; here we give a more efficient algorithm. In Section 5 we apply the two heuristics on various terrains to examine the possibilities of higher-order Delaunay triangulations to reduce local minima, and to test which of the two heuristics is better.

2 NP-hardness

For a set P of n points in the plane, it is easy to compute a triangulation that minimizes the number of local minima if there are no degeneracies. Assume p is the lowest point. Connect every $q \in P\backslash\{p\}$ with p to create a star network with p as the center. Complete this set of edges to a triangulation in any way. Since every point but p has a lower neighbor, no point but p can be a local minimum. Hence, this triangulation is one that minimizes the number of local minima. When many degeneracies are present, minimizing the number of local minima is NP-hard.

Theorem 1 *Let P be a set of n points in the plane, and assume the points have elevations. It is NP-hard to triangulate P with the objective to minimize the number of local minima of the polyhedral terrain.*

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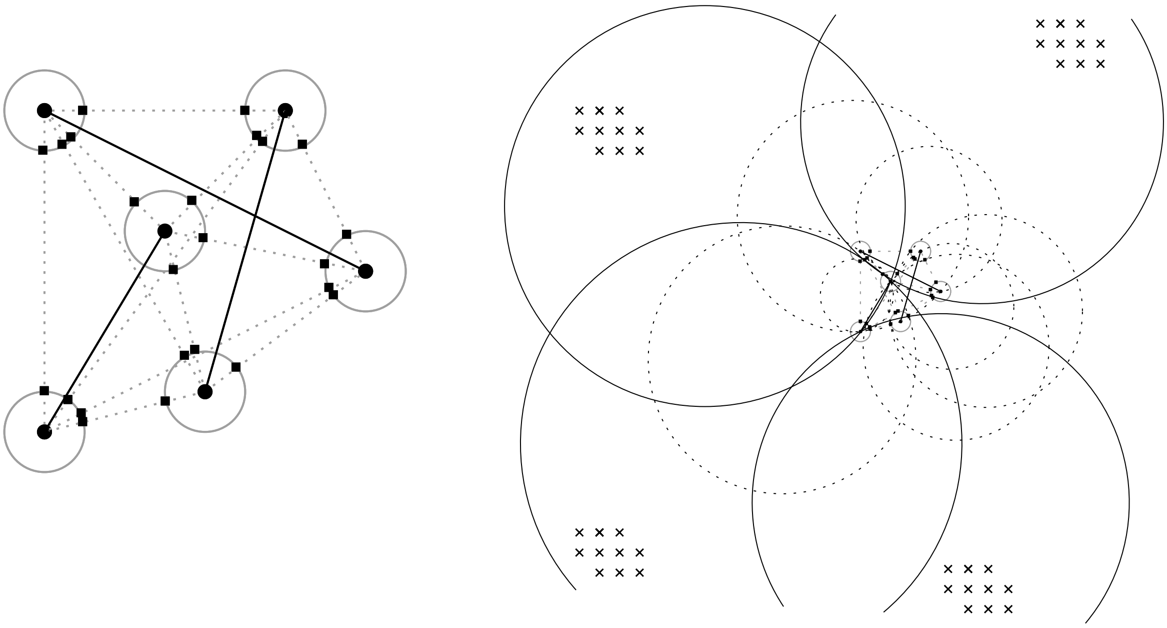

Figure 1: Left, construction for the NP-hardness proof. Square points are in H, cirular points are in P. Right, NP-hardness for higher-order Delaunay triangulations. Crosses are points at least $4r$ from the points in $P \cup H$.

Proof. By reduction from maximum size non-intersecting subset in a set of line segments [1]. Let S be any set of n line segments in the plane, and assume all $2n$ endpoints are disjoint (this can easily be enforced by extending segments slightly). Let P be the set of the $2n$ endpoints. Let ϵ be the smallest distance between two points in P. For every point $p \in P$, let $C(p)$ be a circle centered at p with radius $\epsilon/3$. If p' is the point in P such that pp' is a segment of S, then for every $q \in P \backslash \{p, p'\}$, place a point at the intersection of pq and $C(p)$. We call these points *shields*, because they prevent p and q from being connected by a line segment in any triangulation. Let H be the set of shields.

We assign elevations as follows. For every segment in S, one endpoint isassigned elevation 1 and the other isassigned elevation 2. Every shield in H isassigned elevation 3. By the choice of shields, every point with elevation 1 is a local minimum, and no point with elevation 3 can be a local minimum. A point with elevation 2 is a local minimum if and only if the segment from S that connects p' to the other endpoint p is in the triangulation. Hence, the maximum non-intersecting subset of S corresponds one-to-one with the points with elevation 2 that are local minimum. Since the number of shields is quadratic, NP-hardness follows directly. □

Based on the construction in the proof above, we can show NP-hardness of minimizing the number of local minima for higher-order Delaunay triangulations even when no degeneracies are present.

Corollary 2 *Let P be a set of n points in the plane such that no three points of P lie on a line, and assume that the points have elevations. For any $0 < \epsilon < 1$ and some $c > 0$, it is NP-hard to compute a k-th order Delaunay triangulation that minimizes the number of local minima of the polyhedral terrain for $n^\epsilon \leq k \leq c \cdot n^{1/4}$.*

Proof. Start out with the proof of the theorem above; $|P| = 2n$ and $|H| = 2n(2n-1)$. Let r be the radius of the largest (finite) circle that passes through three points of $P \cup H$ of the construction. Note that r is at least half of the diameter of $P \cup H$. Let $\delta > 0$ be a value chosen such that if three non-colinear points from $P \cup H$ move over a distance δ, then their circle has radius at most $2r$. Both r and δ can be computed in cubic time. For every shield $h \in H$, displace it over a distance d where $0 < d < \delta$, and such that the radius of the circle through h and the two points of P for which h is a shield has radius at least $4r$. Place $|P| + |H| + 1 = 4n^2 + 1$ points inside this circle and at distance at least $4r$ from all points in $P \cup H$. Since the diameter of $P \cup H$ is at most $2r$, this is possible. These points make sure that the triangulation edge for which h was a shield, is still not possible in a $4n^2$-th order Delaunay triangulation: the useful order of the triangulation edge is too high. By construction, other edges between points in P and H are possible. The extra points get elevation 3 as well, and the problem of minimizing the number of local minima in $4n^2$-th order Delaunay triangulations is again the same as maximizing the size of a non-intersecting subset of S.

The number of points in the construction is $O(n^8)$. This gives the proof for $k = c \cdot n^{1/4}$ for some $c > 0$. For smaller values of k we simply place more points at distance at least $4r$. As long as $k = \Omega(n^\epsilon)$ the construction is polynomial. □

3 The flip heuristic

Given a value of k, the flip heuristic repeatedly tests whether the diagonal of a convex quadrilateral in the triangulation can be flipped. It will be flipped if two conditions hold simultaneously: (i) The two new triangles are order-k Delaunay triangles. (ii) The new edge connects the lowest point of the four to the opposite point. A flip does not necessarily remove a local minimum, but cannot create one, and it can make possible that a later flip removes a local minimum.

Our algorithm to perform the flips starts with the Delaunay triangulation and $k' = 1$, then does all flips possible to obtain an order-k' Delaunay triangulation, then increments k' and repeats. This continues until $k' = k$.

We first deal with the maximum number of flips needed, and then we discuss the efficiency of the heuristic.

Lemma 3 *The flip heuristic terminates after at most $O(n^2)$ flips.*

Proof. Normalize the heights of the vertices to be integers in the range $1, \ldots, n$. Observe that this does not influence the flipping criterion. Consider the function $F(T)$ for a triangulation T:

$$F(T) = \sum_{\overline{uv} \in T} \min(u, v)\,.$$

Any flip decreases $F(.)$ with at least one, and $F(.)$ is at most $O(n^2)$ to begin with. □

Lemma 4 *If an edge $\overline{ab}$ is in the triangulation, then the flip heuristic will never have an edge $\overline{cd}$ later with $\min(c, d) \geq \min(a, b)$ that intersects $\overline{ab}$.*

Proof. Assume without loss of generality that $a < b$ and $c < d$. Assume further that $a \leq c$, edge $\overline{ab}$ is in the triangulation T, and that $\overline{cd}$ is the first edge flipped into T that violates the property of the lemma. Before the flip, c and d must be in a convex quadrilateral where c is the lowest point of the four. The other two points, f and g, cannot be a because a is lower by assumption. Possibly, f or g is the same as b. The quadrilateral has edges $\overline{cf}, \overline{cg}, \overline{df}, \overline{dg}$, and two of them intersect $\overline{ab}$. But this contradicts the assumption that $\overline{cd}$ is the first edge violating the property. □

An immediate consequence of the lemma above is that an edge that is flipped out of the triangulation cannot reappear. There are at most $O(nk)$ pairs of points in a point set of n points that give order-k Delaunay edges [4]. Therefore, we conclude:

Lemma 5 *The flip heuristic to reduce the number of local minima performs at most $O(nk)$ flips.*

To implement the flip heuristic efficiently, we maintain the set of all convex quadrilaterals in the current triangulation, with the order of the two triangles that would be created if the diagonal were flipped. The order of a triangle is the number of points in the circumcircle of the vertices of the triangle. Whenever a flip is done, we update the set of convex quadrilaterals. At most four are deleted and at most four new ones are created by the flip. We can find the order of the incident triangles by circular range counting queries. Since we are only interested in the count if the number of points in the circle is at most k, we implement circular range counting queries by point location in the order-k Voronoi diagram [6], taking $O(\log n + k)$ time per query. We conclude:

Theorem 6 *The flip heuristic to minimize the number of local minima in k-th order Delaunay triangulations on n points takes $O(nk^2 + nk \log n)$ time.*

4 The hull heuristic

The second heuristic for reducing the number of local minima is the hull heuristic. It was described in Gudmundsson et al. [4], and has an approximation factor of $\Theta(k^2)$ of the optimum. The hull heuristic adds a useful order-k Delaunay edge if it reduces the number of local minima. This edge may intersect several Delaunay edges, which are removed; the two holes in the triangulation that appear are retriangulated with the constrained Delaunay triangulation. No other higher-order Delaunay edges will be used that intersect the two holes. This guarantees that the final triangulation is order-k. It is known that two useful order-k Delaunay edges in general can give an order-$(2k-2)$ Delaunay triangulation [5], which is higher than allowed.

Here we give a slightly different implementation than in [4]. It is more efficient for larger values of k. Also, we include the adaptation that useful lower-order Delaunay edges are inserted first.

Assume that a point set P and an order k are given. We first compute the Delaunay triangulation $\mathcal{T}$ of P, and then compute the set E of all useful k-th order Delaunay edges, as in [4], in $O(nk \log n + nk^2)$ time. There are $O(nk)$ edges in E, and for each we have the lowest order $k' \leq k$ for which it is a useful order-k' Delaunay edge.

Next we determine the subset $P' \subseteq P$ of points that are a local minimum in the Delaunay triangulation.

	0	1	2	3	4	5	6	7	8	9	10
Calif. Hot Springs	47/47	43/43	33/31	29/26	25/20	24/19	23/18	21/18	18/16	18/16	17/15
Wren Peak	45/45	37/37	31/31	27/27	24/22	23/21	21/20	19/20	19/20	19/19	19/17
Quinn Peak	53/53	44/44	36/36	31/29	26/25	24/23	23/21	21/20	20/19	20/19	19/17
Sphinx Lakes	33/33	27/27	22/22	20/19	19/18	17/16	15/12	12/9	11/9	9/8	9/8
Split Mountain	24/24	17/17	14/14	9/9	9/9	9/9	8/8	7/8	6/7	6/6	5/4

Table 1: Results of the flip/hull heuristic for orders 0–10.

Then we determine the subset $E' \subseteq E$ of edges that connect a point of P' to a lower point. These steps trivially take $O(nk)$ time.

Sort the edges of E' by non-decreasing order. For every edge $e \in E'$, traverse $\mathcal{T}$ to determine the edges of $\mathcal{T}$ that intersect e. If any one of them is not a Delaunay edge or is a marked Delaunay edge, then we stop and continue with the next edge of E'. Otherwise, we remove all intersected Delaunay edges and mark all Delaunay edges of the polygonal hole that appears. Then we insert e and retriangulate the two polygons to the two sides of e using the Delaunay triangulation constrained to the polygons. We also mark these edges. Finally, we remove some edges from E'. If the inserted edge e made that a point $p \in P$ is no longer a local minimum, then we remove all other edges from E' where p is the highest endpoint.

Due to the marking of edges, no edge $e \in E'$ will be inserted if it intersects the hull of a previously inserted edge of E'. Also note that every edge of E' is treated in $O(k)$ time. We conclude:

Theorem 7 *The hull heuristic to minimize the number of local minima in k-th order Delaunay triangulations on n points takes* $O(nk^2 + nk \log n)$ *time.*

5 Experiments

Table 1 shows the number of local minima obtained after applying the flip and hull heuristics to five different terrains. The terrains roughly have 1800 vertices. The vertices were chosen by random sampling from elevation grids with about 100 times more points than the chosen sets.

The values in the table show that higher-order Delaunay triangulations indeed can give significantly fewer local minima than the standard Delaunay triangulation (0-th order). This effect is already clear at low orders, indicating that indeed, many local minima of Delaunay triangulations are caused by having chosen the wrong edges for the terrain (interpolation).

The difference in local minima between the flip and hull heuristics shows that the hull heuristic usually is a bit better, but there are some exceptions.

To test how good the results are, we also tested how many local minima of each terrain cannot be removed simply because there is no useful order-k Delaunay edge possible to a lower point. It turned out that the hull heuristic found the optimal order-k Delaunay triangulation in nearly all cases. Only in six cases we cannot be sure: there was one local minimum left that could potentially be removed.

6 Further research

It remains to be discovered whether minimizing the number of local minima in order-k Delaunay triangulations is already NP-hard for smaller values of k. This is unknown for $k \geq 2$ but significantly less than n^ϵ, for any small constant $\epsilon > 0$.

Another topic for further investigation is the removal of other artifacts from terrains. For example, for drainage applications, it is important to have the drainage network coincide with the triangulation edges, and not go over the middle of triangles.

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