Multi-way Compressed Sensing for Sparse Low-rank Tensors

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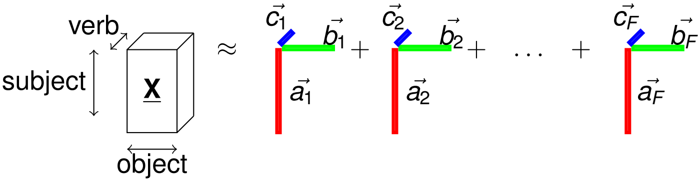
Tensor? What is this?

- Has different formal meaning in Physics (spin, symmetries)
- Informally adopted in CS as shorthand for *three-way* array: dataset $\underline{\mathbf{X}}$ indexed by three indices, (i, j, k)-th entry $\underline{\mathbf{X}}(i, j, k)$.
- For two vectors $\mathbf{a}$ $(I \times 1)$ and $\mathbf{b}$ $(J \times 1)$, $\mathbf{a} \circ \mathbf{b}$ is an $I \times J$ rank-one matrix with (i, j)-th element $\mathbf{a}(i)\mathbf{b}(j)$; i.e., $\mathbf{a} \circ \mathbf{b} = \mathbf{a}\mathbf{b}^T$.
- For three vectors, $\mathbf{a}$ $(I \times 1)$, $\mathbf{b}$ $(J \times 1)$, $\mathbf{c}$ $(K \times 1)$, $\mathbf{a} \circ \mathbf{b} \circ \mathbf{c}$ is an $I \times J \times K$ rank-one three-way array with (i, j, k)-th element $\mathbf{a}(i)\mathbf{b}(j)\mathbf{c}(k)$.
- The *rank of a three-way array* $\underline{\mathbf{X}}$ is the smallest number of outer products needed to synthesize $\underline{\mathbf{X}}$.
- 'Curiosities':
  - Two-way $(I \times J)$: row-rank = column-rank = rank $\leq \min(I, J)$;
  - Three-way: row-rank $\neq$ column-rank $\neq$ "tube"-rank $\neq$ rank
  - Two-way: rank(randn(I,J))=min(I,J) w.p. 1;
  - Three-way: rank(randn(2,2,2)) is a RV (2 w.p. 0.3, 3 w.p. 0.7)

NELL @ CMU / Tom Mitchell

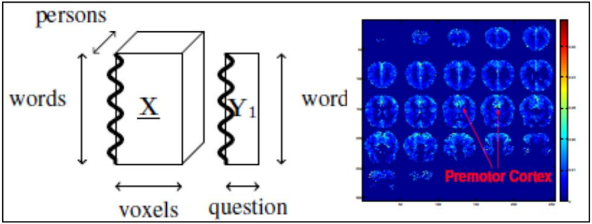
- Crawl web, learn language 'like children do': encounter new concepts, learn from context
- NELL triplets of "subject-verb-object" naturally lead to a 3-mode tensor



- Each rank-one factor corresponds to a *concept*, e.g., 'leaders' or 'tools'
- E.g., say $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$ corresponds to 'leaders': subjects/rows with high score on $\mathbf{a}_1$ will be "Obama", "Merkel", "Steve Jobs", objects/columns with high score on $\mathbf{b}_1$ will be "USA", "Germany", "Apple Inc.", and verbs/fibers with high score on $\mathbf{c}_1$ will be 'verbs', like "lead", "is-president-of", and "is-CEO-of".

Semantic analysis of Brain fMRI data

- fMRI $\rightarrow$ semantic category scores



- fMRI mode is vectorized ($O(10^5 - 10^6)$)
- Could treat as three separate spatial modes $\rightarrow$ 5-way array
- ... or even include time as another dimension $\rightarrow$ 6-way array

Low-rank tensor decomposition / approximation

$$\underline{\mathbf{X}} \approx \sum_{f=1}^{F} \mathbf{a}_f \circ \mathbf{b}_f \circ \mathbf{c}_f,$$

- Parallel factor analysis (PARAFAC) model [Harshman '70-'72], a.k.a. canonical decomposition [Carroll & Chang, '70], a.k.a. CP; cf. [Hitchcock, '27]
- PARAFAC can be written as a system of matrix equations $\mathbf{X}_k = \mathbf{A}\mathbf{D}_k(\mathbf{C})\mathbf{B}^T$, where $\mathbf{D}_k(\mathbf{C})$ is a diagonal matrix holding the k-th row of $\mathbf{C}$ in its diagonal; or in compact matrix form as $\mathbf{X} \approx (\mathbf{B} \odot \mathbf{A})\mathbf{C}^T$, using the Khatri-Rao product.
- In particular, employing a property of the Khatri-Rao product,

$$\mathbf{X} \approx (\mathbf{B} \odot \mathbf{A})\mathbf{C}^T \Longleftrightarrow \text{vec}(\mathbf{X}) \approx (\mathbf{C} \odot \mathbf{B} \odot \mathbf{A})\,\mathbf{1},$$

where $\mathbf{1}$ is a vector of all 1's.

Uniqueness

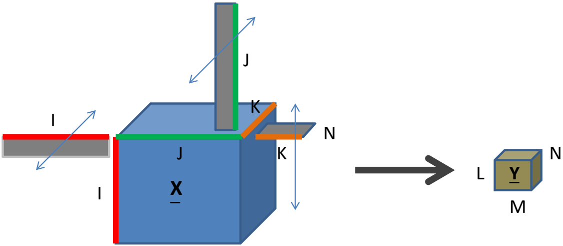
- The distinguishing feature of the PARAFAC model is its essential uniqueness: under certain conditions, $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ can be identified from $\mathbf{X}$, i.e., they are unique up to permutation and scaling of columns [Kruskal '77, Sidiropoulos *et al* '00 - '07, de Lathauwer '04-, Stegeman '06-]
- Consider an $I \times J \times K$ tensor $\underline{\mathbf{X}}$ of rank F. In vectorized form, it can be written as the $IJK \times 1$ vector $\mathbf{x} = (\mathbf{A} \odot \mathbf{B} \odot \mathbf{C})\, \mathbf{1}$, for some $\mathbf{A}$ $(I \times F)$, $\mathbf{B}$ $(J \times F)$, and $\mathbf{C}$ $(K \times F)$ - a PARAFAC model of size $I \times J \times K$ and order F parameterized by $(\mathbf{A}, \mathbf{B}, \mathbf{C})$.
- The *Kruskal-rank* of $\mathbf{A}$, denoted $k_{\mathbf{A}}$, is the maximum k such that *any* k columns of $\mathbf{A}$ are linearly independent ($k_{\mathbf{A}} \leq r_{\mathbf{A}} := \text{rank}(\mathbf{A})$).
- Given $\underline{\mathbf{X}}$ ($\Leftrightarrow \mathbf{x}$), if $k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}} \geq 2F + 2$, then $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ are unique up to a common column permutation and scaling

Big data: need for compression

- Tensors can easily become really big! - size exponential in the number of dimensions (‘ways’, or ‘modes’).
- Cannot load in main memory; can reside in cloud storage.
- Tensor compression?
- Commonly used compression method for ‘moderate’-size tensors: fit orthogonal Tucker3 model, regress data onto fitted mode-bases.
- Lossless if exact mode bases used [CANDELINC]; but Tucker3 fitting is itself cumbersome for big tensors (big matrix SVDs), cannot compress below mode ranks without introducing errors
- If tensor is sparse, can store as $[i, j, k, value]$ + use specialized sparse matrix / tensor alorithms [(Sparse) Tensor Toolbox, Bader & Kolda]. Useful if sparse representation can fit in main memory.

Tensor compression

- Consider compressing $\mathbf{x}$ into $\mathbf{y} = \mathbf{S}\mathbf{x}$, where $\mathbf{S}$ is $d \times IJK$, $d \ll IJK$.
- In particular, consider a specially structured compression matrix $\mathbf{S} = \mathbf{U}^T \otimes \mathbf{V}^T \otimes \mathbf{W}^T$
- Corresponds to multiplying (every slab of) $\underline{\mathbf{X}}$ from the I-mode with $\mathbf{U}^T$, from the J-mode with $\mathbf{V}^T$, and from the K-mode with $\mathbf{W}^T$, where $\mathbf{U}$ is $I \times L$, $\mathbf{V}$ is $J \times M$, and $\mathbf{W}$ is $K \times N$, with $L \leq I$, $M \leq J$, $N \leq K$ and $LMN \ll IJK$



Key

- Due to a property of the Kronecker product

$$\left(\mathbf{U}^T \otimes \mathbf{V}^T \otimes \mathbf{W}^T\right)(\mathbf{A} \odot \mathbf{B} \odot \mathbf{C}) =$$

$$\left((\mathbf{U}^T\mathbf{A}) \odot (\mathbf{V}^T\mathbf{B}) \odot (\mathbf{W}^T\mathbf{C})\right),$$

from which it follows that

$$\mathbf{y} = \left((\mathbf{U}^T\mathbf{A}) \odot (\mathbf{V}^T\mathbf{B}) \odot (\mathbf{W}^T\mathbf{C})\right)\mathbf{1} = \left(\tilde{\mathbf{A}} \odot \tilde{\mathbf{B}} \odot \tilde{\mathbf{C}}\right)\mathbf{1}.$$

i.e., the compressed data follow a PARAFAC model of size $L \times M \times N$ and order F parameterized by $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$, with $\tilde{\mathbf{A}} := \mathbf{U}^T\mathbf{A}$, $\tilde{\mathbf{B}} := \mathbf{V}^T\mathbf{B}$, $\tilde{\mathbf{C}} := \mathbf{W}^T\mathbf{C}$.

Random multi-way compression can be better!

Theorem 1:

- Assume that the columns of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are sparse, and let n_a (n_b, n_c) be an upper bound on the number of nonzero elements per column of $\mathbf{A}$ (respectively $\mathbf{B}$, $\mathbf{C}$).
- Let the mode-compression matrices $\mathbf{U}$ ($I \times L, L \leq I$), $\mathbf{V}$ ($J \times M, M \leq J$), and $\mathbf{W}$ ($K \times N, N \leq K$) be randomly drawn from an absolutely continuous distribution with respect to the Lebesgue measure in $\mathbb{R}^{IL}$, $\mathbb{R}^{JM}$, and $\mathbb{R}^{KN}$, respectively.
- If

$$\min(L, k_{\mathbf{A}}) + \min(M, k_{\mathbf{B}}) + \min(N, k_{\mathbf{C}}) \geq 2F + 2, \quad \text{and}$$

$$L \geq 2n_a, \quad M \geq 2n_b, \quad N \geq 2n_c,$$

then the original factor loadings $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are almost surely identifiable from the compressed data.

Proof rests on two lemmas + Kruskal

- Lemma 1: Consider $\tilde{\mathbf{A}} := \mathbf{U}^T\mathbf{A}$, where $\mathbf{A}$ is $I \times F$, and let the $I \times L$ matrix $\mathbf{U}$ be randomly drawn from an absolutely continuous distribution with respect to the Lebesgue measure in $\mathbb{R}^{IL}$ (e.g., multivariate Gaussian with a non-singular covariance matrix). Then $k_{\tilde{\mathbf{A}}} = \min(L, k_{\mathbf{A}})$ almost surely (with probability 1).
- Lemma 2: Consider $\tilde{\mathbf{A}} := \mathbf{U}^T\mathbf{A}$, where $\tilde{\mathbf{A}}$ and $\mathbf{U}$ are given and $\mathbf{A}$ is sought. Suppose that every column of $\mathbf{A}$ has at most n_a nonzero elements, and that $k_{\mathbf{U}^T} \geq 2n_a$. (The latter holds with probability 1 if the $I \times L$ matrix $\mathbf{U}$ is randomly drawn from an absolutely continuous distribution with respect to the Lebesgue measure in $\mathbb{R}^{IL}$, and $\min(I, L) \geq 2n_a$.) Then $\mathbf{A}$ is the unique solution with at most n_a nonzero elements per column [Donoho & Elad, '03]

Complexity

- First fitting PARAFAC in compressed space and then recovering the sparse **A**, **B**, **C** from the fitted compressed factors entails complexity $O(LMNF + (I^{3.5} + J^{3.5} + K^{3.5})F)$.
- Using sparsity first and then fitting PARAFAC in raw space entails complexity $O(IJKF + (IJK)^{3.5})$ - the difference is huge.
- Also note that the proposed approach does not require computations in the uncompressed data domain, which is important for big data that do not fit in memory for processing.

Further compression - down to $O(\sqrt{F})$ in 2/3 modes

Theorem 2:

- Assume that the columns of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are sparse, and let n_a (n_b, n_c) be an upper bound on the number of nonzero elements per column of $\mathbf{A}$ (respectively $\mathbf{B}$, $\mathbf{C}$).
- Let the mode-compression matrices $\mathbf{U}$ ($I \times L, L \leq I$), $\mathbf{V}$ ($J \times M, M \leq J$), and $\mathbf{W}$ ($K \times N, N \leq K$) be randomly drawn from an absolutely continuous distribution with respect to the Lebesgue measure in $\mathbb{R}^{IL}$, $\mathbb{R}^{JM}$, and $\mathbb{R}^{KN}$, respectively.
- If

$$r_{\mathbf{A}} = r_{\mathbf{B}} = r_{\mathbf{C}} = F$$

$$L(L-1)M(M-1) \geq 2F(F-1), \ N \geq F, \quad \text{and}$$

$$L \geq 2n_a, \quad M \geq 2n_b, \quad N \geq 2n_c,$$

then the original factor loadings $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are almost surely identifiable from the compressed data up to a common column permutation and scaling.

Proof: Lemma 3 + results on a.s. ID of PARAFAC

- Lemma 3: Consider $\tilde{\mathbf{A}} = \mathbf{U}^T\mathbf{A}$, where $\mathbf{A}$ ($I \times F$) is deterministic, tall/square ($I \geq F$) and full column rank $r_{\mathbf{A}} = F$, and the elements of $\mathbf{U}$ ($I \times L$) are i.i.d. Gaussian zero mean, unit variance random variables. Then the distribution of $\tilde{\mathbf{A}}$ is nonsingular multivariate Gaussian.
- From [Stegeman, ten Berge, de Lathauwer 2006] (see also [Jiang, Sidiropoulos 2004], we know that PARAFAC is almost surely identifiable if the loading matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ are randomly drawn from an absolutely continuous distribution with respect to the Lebesgue measure in $\mathbb{R}^{(L+M)F}$, $\tilde{\mathbf{C}}$ is full column rank, and $L(L-1)M(M-1) \geq 2F(F-1)$.

Generalization to higher-way arrays

- Theorem 3: Let $\mathbf{x} = (\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_\delta)\,\mathbf{1} \in \mathbb{R}^{\prod_{d=1}^{\delta} I_d}$, where $\mathbf{A}_d$ is $I_d \times F$, and consider compressing it to $\mathbf{y} = \left(\mathbf{U}_1^T \otimes \cdots \otimes \mathbf{U}_\delta^T\right)\mathbf{x} = \left((\mathbf{U}_1^T\mathbf{A}_1) \odot \cdots \odot (\mathbf{U}_\delta^T\mathbf{A}_\delta)\right)\mathbf{1} = \left(\tilde{\mathbf{A}}_1 \odot \cdots \odot \tilde{\mathbf{A}}_\delta\right)\mathbf{1} \in \mathbb{R}^{\prod_{d=1}^{\delta} L_d}$, where the mode-compression matrices $\mathbf{U}_d$ ($I_d \times L_d, L_d \leq I_d$) are randomly drawn from an absolutely continuous distribution with respect to the Lebesgue measure in $\mathbb{R}^{I_d L_d}$. Assume that the columns of $\mathbf{A}_d$ are sparse, and let n_d be an upper bound on the number of nonzero elements per column of $\mathbf{A}_d$, for each $d \in \{1, \cdots, \delta\}$. If

$$\sum_{d=1}^{\delta} \min(L_d, k_{\mathbf{A}_d}) \geq 2F + \delta - 1, \quad \text{and} \quad L_d \geq 2n_d, \quad \forall d \in \{1, \cdots, \delta\},$$

  then the original factor loadings $\{\mathbf{A}_d\}_{d=1}^{\delta}$ are almost surely identifiable from the compressed data $\mathbf{y}$ up to a common column permutation and scaling.
- Various additional results possible, e.g., generalization of Theorem 2.