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Comment on “Significance tests for the wavelet power and the wavelet power spectrum” by Ge (2007)

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Abstract. As the main result in Ge’s paper, Ge announced that he proved a formula on the distribution of Morlet wavelet power spectrum of continuous-time Gaussian white noise in a rigorous statistical framework. In this paper, we will show that Ge’s formula is wrong and each step of Ge’s proof is wrong. Moreover, we give and prove a correct formula in a rigorous statistical framework.

Keywords. Meteorology and atmospheric dynamics (climatology)

1 Introduction

In the past 20 yr, wavelet analysis has been widely applied in many branches of science and engineering. The continuous wavelet transform possesses the ability to construct a time-frequency representation of a signal that offers very good time and frequency localization, so wavelet transforms can analyze localized intermittent periodicities of potentially great interest in geophysical time series analysis. At the same time, significance tests must be applied to distinguish real features from noise.

A wavelet ψ is denoted as follows:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right).$$

The wavelet transform of a continuous-time signal $x(t)$ is

$$T(a,b) = \int_{-\infty}^{\infty} x(t)\,\psi^*_{a,b}(t)\,\mathrm{d}t, \tag{1}$$

where $\psi^*_{a,b}$ represents the complex conjugate of $\psi_{a,b}$. $|T(a,b)|^2$ is defined as the wavelet power spectrum of $x(t)$. In application, one often uses the Morlet wavelet with parameter ω_0, which is defined as

$$\psi(t) = \pi^{-\frac{1}{4}}\,e^{-\frac{t^2}{2}}\,e^{i\omega_0 t}. \tag{2}$$

A continuous-time Gaussian white noise $x(t)$ satisfies

$$E[x(t)] = 0,$$

$$\mathrm{Cov}\,[x(t), x(t')] = \delta(t-t')\,\sigma^2,$$

where δ is the delta function.

For the distribution of the Morlet wavelet power spectrum of Gaussian white noise, Ge (2007) announced that “the results given by Torrence and Compo (1998) are numerically accurate when adjusted by a factor of the sampling period, while some of their statements require reassessment.” In fact, Ge used a wrong argument to give a wrong formula on the distribution of the Morlet wavelet power spectrum $|T(a,b)|^2$ of continuous-time Gaussian white noise $x(t)$ as follows:

$$|T((a,b)|^2 \Rightarrow \frac{1}{2}\delta t\,\sigma^2\,\chi^2_2 \tag{3}$$

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(see Eq. (18) of Ge's paper), where χ_2^2 is the chi-squared distribution with two degrees of freedom and δt is the sampling period. Hereafter, the symbol $\Rightarrow$ means "is distributed as". From Eqs. (1) and (2), we know that $T(a,b)$ is a function of the parameters a, b, ω_0 and it is independent of sampling period δt. Clearly, Ge's formula is wrong, and so Ge's correction for Torrence and Compo's result is also wrong

In Sect. 2, we will show that each step of Ge's proof is wrong. In Sect. 3, in a rigorous statistical framework, we will give and prove that, for the continuous-time Gaussian white noise $x(t)$, its Morlet wavelet power spectrum is distributed as follows:

$$|T(a,b)|^2 \Longrightarrow \frac{\sigma^2}{2}(1+e^{-\omega_0^2})\, X_1^2 + \frac{\sigma^2}{2}(1-e^{-\omega_0^2})\, X_2^2, \qquad (4)$$

where X_1 and X_2 are independent Gaussian random variables with mean 0 and variance 1. If we use a Morlet wavelet with parameter $\omega_0 = 6$, we have

$$e^{-\omega_0^2} = e^{-36} \approx 0,$$

and so

$$|T(a,b)|^2 \Rightarrow \frac{\sigma^2}{2}\chi_2^2.$$

Torrence and Compo (1998) use heuristic ideas to reason that the distribution of $|T(a,b)|^2$ should be a simple chi-squared function, but they did not derive the above formula from first principles. In Sect. 4, we do some numerical experiments to compare our formula with Ge's formula.

2 Ge's wrong proof

Ge's proof for the distribution of wavelet power spectrum of continuous-time Gaussian white noise (i.e., the proof of Eq. 3) consists of four steps.

Step 1. The real part $\mathrm{Re}[T]$ and the imaginary part $\mathrm{Im}[T]$ of wavelet transform are

$$\mathrm{Re}[T] = \int x(t)\mathrm{Re}[\psi_{a,b}^*(t)]\,\mathrm{d}t,$$

$$\mathrm{Im}[T] = \int x(t)\mathrm{Im}[\psi_{a,b}^*(t)]\,\mathrm{d}t.$$

Ge tried to prove that $\mathrm{Cov}[\mathrm{Re}[T], \mathrm{Im}[T]] = 0$. Ge's whole proof is

$$\mathrm{Cov}[\mathrm{Re}[T], \mathrm{Im}[T]]$$

$$= \mathrm{Cov}\left[\int x(t)\mathrm{Re}[\psi_{a,b}^*(t)]\,\mathrm{d}t \int x(t)\mathrm{Im}[\psi_{a,b}^*(t)]\,\mathrm{d}t\right]$$

$$= \int\int \mathrm{Cov}[x(t), x(t')]\,\mathrm{Re}[\psi_{a,b}^*(t)]\,\mathrm{Im}[\psi_{a,b}^*(t')]\,\mathrm{d}t\,\mathrm{d}t'$$

$$= 0 \qquad \text{for} \quad t \neq t'.$$

Furthermore, it can be verified that

$$\mathrm{Cov}[\mathrm{Re}[T],\ \mathrm{Im}[T]] \equiv 0$$

for $\psi_{a,b}(t)$ being the Morlet wavelets even if $t = t'$.

In the above proof, Ge claimed that

$$\mathrm{Cov}[\mathrm{Re}[T], \mathrm{Im}[T]] = 0 \quad \text{for} \quad t \neq t'.$$

However, by the definition of wavelet transform,

$$\mathrm{Cov}[\mathrm{Re}[T], \mathrm{Im}[T]]$$

is a function of a, b, ω_0 and is independent of t, t'. Ge's proof is obviously wrong.

Step 2. Ge gave two wrong results for variances of $\mathrm{Re}[T]$ and $\mathrm{Im}[T]$. Ge's statement is as follows:

$$\mathrm{Var}[\mathrm{Re}[T]] = \delta t\,\sigma^2 \int \mathrm{Re}^2[\psi_{a,b}^*(t)]\,\mathrm{d}t$$

$$\mathrm{Var}[\mathrm{Im}[T]] = \delta t\,\sigma^2 \int \mathrm{Im}^2[\psi_{a,b}^*(t)]\,\mathrm{d}t$$

The above two formulas were obtained by Ge through a wrong proof (see Appendix A of Ge's paper).

By the definition of wavelet transform, we know that both $\mathrm{Var}[\mathrm{Re}[T]]$ and $\mathrm{Var}[\mathrm{Im}[T]]$ are independent of δt. However, Ge's above result shows that $\mathrm{Var}[\mathrm{Re}[T]]$ and $\mathrm{Var}[\mathrm{Im}[T]]$ are determined by the sampling period δt and are directly proportional to δt. Clearly, this is wrong.

Step 3. Ge announced the following:

we can show that for the Morlet wavelet

$$\int \mathrm{Re}^2[\psi_{a,b}^*(t)]\,\mathrm{d}t = \int \mathrm{Im}^2[\psi_{a,b}^*(t)]\,\mathrm{d}t = \frac{1}{2}$$

This should be the main part of the whole proof. However, Ge does not give any proof. In fact, the computation of the above integrals is very difficult and the values of the above integrals are also not $\frac{1}{2}$.

Step 4. Ge announced the following wrong result that the distribution of wavelet power spectrum of continuous-time Gaussian white noise is

$$|T((a,b)|^2 \Rightarrow \frac{1}{2}\delta t\,\sigma^2\,\chi_2^2.$$

3 Our proof

In this section, we will prove Eq. (4) in a rigorous statistical framework.

A continuous-time Gaussian white noise $x(t)$ satisfies

$$E[\,x(t)\,]=0,\quad E[\,x(t)x(t')\,]=\sigma^2\,\delta(t-t'), \tag{5}$$

where δ is the delta function. Let ψ be a Morlet wavelet with the parameter ω_0:

$$\psi(t)=\pi^{-\frac{1}{4}}\,e^{i\omega_0 t}\,e^{-\frac{t^2}{2}}. \tag{6}$$

Denote

$$\psi_{a,b}(t)=\frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right).$$

The wavelet transform of $x(t)$ is

$$T(a,b)=\int_{-\infty}^{\infty} x(t)\,\psi^*_{a,b}(t)\,\mathrm{d}t, \tag{7}$$

where $\psi^*_{a,b}$ represents the complex conjugate of $\psi_{a,b}$. The wavelet power spectrum is defined as

$$|T(a,b)|^2=(\mathrm{Re}[\,T(a,b)\,])^2+(\mathrm{Im}[\,T(a,b)\,])^2,$$

where $\mathrm{Re}[\,T\,]$ and $\mathrm{Im}[\,T\,]$ represent the real part and imaginary part of T, respectively. By Eq. (7), we have

$$\mathrm{Re}[\,T(a,b)\,]=\int_{-\infty}^{\infty} x(t)\,\mathrm{Re}[\,\psi^*_{a,b}(t)\,]\,\mathrm{d}t,$$

$$\mathrm{Im}[\,T(a,b)\,]=\int_{-\infty}^{\infty} x(t)\,\mathrm{Im}[\,\psi^*_{a,b}(t)\,]\,\mathrm{d}t, \tag{8}$$

and so $\mathrm{Re}[\,T(a,b)\,]$ and $\mathrm{Im}[\,T(a,b)\,]$ are both Gaussian random variables with mean 0.

Proposition 1. For any $a>0$ and b, $\mathrm{Re}[\,T(a,b)\,]$ and $\mathrm{Im}[\,T(a,b)\,]$ are independent.

Proof. By Eq. (8), we have

$$\mathrm{Re}[\,T(a,b)\,]\,\mathrm{Im}[\,T(a,b)\,]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x(t')\,\mathrm{Re}[\,\psi^*_{a,b}(t)\,]\,\mathrm{Im}[\,\psi^*_{a,b}(t)\,]\,\mathrm{d}t\,\mathrm{d}t'.$$

So

$$E[\,\mathrm{Re}[\,T(a,b)\,]\,\mathrm{Im}[\,T(a,b)\,]\,]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\,x(t)x(t')\,]\,\mathrm{Re}[\psi^*_{a,b}(t)]\,\mathrm{Im}[\,\psi^*_{a,b}(t')\,]\,\mathrm{d}t\,\mathrm{d}t'.$$

From this and Eq. (5), we have

$$E[\,\mathrm{Re}[\,T(a,b)\,]\,\mathrm{Im}[\,T(a,b)\,]\,]$$

$$=\int_{-\infty}^{\infty}\mathrm{Im}[\,\psi^*_{a,b}(t')\,]\left(\int_{-\infty}^{\infty}\sigma^2\,\delta(t-t')\,\mathrm{Re}[\,\psi^*_{a,b}(t)\,]\,\mathrm{d}t\right)\mathrm{d}t'. \tag{9}$$

By the definition of δ function, we know that the intermediate integral is equal to $\sigma^2\,\mathrm{Re}[\,\psi^*_{a,b}(t')\,]$:

$$\int_{-\infty}^{\infty}\sigma^2\,\delta(t-t')\,\mathrm{Re}[\,\psi^*_{a,b}(t)\,]\,\mathrm{d}t=\sigma^2\,\mathrm{Re}[\,\psi^*_{a,b}(t')\,].$$

From this and Eq. (9), we get

$$E\,[\,\mathrm{Re}[\,T(a,b)\,]\,\mathrm{Im}[\,T(a,b)\,]\,]$$

$$=\sigma^2\int_{-\infty}^{\infty}\mathrm{Im}[\,\psi^*_{a,b}(t')\,]\,\mathrm{Re}[\,\psi^*_{a,b}(t')\,]\,\mathrm{d}t'$$

$$=\frac{\sigma^2}{a}\int_{-\infty}^{\infty}\mathrm{Im}\left[\psi^*\left(\frac{t'-b}{a}\right)\right]\mathrm{Re}\left[\psi^*\left(\frac{t'-b}{a}\right)\right]\mathrm{d}t'$$

$$=\sigma^2\int_{-\infty}^{\infty}\mathrm{Im}[\psi^*(t')\,]\,\mathrm{Re}[\,\psi^*(t')\,]\mathrm{d}t'. \tag{10}$$

Again, by the Euler formula,

$$e^{it\omega_0}=\cos(t\omega_0)+i\sin(t\omega_0).$$

By Eq. (6), we have

$$\mathrm{Re}[\,\psi^*(t')\,]=\pi^{-\frac{1}{4}}\,e^{-\frac{(t')^2}{2}}\,\cos(t'\,\omega_0),$$

$$\mathrm{Im}[\,\psi^*(t')\,]=-\pi^{-\frac{1}{4}}\,e^{-\frac{(t')^2}{2}}\,\sin(t'\,\omega_0).$$

From this and Eq. (10), we get

$$\sigma^{-2}\,E\,[\,\mathrm{Re}[\,T(a,b)\,]\,\mathrm{Im}[\,T(a,b)\,]\,]$$

$$=-\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-(t')^2}\,\sin(t'\omega_0)\,\cos(t'\omega_0)\,\mathrm{d}t'$$

$$= -\frac{\pi^{-\frac{1}{2}}}{2} \int_{-\infty}^{\infty} e^{-(t')^2} \sin(2t'\omega_0)\,\mathrm{d}t'.$$

Since the integrand $e^{-(t')^2} \sin(2t'\omega_0)$ is an odd function, the above integral vanishes:

$$\int_{-\infty}^{\infty} e^{-(t')^2} \sin(2t'\omega_0)\,\mathrm{d}t' = 0.$$

So

$$E\,[\,\mathrm{Re}\,[\,T(a,b)\,]\,\mathrm{Im}\,[\,T(a,b)\,]\,] = 0. \tag{11}$$

Noticing that

$$E[\,\mathrm{Re}\,[\,T(a,b)\,]\,] = 0, \quad E[\,\mathrm{Im}\,[\,T(a,b)\,]\,] = 0, \tag{12}$$

by Eq. (11), we have

$$\mathrm{Cov}\,(\mathrm{Re}\,[\,T(a,b)\,],\ \mathrm{Im}\,[\,T(a,b)\,])$$
$$= E[\,\mathrm{Re}\,[\,T(a,b)\,]\,\mathrm{Im}\,[\,T(a,b)\,]\,]$$
$$-E[\,\mathrm{Re}\,[\,T(a,b)\,]\,]\,E[\,\mathrm{Im}\,[\,T(a,b)\,]\,]$$
$$= 0.$$

Therefore, $\mathrm{Re}\,[\,T(a,b)\,]$ and $\mathrm{Im}\,[\,T(a,b)\,]$ are independent. □

Next, we find the distributions of $\mathrm{Re}\,[\,T(a,b)\,]$ and $\mathrm{Im}\,[\,T(a,b)\,]$. Since we have known that $\mathrm{Re}\,[\,T(a,b)\,]$ and $\mathrm{Im}\,[\,T(a,b)\,]$ are both Gaussian random variables with mean 0, we only need to compute their variances. For this purpose, we first prove the following proposition.

Proposition 2. The following results hold:

1. $\mathrm{Var}\,(\mathrm{Re}\,[\,T(a,b)\,]) = \sigma^2 \int_{-\infty}^{\infty} (\mathrm{Re}\,[\,\psi^*_{a,b}(t)\,])^2\,\mathrm{d}t;$
2. $\mathrm{Var}\,(\mathrm{Im}\,[\,T(a,b)\,]) = \sigma^2 \int_{-\infty}^{\infty} (\mathrm{Im}\,[\,\psi^*_{a,b}(t)\,])^2\,\mathrm{d}t.$

Proof. By the definition of variances and Eq. (8), we have

$$\mathrm{Var}\,(\mathrm{Re}\,[\,T(a,b)\,])$$
$$= E[\,(\mathrm{Re}\,[\,T(a,b)\,])^2\,] - (\,E[\,\mathrm{Re}\,[\,T(a,b)\,]\,])^2$$
$$= E[\,(\mathrm{Re}\,[\,T(a,b)\,])^2\,]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\,x(t)x(t')\,]\,\mathrm{Re}[\,\psi^*_{a,b}(t)\,]\,\mathrm{Re}[\,\psi^*_{a,b}(t')\,]\,\mathrm{d}t\,\mathrm{d}t'$$
$$= \sigma^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t-t')\,\mathrm{Re}[\,\psi^*_{a,b}(t)\,]\,\mathrm{Re}[\,\psi^*_{a,b}(t')\,]\,\mathrm{d}t\,\mathrm{d}t'$$
$$= \sigma^2 \int_{-\infty}^{\infty} \mathrm{Re}[\,\psi^*_{a,b}(t)\,] \left(\int_{-\infty}^{\infty} \delta(t-t')\mathrm{Re}[\,\psi^*_{a,b}(t')\,]\,\mathrm{d}t' \right) \mathrm{d}t$$
$$= \sigma^2 \int_{-\infty}^{\infty} \left(\mathrm{Re}[\,\psi^*_{a,b}(t)\,]\right)^2 \mathrm{d}t$$

and

$$\mathrm{Var}\,(\mathrm{Im}\,[\,T(a,b)\,])$$
$$= E[\,(\mathrm{Im}\,[\,T(a,b)\,])^2\,] - (\,E[\,\mathrm{Im}\,[\,T(a,b)\,]\,])^2$$
$$= E[\,(\mathrm{Im}\,[\,T(a,b)\,])^2\,]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\,x(t)x(t')\,]\,\mathrm{Im}[\,\psi^*_{a,b}(t)\,]\,\mathrm{Im}[\,\psi^*_{a,b}(t')\,]\,\mathrm{d}t\,\mathrm{d}t'$$
$$= \sigma^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t-t')\,\mathrm{Im}[\,\psi^*_{a,b}(t)\,]\,\mathrm{Im}[\,\psi^*_{a,b}(t')\,]\,\mathrm{d}t\,\mathrm{d}t'$$
$$= \sigma^2 \int_{-\infty}^{\infty} \mathrm{Im}[\,\psi^*_{a,b}(t)\,] \left(\int_{-\infty}^{\infty} \delta(t-t')\mathrm{Im}[\,\psi^*_{a,b}(t')\,]\,\mathrm{d}t' \right) \mathrm{d}t$$
$$= \sigma^2 \int_{-\infty}^{\infty} \left(\mathrm{Im}[\,\psi^*_{a,b}(t)\,]\right)^2 \mathrm{d}t.$$

Proposition 2 is proved. □

We again compute two integrals:

$$\int_{-\infty}^{\infty} \left(\mathrm{Re}[\,\psi^*_{a,b}(t)\,]\right)^2 \mathrm{d}t$$

and

$$\int_{-\infty}^{\infty} \left(\mathrm{Im}[\,\psi^*_{a,b}(t)\,]\right)^2 \mathrm{d}t.$$

Proposition 3. The following results hold:

1. $\int_{-\infty}^{\infty} \left(\mathrm{Re}[\,\psi^*_{a,b}(t)\,]\right)^2 \mathrm{d}t = \frac{1}{2}\left(1 + e^{-\omega_0^2}\right);$
2. $\int_{-\infty}^{\infty} \left(\mathrm{Im}[\,\psi^*_{a,b}(t)\,]\right)^2 \mathrm{d}t = \frac{1}{2}\left(1 - e^{-\omega_0^2}\right),$
   where ω_0 is the parameter of the Morlet wavelets (see Eq. 6).

Proof. By Eq. (6), we deduce that

$$\mathrm{Re}\,[\,\psi^*_{a,b}(t)\,] = \frac{1}{\sqrt{a}}\,\mathrm{Re}\left[\psi^*\left(\frac{t-b}{a}\right)\right]$$
$$= \frac{\pi^{-\frac{1}{4}}}{\sqrt{a}} \cos\frac{\omega_0(t-b)}{a}\, e^{-\frac{(t-b)^2}{2a^2}},$$

and so

$$\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[\psi_{a,b}^{*}(t)\right]\right)^2 \mathrm{d}t = \frac{\pi^{-\frac{1}{2}}}{a}\int_{-\infty}^{\infty}\cos^2\frac{\omega_0(t-b)}{a}\, e^{-\frac{(t-b)^2}{a^2}}\,\mathrm{d}t.$$

Letting $u=\frac{t-b}{a}$, we have

$$\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[\psi_{a,b}^{*}(t)\right]\right)^2 \mathrm{d}t = \pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\cos^2(\omega_0 u)\, e^{-u^2}\,\mathrm{d}u.$$

By the formulas

$$\cos^2(\omega_0 u) = \frac{1+\cos(2\omega_0 u)}{2} \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-u^2}\,\mathrm{d}u = \sqrt{\pi},$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[\psi_{a,b}^{*}(t)\right]\right)^2 \mathrm{d}t &= \tfrac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}(1+\cos(2\omega_0 u))\, e^{-u^2}\,\mathrm{d}u \\ &= \tfrac{1}{2}+\tfrac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\cos(2\omega_0 u)\, e^{-u^2}\,\mathrm{d}u. \end{aligned} \tag{13}$$

Now we compute the final integral in the above formula. By the Euler formula,

$$\begin{aligned}\int_{-\infty}^{\infty}\cos(2\omega_0 u)\, e^{-u^2}\,\mathrm{d}u &= \int_{-\infty}^{\infty}\left(\mathrm{Re}\, e^{2i\omega_0 u}\right) e^{-u^2}\,\mathrm{d}u \\ &= \mathrm{Re}\left(\int_{-\infty}^{\infty} e^{-u^2+2i\omega_0 u}\,\mathrm{d}u\right).\end{aligned}$$

Since

$$\begin{aligned}-u^2+2i\omega_0 u &= -(u^2-2i\omega_0 u) \\ &= -(u^2-2i\omega_0 u+(i\omega_0)^2-(i\omega_0)^2) \\ &= -(u-i\omega_0)^2-\omega_0^2,\end{aligned}$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty}\cos(2\omega_0 u)\, e^{-u^2}\,\mathrm{d}u &= \mathrm{Re}\left(\int_{-\infty}^{\infty} e^{-(u-i\omega_0)^2}\, e^{-\omega_0^2}\,\mathrm{d}u\right) \\ &= e^{-\omega_0^2}\,\mathrm{Re}\left(\int_{-\infty}^{\infty} e^{-(u-i\omega_0)^2}\,\mathrm{d}u\right).\end{aligned} \tag{14}$$

The problem is reduced to computation of the integral of the complex-valued function $e^{-(u-i\omega_0)^2}$.

Now we compute the complex integral:

$$\int_{-\infty}^{\infty} e^{-(x-i\omega_0)^2}\,\mathrm{d}x.$$

Denote $z=x+iy$. The function e^{-z^2} is an analytic function in the whole complex plane. This integral is just the integral of

$$e^{-z^2}$$

on the straight line:

$$z = x - i\omega_0 \quad (-\infty < x < \infty).$$

We use the method of the contour integral to compute it.

Let Γ be a rectangle with four vertices:

$$-R-i\omega_0, \quad R-i\omega_0, \quad R, \quad \text{and} \quad -R,$$

i.e.,

$$\Gamma = \Gamma_1+\Gamma_2-\Gamma_3-\Gamma_4,$$

where

$$\begin{aligned}\Gamma_1 &: \quad z = x-i\omega_0 \quad (-R\le x\le R), \\ \Gamma_2 &: \quad z = R+iy \quad (-\omega_0\le y\le 0), \\ \Gamma_3 &: \quad z = x \quad (-R\le x\le R), \\ \Gamma_4 &: \quad z = -R+iy \quad (-\omega_0\le y\le 0).\end{aligned}$$

Since e^{-z^2} is analytic in the whole complex plane, by the Cauchy theorem, we have

$$\int_{\Gamma} e^{-z^2}\,\mathrm{d}z = 0.$$

From this, we deduce that

$$\begin{aligned}&\int_{-R}^{R} e^{-(x-i\omega_0)^2}\,\mathrm{d}x + \int_{-\omega_0}^{0} e^{-(R+iy)^2}\,\mathrm{d}y \\ &-\int_{-R}^{R} e^{-x^2}\,\mathrm{d}x - \int_{-\omega_0}^{0} e^{-(-R+iy)^2}\,\mathrm{d}x \\ &= I_1(R)+I_2(R)-I_3(R)-I_4(R) = 0.\end{aligned} \tag{15}$$

Noticing that $|e^{-2iRy}|=1$ and for $-\omega_0\le y\le 0$,

$$\begin{aligned}\left|e^{-(R+iy)^2}\right| &= \left|e^{-R^2-2iRy+y^2}\right| \\ &= e^{-R^2+y^2}\left|e^{-2iRy}\right| \\ &\le e^{-R^2+\omega_0^2},\end{aligned}$$

we deduce that

$$|I_2(R)| = \left| \int_{-\omega_0}^{0} e^{-(R+iy)^2}\, \mathrm{d}y \right|$$

$$\leq \int_{-\omega_0}^{0} \left| e^{-(R+iy)^2} \right| \mathrm{d}y \leq \omega_0\, e^{-R^2+\omega_0^2}$$

$$= \omega_0\, e^{\omega_0^2}\, e^{-R^2}$$

and

$$e^{-R^2} \to 0\ (R \to +\infty).$$

So we have

$$I_2(R) \to 0 \qquad (R \to +\infty). \tag{16}$$

Noticing that $|e^{2iRy}| = 1$ and

$$\left| e^{-(-R+iy)^2} \right| = \left| e^{-R^2+2iRy+y^2} \right|$$

$$= e^{-R^2+y^2} \left| e^{2iRy} \right| \leq e^{-R^2+\omega_0^2},$$

we deduce that

$$|I_4(R)| = \left| \int_{-\omega_0}^{0} e^{-(-R+iy)^2}\, \mathrm{d}y \right|$$

$$\leq \int_{-\omega_0}^{0} \left| e^{-(-R+iy)^2} \right| \mathrm{d}y \leq \omega_0\, e^{-R^2+\omega_0^2}$$

$$= \omega_0\, e^{\omega_0^2}\, e^{-R^2}$$

and

$$e^{-R^2} \to 0\ (R \to +\infty).$$

So we have

$$I_4(R) \to 0 \qquad (R \to +\infty). \tag{17}$$

With a known result of

$$\int_{-\infty}^{\infty} e^{-x^2}\, \mathrm{d}x = \sqrt{\pi},$$

we get

$$I_3(R) \to \sqrt{\pi} \qquad (R \to +\infty). \tag{18}$$

Letting $R \to +\infty$ in Eq. (15), we deduce by Eqs. (16)–(18) that

$$\lim_{R\to\infty} I_1(R) = \sqrt{\pi}.$$

So we have

$$\int_{-\infty}^{\infty} e^{-(x-i\omega_0)^2}\, \mathrm{d}x = \sqrt{\pi}.$$

By Eq. (14), we have

$$\int_{-\infty}^{\infty} \cos(2\omega_0 t)\, e^{-t^2}\, \mathrm{d}t = \sqrt{\pi}\, e^{-\omega_0^2}. \tag{19}$$

By Eq. (13), we get

$$\int_{-\infty}^{\infty} \left(\mathrm{Re}[\,\psi_{a,b}^*(t)\,]\right)^2 = \frac{1}{2} + \frac{1}{2} e^{-\omega_0^2} = \frac{1}{2}(1 + e^{-\omega_0^2}).$$

(i) is proved.

We will prove (ii). By Eq. (6), we deduce that

$$\mathrm{Im}[\,\psi_{a,b}^*(t)\,] = \tfrac{1}{\sqrt{a}}\, \mathrm{Im}\left[\,\psi_{a,b}^*\left(\tfrac{t-b}{a}\right)\right]$$

$$= -\tfrac{\pi^{-\frac{1}{4}}}{\sqrt{a}}\, \sin \tfrac{\omega_0(t-b)}{a}\, e^{-\frac{(t-b)^2}{2a^2}},$$

and so

$$\int_{-\infty}^{\infty} \left(\mathrm{Im}[\,\psi_{a,b}^*(t)\,]\right)^2 \mathrm{d}t = \frac{\pi^{-\frac{1}{2}}}{a} \int_{-\infty}^{\infty} \sin^2 \frac{\omega_0(t-b)}{a}\, e^{-\frac{(t-b)^2}{a^2}}\, \mathrm{d}t.$$

Letting $u = \frac{t-b}{a}$, we have

$$\int_{-\infty}^{\infty} \left(\mathrm{Im}[\,\psi_{a,b}^*(t)\,]\right)^2 \mathrm{d}t = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \sin^2(\omega_0 u)\, e^{-u^2}\, \mathrm{d}u.$$

By the formulas

$$\sin^2(\omega_0 u) = \frac{1-\cos(2\omega_0 u)}{2}$$

and

$$\int_{-\infty}^{\infty} e^{-u^2}\, \mathrm{d}u = \sqrt{\pi},$$

we get

$$\int_{-\infty}^{\infty} \left(\mathrm{Im}[\,\psi_{a,b}^*(t)\,]\right)^2 = \tfrac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} (1-\cos(2\omega_0 u))\, e^{-u^2}\, \mathrm{d}u$$

$$= \tfrac{1}{2} - \tfrac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \cos(2\omega_0 u)\, e^{-u^2}\, \mathrm{d}u.$$

By Eq. (19), we get (ii). Proposition 3 is proved. □

Table 1. Mean of wavelet power spectrum ($b = 0.5$, $\delta t = \frac{1}{1024}$).

Scale	$a = 0.3$	$a = 0.5$	$a = 0.7$
Monte-Carlo method	0.9996	0.9991	1.0003
Our formula	1.000	1.000	1.000
Ge's formula	0.001	0.001	0.001

From Propositions 2 and 3, we deduce that

$$\mathrm{Var}(\mathrm{Re}[T(a,b)]) = \sigma^2 \int_{-\infty}^{\infty} \left(\mathrm{Re}[\psi^*_{a,b}(t)]\right)^2 \mathrm{d}t = \tfrac{\sigma^2}{2}\left(1+e^{-\omega_0^2}\right)$$

and

$$\mathrm{Var}(\mathrm{Im}[T(a,b)]) = \sigma^2 \int_{-\infty}^{\infty} \left(\mathrm{Im}[\psi^*_{a,b}(t)]\right)^2 \mathrm{d}t = \tfrac{\sigma^2}{2}\left(1-e^{-\omega_0^2}\right).$$

Again, since $\mathrm{Re}[T(a,b)]$ and $\mathrm{Im}[T(a,b)]$ are both Gaussian random variables with mean 0, we know that $\mathrm{Re}[T(a,b)]$ is distributed as follows:

$$\mathrm{Re}[T(a,b)] \Longrightarrow \left(\frac{\sigma^2}{2}(1+e^{-\omega_0^2})\right)^{\frac{1}{2}} X_1, \quad (20)$$

and $\mathrm{Im}[T(a,b)]$ is distributed as follows:

$$\mathrm{Im}[T(a,b)] \Longrightarrow \left(\frac{\sigma^2}{2}(1-e^{-\omega_0^2})\right)^{\frac{1}{2}} X_2, \quad (21)$$

where X_1 and X_2 are Gaussian random variables with mean 0 and variance 1.

Since the wavelet power spectrum of $x(t)$ is

$$|T(a,b)|^2 = |\mathrm{Re}\,T(a,b)|^2 + |\mathrm{Im}\,T(a,b)|^2,$$

by Proposition 1 and Eqs. (20), and (21), we get the following theorem.

Theorem 1. Let $x(t)$ be the continuous-time Gaussian white noise, which is stated in Eq. (5), and ψ be a Morlet wavelet with parameter ω_0. Denote the Morlet wavelet transform of $x(t)$ by $T(a,b)$. Then the wavelet power spectrum $|T(a,b)|^2$ is distributed as follows:

$$|T(a,b)|^2 \Longrightarrow \frac{\sigma^2}{2}(1+e^{-\omega_0^2})X_1^2 + \frac{\sigma^2}{2}(1-e^{-\omega_0^2})X_2^2,$$

where X_1 and X_2 are independent Gaussian random variables with mean 0 and variance 1.

Table 2. 95 % confidence level of wavelet power spectrum ($b = 0.5$, $\delta t = \frac{1}{1024}$).

Scale	$a = 0.3$	$a = 0.5$	$a = 0.7$
Monte-Carlo method	3.002	3.003	3.001
Our formula	2.996	2.996	2.996
Ge's formula	0.003	0.003	0.003

For large ω_0 (e.g., $\omega_0 = 6$), we have $e^{-\omega_0^2} \approx 0$. By Theorem 4, we get

$$|T(a,b)|^2 \Rightarrow \frac{\sigma^2}{2}(X_1^2 + X_2^2),$$

i.e.,

$$|T(a,b)|^2 \Longrightarrow \frac{\sigma^2}{2}\chi_2^2,$$

where χ_2^2 is the chi-squared distributed with 2 degrees of freedom.

4 Numerical experiments

In this section, we will do some numerical experiments. For a continuous-time Gaussian white noise $x(t)$ with autocovariance,

$$\mathrm{Cov}[x(t), x(t')] = \delta(t-t')\sigma^2, \quad (22)$$

we will compute its wavelet power spectrum $|T(a,b)|^2$ by using Monte-Carlo method, our formula and Ge's formula. The wavelet we use is the Morlet wavelet with parameter $\omega_0 = 6$.

When we use Monte-Carlo method to approximately compute wavelet power spectrum, we will use the sampling period $\delta t = \frac{1}{1024}$ to sample continuous-time Gaussian white noise $x(t)$. The autocovariance of Gaussian white noise is defined by the delta function $\delta(t)$, where $\delta(t)$ is defined as

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

for any smooth function $f(t)$. It is well known that, for the sequence of functions $g_\epsilon(t)$,

$$g_\epsilon(t) = \frac{1}{\epsilon}, \quad t \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$$

and

$$g_\epsilon(t) = 0, \quad t \in [-\infty, -\frac{\epsilon}{2}) \bigcup (\frac{\epsilon}{2}, \infty], \quad (23)$$

one has

$$\lim_{\epsilon \to 0} g_\epsilon(t) = \delta(t)$$

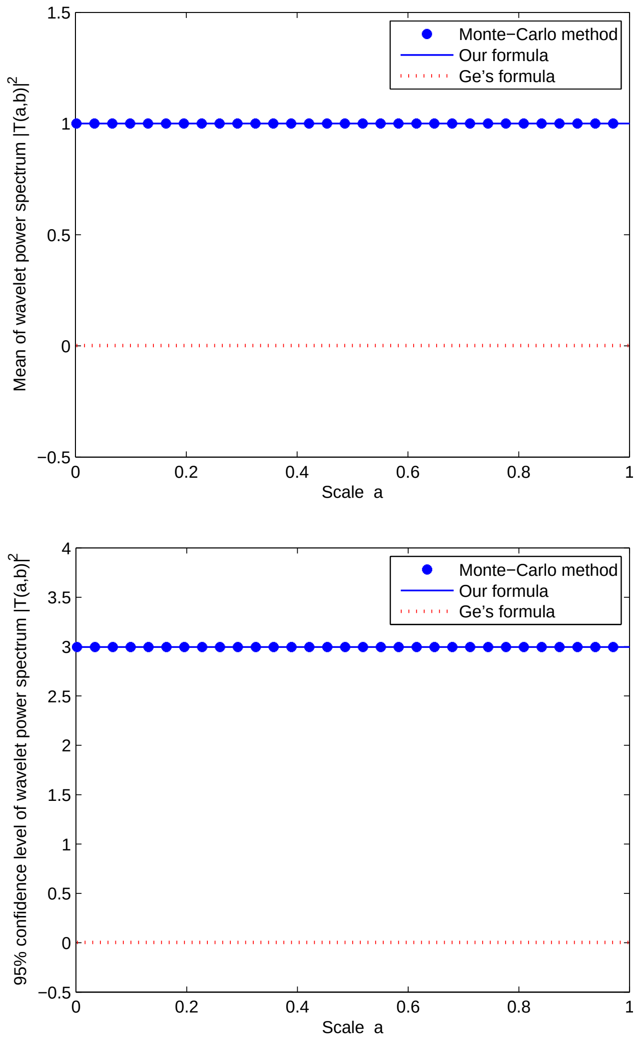

Fig. 1. The mean and 95 % confidence level of wavelet power spectrum $|T(a,b)|^2$ are computed by Monte-Carlo method, our formula and Ge's formula, where $0 < a < 1$, $b = 0.5$, and $\delta t = \frac{1}{1024}$.

in the sense of generalized functions. Take

$$\epsilon = \delta t,$$

where δt is the sample period, so

$$\delta(t) \approx g_{\delta t}(t).$$

Again, by Eqs. (22) and (23), we have

$$\begin{aligned} \mathrm{Cov}[x(n\delta t), x(m\delta t)] &= \delta(n\,\delta t - m\,\delta t)\,\sigma^2 \\ &\approx g_{\delta t}(n\,\delta t - m\,\delta t)\,\sigma^2 \\ &= \begin{cases} \frac{1}{\delta t}\sigma^2 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases} \end{aligned}$$

Finally, with the help of Monte-Carlo method and numerical integral, we can approximately obtain the distribution of wavelet power spectrum of continuous-time Gaussian white noise.

Ge (2007) indicated that wavelet power spectrum of continuous-time Gaussian white noise is distributed as follows:

$$|T((a,b)|^2 \Rightarrow \frac{\sigma^2}{2}\delta t\,\sigma^2\,\chi_2^2$$

(see Eq. 18 of Ge's paper).

In Theorem 1, we have proved that the wavelet power spectrum of continuous-time Gaussian white noise is distributed as follows:

$$|T(a,b)|^2 \Longrightarrow \frac{\sigma^2}{2}(1+e^{-\omega_0^2})\,X_1^2 + \frac{\sigma^2}{2}(1-e^{-\omega_0^2})\,X_2^2.$$

If we use a Morlet wavelet with parameter $\omega_0 = 6$, we have

$$|T(a,b)|^2 \Rightarrow \frac{\sigma^2}{2}\chi_2^2.$$

Without loss of generalization, we take $\sigma = 1$ and $b = 0.5$. Tables 1 and 2, and Fig. 1 show the mean and 95 % confidence level of wavelet power spectrum of Gaussian white noise obtained by Monte-Carlo method, our formula and Ge's formula. It is clear that our result on wavelet power spectrum is consistent with that obtained by Monte-Carlo method. Ge's formula is quite different from that obtained by Monte-Carlo method or our method. From here, we can also see that Ge's formula is wrong.

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