The Minimum Area Spanning Tree Problem

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Abstract

We define and study the Minimum Area Spanning Tree (MAST) problem. Given a set $\mathcal{P}$ of n points in the plane, find a spanning tree $\mathcal{T}$ of $\mathcal{P}$ of minimum area, where the area of a spanning tree is the area of the union of the $n-1$ disks whose diameters are the edges in $\mathcal{T}$. We prove that the minimum spanning tree of $\mathcal{P}$ is a constant-factor approximation for MAST. We then apply this result to obtain a constant-factor approximation for the Minimum Area Range Assignment (MARA) problem and for the Minimum Area Connected Disk Graph (MACDG) problem. The former problem is a variant of the power assignment problem in radio networks, and the latter problem is a related natural problem.

1 Introduction

We introduce and study the Minimum Area Spanning Tree (MAST) problem. Given a set $\mathcal{P}$ of n points in the plane, find a spanning tree of $\mathcal{P}$ of minimum area, where the area of a spanning tree $\mathcal{T}$ of $\mathcal{P}$ is defined as follows. For each edge e in $\mathcal{T}$ draw the disk whose diameter is e. The *area* of $\mathcal{T}$ is then the area of the union of these $n-1$ disks. Although this problem seems natural (see also applications below), we are not aware of any previous work on this problem.

One of the main results of this paper (presented in Section 2) is that the minimum spanning tree of $\mathcal{P}$ is a constant-factor approximation for MAST. This is an important property of the minimum spanning tree as is shown below. (See, e.g., [3, 5] for background on the minimum spanning tree.)

We apply the result above to two problems from a class of problems that has received considerable attention. The first problem is a variant of the power assignment problem (also called the range assignment problem). Let $\mathcal{P}$ be a set of n points in the plane, representing n transmitters-receivers (or transmitters for short). In the standard version of the power assignment problem one needs to assign transmission ranges to the transmitters in $\mathcal{P}$, so that (i) the resulting communication graph is strongly connected (that is, the graph in which there exists a directed edge from $p_i \in \mathcal{P}$ to $p_j \in \mathcal{P}$ if and only if p_j lies in the disk D_{p_i} is strongly connected, where the radius of D_{p_i} is the transmission range, r_i, assigned to p_i), and (ii) the total power consumption (i.e., the cost of the assignment of ranges) is minimal, where the total power consumption is $\sum_{p_i \in \mathcal{P}} \text{area}(D_{p_i})$.

The power assignment problem is known to be NP-hard (see Kirousis et al. [6] and Clementi et al. [2]). Kirousis et al. [6] also obtain a 2-approximation for this problem, based on the minimum spanning tree of $\mathcal{P}$, and this is the best approximation known.

Consider now the variant of the power assignment problem where the second requirement above is replaced by (ii') the area of the union of the disks $D_{p_1}, \ldots, D_{p_n}$ is minimal. We refer to this problem as the Minimum Area Range Assignment (MARA) problem. In general, the presence of a foreign receiver (whether friendly or hostile) in the region $D_{p_1} \cup \cdots \cup D_{p_n}$ is undesirable, and the smaller the area of this region, the lower the probability that such a foreign receiver is present. In Section 3 we prove that the range assignment of Kirousis et al. (that is based on the minimum spanning tree) is also a constant-factor approximation for MARA.

Another related and natural problem for which we obtain a constant-factor approximation (in the full version of this paper) is the following. Let $\mathcal{P}$ be a set of n points in the plane. For each point $p \in \mathcal{P}$, draw a disk D_{p_i} of radius 0 or more, such that, (i) the resulting disk graph is connected (that is, the graph in which there exists an edge between $p_i \in \mathcal{P}$ and $p_j \in \mathcal{P}$ if and only if $D_{p_i} \cap D_{p_j} \neq \emptyset$ is connected), and (ii) the area of the union of the disks $D_{p_1}, \ldots, D_{p_n}$ is minimal. We refer to this problem as the Minimum Area Connected Disk Graph (MACDG) problem. (See, e.g., [4, 7] for background on intersection graphs and disk graphs in particular.)

A potentially interesting property concerning the area of the minimum spanning tree that is obtained as an intermediate result in Section 2 is that the depth of the arrangement of the disks corresponding to the edges of the minimum spanning tree is bounded by some constant. Notice that this property does not follow immediately from the fact that the degree of the minimum spanning tree is at most 6, as is shown in Figure 2.

Finally, all the above results hold in any fixed dimension d (with obvious modifications).

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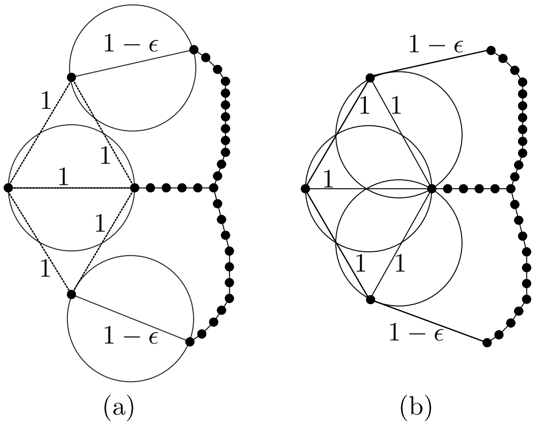


Figure 1: A minimum spanning tree is not necessarily a minimum area spanning tree. (a) The minimum spanning tree. (b) A minimum area spanning tree.

2 MST is a constant-factor approximation for MAST

Let $\mathcal{T}$ be any spanning tree of $\mathcal{P}$. For an edge e in $\mathcal{T}$, let $D(e)$ denote the disk whose diameter is e. Put $D(\mathcal{T}) = \{D(e) \mid e \text{ is an edge in } \mathcal{T}\}$, $\bigcup_{\mathcal{T}} = \bigcup_{e\in\mathcal{T}} D(e)$, and $\sigma_{\mathcal{T}} = \sum_{e\in\mathcal{T}} \text{area}(D(e))$. Let MST be a minimum spanning tree of $\mathcal{P}$. MST is not necessarily a solution for the Minimum Area Spanning Tree (MAST) problem; see Figure 1. In this section we prove that MST is a constant-factor approximation for MAST, that is, $\text{area}(\bigcup_{\text{MST}}) = O(\text{area}(\bigcup_{\text{OPT}}))$, where OPT is an optimal spanning tree, i.e., a solution to MAST.

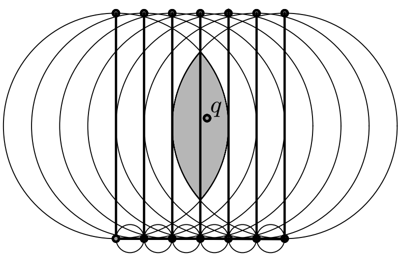


Figure 2: A spanning tree $\mathcal{T}$ of degree 3, and a point q (in the interior of a cell of the arrangement of the disks in $D(\mathcal{T})$) of depth $O(n)$.

We begin by showing another interesting property of MST, namely, that the depth of any point p in the interior of a cell of the arrangement of the disks in $D(\text{MST})$ is bounded by a small constant. This property does not follow directly from the fact that the degree of MST is bounded by 6; see Figure 2. Let MST_p be a minimum spanning tree for $\mathcal{P} \cup \{p\}$. We need the following known and easy claim.

Claim 1 *We may assume that there is no edge (a, b) in MST_p, such that, (a, b) is not in MST and both a and b are points of $\mathcal{P}$.*

Proof. Assume there is such an edge (a, b) in MST_p. Consider the path in MST between a and b. At least one of the edges along this path is not in MST_p. Let e be such an edge. $|e| \le |(a, b)|$, since otherwise (a, b) would have been chosen by the algorithm that computed MST (e.g., Kruskal's minimum spanning tree algorithm [1]). Therefore, we may replace the edge (a, b) in MST_p by e, without increasing the total weight of the tree. □

An immediate corollary of this claim is that we may assume that if e is an edge in MST_p but not in MST, then one of e's endpoints is p.

The proof of the following lemma appears in the full version of this paper.

Lemma 1 $\sigma_{\text{MST}} \le 5\,\text{area}(\bigcup_{\text{MST}})$.

Remark. A more careful analysis allows one to replace the 5 in the statement of the lemma above by 4 or perhaps even by 3. However, for our purpose 5 is good enough.

Let OPT be an optimal spanning tree of $\mathcal{P}$, i.e., a solution to MAST. We use OPT to construct another spanning tree, ST, of $\mathcal{P}$. Initially ST is empty. Let e_1 be the longest edge in OPT. Draw two concentric disks C_1 and C_1^3 around the mid point of e_1 of diameters $|e_1|$ and $3|e_1|$, respectively. Compute a minimum spanning tree of the points of $\mathcal{P}$ lying in C_1^3, using Kruskal's algorithm [1]. Whenever an edge is chosen by Kruskal's algorithm, it is immediately added to ST. See Figure 3. Let S_1 denote the set of edges that have been added to ST in this (first) iteration.

Next, let e_2 be the longest edge in OPT, such that at least one of its endpoints lies outside C_1^3. As for e_1, draw two concentric disks C_2 and C_2^3 around the mid point of e_2 of diameters $|e_2|$ and $3|e_2|$, respectively. Apply Kruskal's minimum spanning algorithm to the points of $\mathcal{P}$ lying in C_2^3 with the following modification. The next edge in the sorted list of potential edges is chosen by the algorithm if and only if it is not in ST and its addition to ST does not create a cycle in ST. Moreover, when an edge is chosen by the algorithm it is immediately added to ST; see Figure 4 (a) and (b). Let S_2 denote the set of edges that have been added to ST in this iteration.

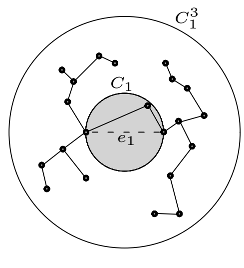
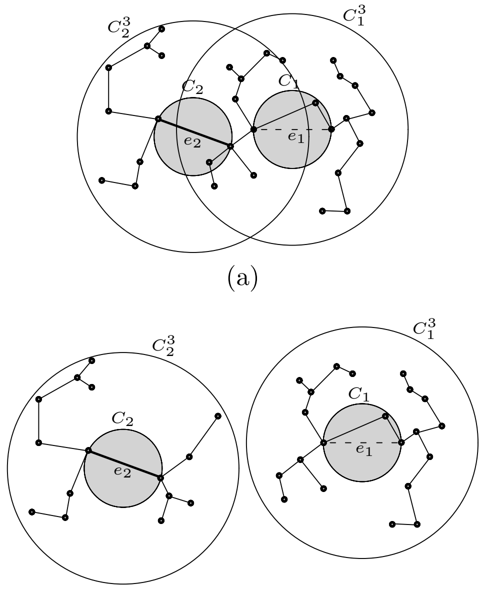


Figure 3: ST after choosing e_1.



(a)

(b)

Figure 4: ST after choosing e_1 and e_2. (a) One of the end points of e_2 is in C_1^3. (b) Both endpoints of e_2 are not in C_1^3.

In the i'th iteration, let e_i be the longest edge in OPT, such that there is no path yet in ST between its endpoints. Draw two concentric circles C_i and C_i^3 around the mid point of e_i, and apply Kruskal's minimum spanning tree algorithm with the modification above to the points of $\mathcal{P}$ lying in C_i^3. Let S_i denote the set of edges that have been added to ST in this iteration. The process ends when for each edge e in OPT there already exists a path in ST between the endpoints of e.

Claim 2 *For each i, S_i is a subset of the edge set of the minimum spanning tree* MST_i *that is obtained by applying Kruskal's algorithm,* without *the modification above, to the points in C_i^3.*

Proof. Let e be an edge that was added to ST during the i'th iteration. If e is not chosen by Kruskal's algorithm (without the modification above), it is only because, when considering e, a path between its two endpoints already exists in MST_i. But this implies that e could not have been added to ST, since, any edge already in MST_i was either also added to ST or was not added since there already existed a path in ST between its two endpoints. Thus, when e was considered by the modified algorithm it should have been rejected. We conclude that e must be in MST_i. □

Claim 3 ST *is a spanning tree of $\mathcal{P}$.*

Proof. Since only edges that do not create a cycle in ST were added to ST, there are no cycles in ST. Also ST is connected, since otherwise there still exists an edge in OPT that forces another iteration of the construction algorithm. □

Let $\mathcal{C}$ denote the set of the disks $C_1, \ldots, C_k$, and let $\mathcal{C}^3$ denote the set of the disks $C_1^3, \ldots, C_k^3$, where k is the number of iterations in the construction of ST.

Claim 4 *For any pair of disks C_i, C_j in $\mathcal{C}$, $i \neq j$, it holds that $C_i \cap C_j = \emptyset$.*

Proof. Let C_i be any disk in $\mathcal{C}$. We show that for any disk $C_j \in \mathcal{C}$ such that $j > i$, $C_i \cap C_j = \emptyset$. From the construction of ST it follows that $|e_j|$, the diameter of C_j, is smaller or equal to $|e_i|$, the diameter of C_i. Moreover, at least one of the endpoints of e_j lies outside C_i^3 (since if both endpoints of e_j lie in C_i^3, then, by the end of the i'th iteration, a path connecting between these endpoints must already exist in ST). Therefore, C_j whose center coincides with the mid point of e_j, cannot intersect C_i. □

Claim 5 $\sigma_{\text{ST}} = O(\text{area}(\bigcup_{\text{OPT}}))$.

Proof. Recall that $\sigma_{\text{ST}} = \Sigma_i \sigma_{S_i}$, where $\sigma_{S_i} = \Sigma_{e \in S_i} \text{area}(D(e))$. We first show by the sequence of inequalities below that $\sigma_{S_i} = O(\text{area}(C_i))$.

$$\sigma_{S_i} \leq^1 \sigma_{\text{MST}_i} \leq^2 5\,\text{area}(\bigcup_{\text{MST}_i}) =^3 O(\text{area}(C_i^3)) =^4 O(\text{area}(C_i))\,.$$

The first inequality follows immediately from Claim 2. The second inequality is true by Lemma 1. Consider Equality 3. Since all edges in MST_i are contained in C_i^3, it holds that $\bigcup_{\text{MST}_i}$ is contained in a disk that is obtained by expanding C_i^3 by some constant factor. It follows that $\text{area}(\bigcup_{\text{MST}_i}) = O(\text{area}(C_i^3)) = O(\text{area}(C_i))$.

Therefore,

$$\sigma_{\text{ST}} = \Sigma_i \sigma_{S_i} = \Sigma_i O(\text{area}(C_i))\,.$$

But according to Claim 4, the latter expression is equal to $O(\text{area}(\bigcup_{\mathcal{C}}))$, and, since $\mathcal{C}$ is a subset of $D(\text{OPT})$, we conclude that $\sigma_{\text{ST}} = O(\text{area}(\bigcup_{\text{OPT}}))$. □

We are now ready to prove the main result of this section.

Theorem 2 MST *is a constant-factor approximation for* MAST, *that is,* $\text{area}(\bigcup_{\text{MST}}) \leq c \cdot \text{area}(\bigcup_{\text{OPT}})$, *for some constant c.*

Proof.

$$\text{area}(\bigcup_{\text{MST}}) \leq^1 \sigma_{\text{MST}} \leq^2 \sigma_{\text{ST}} \leq^3 c \cdot \text{area}(\bigcup_{\text{OPT}})\,.$$

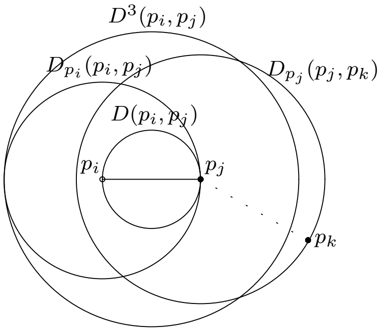


Figure 5: $(p_i,p_j) \in$ MST; $D(p_i,p_j) \in D(\text{MST})$; $D_{p_i}(p_i,p_j), D_{p_j}(p_j,p_k) \in$ RA; $D^3(p_i,p_j) \in D^3(\text{MST})$.

The first inequality is trivial. The second inequality holds for any spanning tree of $\mathcal{P}$; that is, for any spanning tree $\mathcal{T}$, $\sigma_{\text{MST}} \leq \sigma_{\mathcal{T}}$. (Since if the lengths $|e|$ of the edges are replaced with weights $\pi|e|^2/2$, we remain with the same minimum spanning tree.) The third inequality is proven in Claim 5. □

3 A constant-factor approximation for MARA

MST induces an assignment of ranges to the points of $\mathcal{P}$. Let $p_i \in \mathcal{P}$ and let r_i be the length of the longest edge in MST that is connected to p_i, then the range that is assigned to p_i is r_i. Put RA $= \{D_{p_1}, \ldots, D_{p_n}\}$, where D_{p_i} is the disk of radius r_i centered at p_i. In this section we apply the main result of the previous section (i.e., MST is a constant-factor approximation for MAST), in order to prove that the range assignment that is induced by MST is a constant-factor approximation for the Minimum Area Range Assignment (MARA) problem. That is, (i) the corresponding (directed) communication graph is strongly connected, and (ii) the area of the union of the disks in RA is bounded by some constant times the area of the union of the transmission disks in an optimal range assignment, i.e., a solution to MARA.

The first requirement above was already proven by Kirousis et al. [6], who showed that the range assignment induced by MST is a 2-approximation for the standard range assignment problem. Let OPT^R denote an optimal range assignment, i.e., a solution to MARA. It remains to prove the second requirement above.

Claim 6 $\text{area}(\bigcup_{RA}) \leq 9\,\text{area}(\bigcup_{\text{MST}})$.

Proof. We define an auxiliary set of disks. For each edge e in MST, draw a disk of diameter $|3e|$ centered at the mid point of e. Let $D^3(\text{MST})$ denote the set of these $n-1$ disks; see Figure 5. We now observe that $\text{area}(\bigcup_{RA}) \leq \text{area}(\bigcup_{D^3(\text{MST})})$. This is true since for each $p_i \in \mathcal{P}$, $D_{p_i} = D_{p_i}(p_i,p_j)$ for some point $p_j \in \mathcal{P}$ that is connected to p_i (in MST) by an edge, and $D_{p_i}(p_i,p_j)$ is contained in the disk of $D^3(\text{MST})$ corresponding to the edge (p_i,p_j). Finally, clearly $\text{area}(\bigcup_{D^3(\text{MST})}) \leq 9\,\text{area}(\bigcup_{\text{MST}})$. □

Theorem 3 *RA is a constant-factor approximation for MARA, that is,* $\text{area}(\bigcup_{\text{RA}}) \leq c' \cdot \text{area}(\bigcup_{\text{OPT}^R})$, *for some constant* c'.

Proof. The proof is based on the observation that the (directed) communication graph corresponding to OPT^R contains a spanning tree, and on the main result of Section 2. Let p be any point in $\mathcal{P}$. We construct a spanning tree $\mathcal{T}$ of $\mathcal{P}$ as follows. For each point $q \in \mathcal{P}$, $q \neq p$, compute a shortest (in terms of number of hops) directed path from q to p, and add the edges in this path to $\mathcal{T}$. Now make all edges in $\mathcal{T}$ undirected. $\mathcal{T}$ is a spanning tree of $\mathcal{P}$. For each edge (p_i,p_j) in $\mathcal{T}$, the disk $D(p_i,p_j)$ is contained either in the transmission disk of p_i (in OPT^R), or in the transmission disk of p_j (in OPT^R). Hence, $\bigcup_{\mathcal{T}} \subseteq \bigcup_{\text{OPT}^R}$.

The following sequence of inequalities completes the proof. (OPT denotes a solution to MAST.)

$$\text{area}(\bigcup_{\text{RA}}) \leq^1 9\,\text{area}(\bigcup_{\text{MST}}) \leq^2 9c \cdot \text{area}(\bigcup_{\text{OPT}}) \leq^3 9c \cdot \text{area}(\bigcup_{\mathcal{T}}) \leq^4 9c \cdot \text{area}(\bigcup_{\text{OPT}^R}).$$

The first inequality follows from Claim 6; the second inequality follows from Theorem 2; the third inequality follows from the definition of OPT; the fourth inequality was shown above. □

References

[1] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. Introduction to Algorithms. In *MIT Press*, Second Edition, 2001.

[2] A. E. F. Clementi, P. Penna, and R. Silvestri. On the Power Assignment Problem in Radio Networks. In *Electronic Colloquium on Computational Complexity*, 2000.

[3] D. Eppstein. Spanning Trees and Spanners. In *Handbook of Computational Geometry*, pages 425–461. J.-R. Sack and J. Urrutia, eds., Elsevier, 1999.

[4] M. C. Golumbic. Algorithmic Graph Theory and Perfect Graphs. In *Academic Press*, New York, 1980.

[5] R. L. Graham and P. Hell. On the History of the Minimum Spanning Tree Problem. In *Annals of the History of Computing*, 7:43–57, 1985.

[6] L.M. Kirousis, E. Kranakis, D. Krizanc, and A. Pelc. Power Consumption in Packet Radio Networks. In *14th Annual Sympos. Theoretical Aspects of Computer Science (STACS)*, pages 363–374. LNCS 1200, 1997.

[7] T.A. McKee and F.R. McMorris. Topics in Intersection Graph Theory. In Volume 2 of *Monographs on Discrete Mathematics and Applications*, SIAM, 1999.