Min-Time Coverage in Constricted Environments with Arbitrary Guidepath Networks

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***Abstract*— In a recent research program, we have undertaken the investigation of robotic traffic management problems arising when a fleet of networked mobile robots is employed in the support of certain coverage tasks that take place in physically constricted environments. But our past investigation of these problems is restricted to the case where the guidepath networks supporting the robot traffic have a dendritic topology. In the current work, we extend the investigation of the considered problems to the case where the underlying guidepath networks have an arbitrary topology. We provide (i) detailed descriptions of the considered problems in this new operational setting, (ii) analytical characterizations of these problems that take the form of integer programming formulations, and (iii) strong combinatorial relaxations for the derived formulations that are applicable to larger problem instances. A numerical experiment presented in the last part of the manuscript demonstrates and assesses the efficacy and the tractability of the analytical developments. We also notice that the undertaken extension of the past results is a nontrivial task, for the reasons that are explained in the manuscript**[1]

Keywords: Networked mobile robotic systems; multi-robot coordination; combinatorial scheduling; coverage problems

I. Introduction

A research topic of increasing interest and importance within the robotics & automation community concerns the employment of networked multiple mobile robot systems (MMRS) [1] for the support of operations taking place in physically constricted environments. Some characteristic examples of such environments are the subterranean water supply, sewage and other similar utility networks, mines, and the extensive pipeline networks that are used for the transport of oil, gas and other similar commodities [2], [3], [4], [5], [6], [7]. In fact, the aforementioned research topic has also been the focus of a major DARPA challenge known as the "Subterranean – or SubT – Challenge" [8]. The corresponding research activity has enabled (i) stable and safe navigation of the deployed robots in the spatially constricted – and maybe otherwise challenging – corridors (i.e., tunnels, pipes, etc.) that support the robotic traffic, and (ii) reliable communication among the robots and the command-&-control (C-&-C) center that manages the overall operation. Also, the deployed testbeds have allowed the extensive testing and significant enhancements of the exploration and the mapping strategies for unknown environments that have been pursued, for many years, by the robotics community [8].

But as the aforementioned technologies and capabilities get more proven and more robust, there is an increasing interest to employ networked MMRS-based solutions for the support of more routine tasks in the considered environments, like the continuous inspection and monitoring of various locations in them for security or maintenance purposes [9]. In this new operational regime, the operational environment of the robots is well structured and known *a priori*, and the emphasis in the management of the corresponding operations shifts away from the need for mapping and exploration to issues like *operational correctness* and *expediency*.

A first attempt to address the planning and control problems that arise in this new class of MMRS applications, has been undertaken in an ongoing research program of ours. This program addresses the expedient visitation of a set of target locations in a subterranean or pipeline network by a fleet of robots that are initially concentrated at a C&C center that constitutes the entry point of this network. The problems of robot safety and collision avoidance that arise due to the constricted nature of the considered networks, are addressed through the imposition of a zoning scheme that splits the various aisles of the underlying guidepath network in a number of zones of unit buffering capacity, and requires the robots to negotiate their advancement through the established zones with a traffic coordinator. From a more methodological standpoint, the imposition of this zoning scheme embeds the robot traffic in a graph that is defined by the established zones and their neighboring relationship, and brings the corresponding traffic management problems to the class of problems concerning the optimal multi-robot path planning on graphs (OMRP^{2}G) [10], [11], [12], [13], [14].

Furthermore, the remote nature of the considered guidepath networks necessitates the proactive preservation by the operating robots of an *ad hoc* multi-hop wireless communication network that ensures continuous connectivity of the robots with the C&C center and with each other. This additional requirement differentiates very substantially the traffic dynamics, and the corresponding traffic management problems for the operating robots, from the past problems

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investigated in the OMRP^{2}G literature.

A detailed characterization of this new class of problems, and a thorough positioning of them in the corresponding literatures concerning (i) the deployment and the operation of networked MMRS and (ii) the optimal multi-robot path planning on graphs, have been provided in [15]. That work also provided analytical problem formulations in the form of mixed integer programs (MIPs), and proved that, in the general case, the considered problems are NP-hard combinatorial optimization problems [16]. Furthermore, in some sequel works, we have established some additional properties for the optimal solution space of these problems [17], and we have developed (i) a strong combinatorial relaxation for their MIP formulations [18] and (ii) a heuristic algorithm that can be used for larger problem instances that cannot be addressed effectively by the MIP-based methods [19].

On the other hand, all this earlier work has assumed a *dendritic* topology for the guidepath network; i.e., the guidepath network is a rooted tree with its root representing the C&C center and its leaves being the locations to be visited by the robots. The focus on this particular structure was partly motivated by the practical observation that, in a large subset of the targeted operations, the corresponding guidepath networks possess, indeed, a tree structure [3]. But also it simplified the modeling and the analysis of the considered problems thanks to the resulting lack of routing flexibility for the traveling robots as they try to reach their various destinations.

The work presented in this paper extends our past research to instantiations of the considered OMRP^{2}G problems where the underlying guidepath network has an arbitrary topology and the locations to be visited by the robots are an arbitrary subset of nodes in this network. The pursued extension is nontrivial since the routing flexibility that is present in this new case (i) introduces additional choice in the robot effort to access simultaneously the various target locations, and (ii) complicates significanty the enforcement of the communication connectivity requirement during the analytical formulation of the new problem versions.

A more detailed enumeration of the intended contributions of the current manuscript, in the context of the aforementioned past developments, is as follows:

1) We provide a detailed characterization of the considered MMRS operations and the ensuing traffic management problem in the new setting of guidepath networks with arbitrary topology.
2) We also develop integer programming (IP) formulations for the new problem versions that retain the desirable property of employing sets of variables and constraints which are polynomially sized with respect to the size of the underlying guidepath network. The preservation of this property is a challenging problem since we need to express the requirement for the preservation of the communication connectivity among the robots and the C&C center while accounting for the routing flexibility that is enabled by the arbitrary topology of the guidepath network.
3) We further establish that it is possible to relax the integrality requirement for a very large subset of variables of the derived IPs without compromising the feasibility and the optimality of the obtained solutions. This result is similar in spirit to the corresponding result of [18], but the subsets of the relaxed variables for each case are different, and furthermore, the analytical developments and arguments that enable the corresponding relaxations are different, as well.
4) Finally, we present the results of a numerical experiment that (i) demonstrates and assesses the capability of the developed formulations to provide optimal or near-optimal solutions to the considered traffic-management problems within some reasonable computational times, and also (ii) reveals some interesting dependencies of this capability on certain parameters of the underlying operational environment.

The rest of the paper is organized as follows: Section II introduces the new coverage problem versions for the considered MMRS, and Section III provides analytical formulations of these new problems as IPs. Section IV presents the strong combinatorial relaxations of these IPs, and Section V presents the numerical experiment regarding their computational efficacy. Finally, Section VI concludes the paper and outlines some directions for future work.

II. The considered MMRS and the corresponding coverage problems

A detailed description of the considered MMRS operations: Consider a network of underground tunnels that is managed by a C&C center located at an entry point of this network. The facility employs a fleet of mobile robots for performing certain inspection and monitoring tasks in the network. Idling robots are stationed at the C&C center, and any of these robots can carry out any inspection task that might arise at any location of the network (i.e., the robots are interchangeable in the execution of the various inspection tasks).

The tunnels are narrow, and since the robots have limited sensing and maneuvering capability, the robot traffic in the considered network must be controlled for ensuring the robot safety and collision avoidance. As pointed out in the introductory section, a typical way for meeting this requirement is by splitting each tunnel and their intersections into a number of zones, and stipulating that each zone can be occupied by at most one robot at any point in time. The imposition of such a zoning scheme embeds the robot traffic in a *guidepath network* where the nodes of this network correspond to the zones defined in the underlying tunnel system, and the edges represent the neighboring relationship of these zones. Also, in this abstracted representation, the C&C center constitutes a special node of the guidepath network with infinite buffering capacity. Each robot must negotiate its advancement from its current zone to a neighboring zone with a traffic coordinator, and the corresponding zone allocation protocol turns the advancement of the deployed robots towards their corresponding destinations, into a "sequential resource allocation process" [20] where the contested resources are the zones of the guidepath network.

But in the considered MMRS applications, the imposed zoning scheme and the corresponding zone allocation protocol are further employed for ensuring the necessary communication connectivity among the deployed robots and the C&C center. More specifically, in the considered MMRS, the robots possess wireless communication capability that can be used for communicating with each other and with the C&C center, a feature that places the considered MMRS in the class of *networked* MMRS [1], [21]. However, in the context of the targeted applications, the communication capability of the deployed robots is severely limited by the nature and the geometry of the surrounding environment. This problem is addressed through the following additional stipulations:

1) The specification of the imposed zones must also ensure that robots located in neighboring zones can communicate easily and reliably with each other.
2) At every time point, the subgraph of the guidepath network that is induced by the nodes occupied by robots, must be connected and it must contain the node corresponding to the C&C center.

Clearly, the satisfaction of these two requirements enables the relay of messages between any deployed robot and the C&C center. But it turns the inspection of a location in any zone of the guidepath network into a task requiring an entire team of robots that must be deployed on a path of the underlying guidepath network leading from the node corresponding to the C&C center to the node corresponding to the zone containing the targeted location. On the other hand, robotic teams that pursue the inspection of different locations, can share the robots that are located in a common subpath towards their destinations. In other words, the support of an inspection task in the considered MMRS application requires the deployment of an entire team of robots, but also every deployed robot may support the execution of more than one task. MMRS applications with such multi-robot multi-tasking assignments are acknowledged as some of the hardest classes of such applications with respect to the effective and the efficient management of the corresponding workflows [22]. In the considered case, things are further complicated by (i) the limited buffering capacity of the various zones of the imposed zoning scheme, that can turn these zones into "bottlenecks" for the robotic flows towards the various destinations, and (ii) the potential availability of alternate routes towards the different target zones of the guidepath network.

A formal modeling of the considered coverage problems: Next, we provide a formal definition of the coverage tasks for the aforementioned environments, and the corresponding traffic-management problems, that are addressed in this work. These problems are defined on the guidepath network that models the imposed zoning scheme. Hence, let us denote this guidepath network by the graph $\mathcal{G} = (V, E)$, where the node set V models the zones of the underlying tunnel system,[2] and the edge set E models the neighboring relationship of these zones. Naturally, graph $\mathcal{G}$ is undirected and connected. A particular node $o \in V$ models the C&C center of the underlying facility, and a subset L of $V \setminus \{o\}$ represents the set of zones that must be visited for inspecting some location in them. Also, let $\mathcal{R}$ denote the set of (identical) robots possessed by this facility, and further assume that at the beginning of the considered operations, all these robots are located at node o. Then, the tuple $\langle \mathcal{G}, L, \mathcal{R} \rangle$ defines an instantiation of the considered MMRS operations.

It is evident from the above description that the robot-accommodating – or buffering – capacity of node o is $|\mathcal{R}|$ (essentially, arbitrarily large). On the other hand, the above definition of graph $\mathcal{G}$ and the assumed zone allocation protocol imply a unit buffering capacity for every node $v \in V \setminus \{o\}$.

Following standard practice, we further assume that zone traversal times are uniform for all (zone, robot) pairs, and we use this property in order to discretize time.[3] In the following, we shall use t to denote the various periods – or epochs – in the resulting operational regime. Furthermore, we assume that the requested inspections can be carried out during a single epoch, while the inspecting robot is visiting the corresponding zone.

The set of neighbors of a zone $v \in V$ is denoted by $\mathcal{N}(v)$. Let $z(r,t)$ denote the zone $v \in V$ occupied by robot r at period t. Then, $z(r, t+1) \in \{z(r,t)\} \cup \mathcal{N}(z(r,t))$; i.e., robot r can either remain in the same zone at period $t+1$, or advance to a neighboring zone $v' \in \mathcal{N}(v)$. As already noticed, at any period t, zones $v \neq o$ cannot contain more than one robot. On the other hand, a group of robots can coordinate their advancement over a path of neighboring zones; i.e., for a group of robots $r_1, r_2, \ldots, r_n$ with $z(r_i, t) \in \mathcal{N}(z(r_{i-1}, t))$, for $i = 2, \ldots, n$, we allow $z(r_i, t+1) = z(r_{i+1}, t)$, $i = 1, \ldots, n-1$, provided that robot r_n moves itself to a free zone or to zone o at period $t+1$. We characterize a maximal string of such robot moves as a robot *flow* occurring at time t. The net effect of this flow is the transfer of a robot from zone $z(r_1, t)$ to zone $z(r_n, t+1)$. Also, the unit buffering capacity of the zones $v \neq o$ imply that two simultaneously occurring flows are conflicting if the supporting paths of these two flows have a common internal node $v \neq o$.

Robots can reverse the direction of their motion within their zone. This assumption is reasonable in the context of the considered MMRS applications, and furthermore, it is necessary if the underlying graph $\mathcal{G}$ possesses leaf nodes.

At any epoch t, let $\hat{V}(t)$ denote the set of zones of $\mathcal{G}$ that are occupied by robots. Then, as discussed in the previous part of this section, the necessary communication connectivity among the robots and the C&C center is established by

[2]In the following, we shall refer to the elements of V either as "nodes" or as "zones".

[3]This assumption can always be satisfied by an appropriate refinement of the zones. Furthermore, the assumption can be relaxed, but treating this more general case in the context of this document would overload the employed notation and complicate the details of the presented analysis, without adding anything substantial to the main points of this discussion.

stipulating that (i) $o \in \widehat{V}(t)$ and (ii) the subgraph $\widehat{\mathcal{G}}(t)$ of $\mathcal{G}$ that is induced by the node subset $\widehat{V}(t)$, is connected. We also notice that such connectivity is possible for an arbitrary zone $v \in V$ only if $|\mathcal{R}| > \max_{v \in V}\{d(o, v)\}$, where $d(o, v)$ denotes the distance of node v from node o in $\mathcal{G}$; this distance is defined by the number of edges on any shortest path connecting node o with node v in $\mathcal{G}$.

The planning and control problems addressed in this work seek to determine an expedient plan that will advance the robots $r \in \mathcal{R}$ in a way that is consistent with the aforestated assumptions regarding the robot capabilities and the zone allocation protocol, and at the end of its execution, each target zone $v \in L$ will have been visited by some robot. In more technical terms, a *plan* – or, alternatively, a *(traffic) schedule* – is a sequence of distributions, $\mathcal{D}_t, t = 0, 1 \ldots$, of the system robots to the various zones of the guidepath network $\mathcal{G}$, with the initial distribution $\mathcal{D}_0$ having all the robots located at node o. Also, distribution $\mathcal{D}_{t+1}$, for period $t+1$, is obtained from the distribution $\mathcal{D}_t$ by relocating a number of robots from their zones at period t to some neighboring zone, while abiding to the introduced assumptions about the maneuvering capabilities of the robots and the zone allocation protocol. We specify further the "plan" concept in the next section, where we discuss the computation of an optimal plan through some mathematical programming formulations of the addressed problems.

Let $\mathcal{P}$ denote the set of feasible plans, and for every $P \in \mathcal{P}$ and $v \in L$, let $C(v; P)$ denote the first period that plan P places a robot in zone v. In the considered research program, we are especially interested in plans P^* such that

$$P^* = \arg\min_{P \in \mathcal{P}} \max_{v \in L} C(v; P) \tag{1}$$

or

$$P^* = \arg\min_{P \in \mathcal{P}} \sum_{v \in L} C(v; P) \tag{2}$$

As in [15], we characterize the combinatorial optimization problem that is induced by Eq. 1 as the *Makespan-Minimization problem*, or the *M-problem*, and the combinatorial optimization problem that is induced by Eq. 2 as the *Total Visitation Time problem*, or the *TVT-problem*.

In [15] it has been shown that the M- and TVT-problems are in a Pareto optimal relationship [10], and therefore, they require separate treatments. Also, in the same work it was established that the restrictions of the M- and TVT-problems on guidepath networks with a dendritic topology are NP-Hard, and therefore, this is also the case for the more general problems versions that are considered in this work.

III. IP FORMULATION OF THE CONSIDERED COVERAGE PROBLEMS

Next we provide analytical characterizations for the M- and TVT-problems that were defined in the previous section, in the form of IP formulations. In the subsequent discussion, we employ the notation introduced in Section II. We also let $\bar{T}$ denote an upper bound for the completion time of an optimal plan P^* for each problem. One way to obtain such an upper bound is by considering the completion time of the plan P that tries to reach one target zone $v \in L$ at a time, while using a shortest path from node o to node v.

As in [15], the decision variables employed by the presented formulations can be classified into:

- State variables
  - $x_{v,t},\ v \in V,\ t \in \{0, 1, \ldots, \bar{T}\}$: a nonnegative integer variable indicating the number of robots in zone v at period t.
- Control variables
  - $u_{v,v',t},\ v \in V,\ v' \in \mathcal{N}(v),\ t \in \{1, \ldots, \bar{T}\}$: a nonnegative integer variable representing the number of robots moving from zone v to neighboring zone v' at period t.
- Auxiliary variables
  - $y_{v,t},\ v \in L,\ t \in \{1, \ldots, \bar{T}\}$: a binary variable for testing whether target zone v has been visited by period t.
  - $s_t,\ t \in \{1, \ldots, \bar{T}\}$: a binary variable for testing whether all target zones have been visited by period t.

We also carry over, to this new setting, the following technological constraints from the corresponding IP formulations of the M- and TVT-problems on dendritic topologies that were investigated in [15]:

$$x_{o,0} = |\mathcal{R}| \tag{3}$$

$$\forall v \in V \setminus \{o\}, \quad x_{v,0} = 0 \tag{4}$$

$$\begin{gathered} \forall v \in V,\ \forall t \in \{1, \ldots, \bar{T}\}, \\ x_{v,t} = x_{v,t-1} + \sum_{v' \in \mathcal{N}(v)} \left(u_{v',v,t} - u_{v,v',t}\right) \end{gathered} \tag{5}$$

$$\forall v \in V,\ \forall t \in \{1, \ldots, \bar{T}\}, \quad \sum_{v' \in \mathcal{N}(v)} u_{v,v',t} \leq x_{v,t-1} \tag{6}$$

$$\forall v \in V \setminus \{o\},\ \forall t \in \{1, \ldots, \bar{T}\}, \quad x_{v,t} \leq 1 \tag{7}$$

$$\forall v \in L,\ \forall t \in \{1, \ldots, \bar{T}\}, \quad y_{v,t} \leq \sum_{q \in \{1, \ldots, t\}} x_{v,q} \tag{8}$$

$$\forall v \in L,\ \forall t \in \{1, \ldots, \bar{T}\}, \quad s_t \leq y_{v,t} \tag{9}$$

Constraints 3 and 4 define the initial distribution of the robots by means of the state variables $x_{v,0},\ v \in V$. Constraint 5 expresses the evolution of the robot distribution to the system zones at period t based on the control decisions that are expressed by the variables $u_{v,v't}$. Constraint 6 stipulates that the control decisions at period t must be feasible with respect to the robot distribution over the system zones at period $t-1$. Constraint 7 enforces the unit buffering capacity of the zones $v \neq o$. Constraint 8 forces the binary variable $y_{v,t}$ to zero if target zone $v \in L$ has not been visited by period t. Finally, Constraint 9 forces the binary variable s_t to zero if there is a target zone $v \in L$ that has not been visited by period t.

On the other hand, the constraint that ensures the communication connectivity of the robots and the C&C center at any period $t \in \bar{T}$ in the IP formulations of the M- and TVT-problems presented in [15], relies substantially on the dendritic topology of the underlying guidepath network that was presumed in that work, and therefore, it is not applicable to the more general versions of the M- and TVT-problems that are addressed in the current manuscript. We remind the reader that in the new operational regime, the communication connectivity requirement for the robots and the C&C center can be ensured, for every period $t \in \bar{T}$, by enforcing the requirement that the subgraph $\widehat{\mathcal{G}}(t)$ of the guidepath network $\mathcal{G}$, that is induced by the zones occupied by robots in period t, is connected and contains node o.

Next, we present a new constraint set that will introduce in the developed IPs the connectivity requirement that was stated in the previous paragraph. For this, we need the notion of the *eccentricity* Θ of node o in the guidepath network $\mathcal{G}$, which is defined by $\Theta = \max_{v \in V}\{d(o,v)\}$. We also define the binary variables $\psi_{v,t,\theta}$, for all $v \in V,\ t \in \{0,\ldots,\bar{T}\}$, and $\theta \in \{0,\ldots,\Theta\}$. Then, the sought constraints are as follows:

$$\forall t \in \{0,\ldots,\bar{T}\}, \quad \psi_{o,t,0} = 1 \tag{10}$$

$$\forall v \in V \setminus \{o\},\ \forall t \in \{0,\ldots,\bar{T}\}, \quad \psi_{v,t,0} = 0 \tag{11}$$

$$\forall v \in V,\ \forall t \in \{0,\ldots,\bar{T}\},\ \forall \theta \in \{0,\ldots,\Theta\}, \quad \psi_{v,t,\theta} \le x_{v,t} \tag{12}$$

$$\forall v \in V,\ \forall t \in \{0,\ldots,\bar{T}\},\ \forall \theta \in \{1,\ldots,\Theta\}, \quad \psi_{v,t,\theta} \le \psi_{v,t,\theta-1} + \sum_{v' \in \mathcal{N}(v)} \psi_{v',t,\theta-1} \tag{13}$$

$$\forall v \in V \setminus \{o\},\ \forall t \in \{0,\ldots,\bar{T}\}, \quad x_{v,t} \le \psi_{v,t,\Theta} \tag{14}$$

For each epoch $t \in \{0,\ldots,\bar{T}\}$, Constraints 10 and 12 imply that $x_{o,t} = 1$. Then, Constraints 10–13 essentially implement a forward-reaching scheme that, starting from node o, iteratively marks every other node $v \neq o$ of the guidepath network $\mathcal{G}$ that is reachable from node o through a path occupied by robots in period t, by allowing the corresponding variables $\psi_{v,t,\Theta}$ to take the value of 1 and forcing to zero the variable $\psi_{v',t,\Theta}$ for every node v' that does not possess this property. Furthermore, Constraint 14 stipulates that *every* deployed robot must be reachable from node o through a path occupied by robots. Finally, we also notice that the sets of variables and constraints involved in Equations 10–14 are polynomially sized with respect to the size of the guidepath network $\mathcal{G}$.

With the availability of the new Constraints 10–14, the M-problem can be expressed by the following IP formulation:

$$\max \sum_{t \in \{1,\ldots,\bar{T}\}} s_t \tag{15}$$

s.t. Constraints 3 – 14 plus the sign restrictions for the employed variables that were specified during the introduction of these variables.

Similarly, the TVT-problem can be expressed by the following IP formulation:

$$\max \sum_{v \in L} \sum_{t \in \{1,\ldots,\bar{T}\}} y_{v,t} \tag{16}$$

s.t. Constraints 3 – 8, 10–14 plus the sign restrictions for the employed variables.

Finally, an optimal solution of each of these two formulations determines an optimal plan P^* for the corresponding scheduling problem through the quantities $[u_{v,v',t} - u_{v',v,t}]^+$ for every pair (v, v') of neighboring zones and period t.[4]

IV. Strong combinatorial relaxations for the IP formulations of Section III

In this section we consider the combinatorial relaxations [23] of the IP formulations for the M- and the TVT-problems of Section III that are obtained by relaxing the integrality requirements of the variables $u_{v,v',t}$, $y_{v,t}$, s_t, and $\psi_{v,t,\theta}$ to the following constraints for these variables:

$$\forall v \in V,\ \forall v' \in \mathcal{N}(v),\ \forall t \in \{1,\ldots,\bar{T}\}, \quad u_{v,v',t} \ge 0 \tag{17}$$

$$\forall v \in L,\ \forall t \in \{1,\ldots,\bar{T}\}, \quad 0 \le y_{v,t} \le 1.0 \tag{18}$$

$$\forall t \in \{1,\ldots,\bar{T}\}, \quad 0 \le s_t \le 1.0 \tag{19}$$

$$\forall v \in V,\ \forall t \in \{0,\ldots,\bar{T}\},\ \forall \theta \in \{0,\ldots,\Theta\}, \quad \psi_{v,t,\theta} \ge 0 \tag{20}$$

Hence, the considered relaxations are MIPs with the only integer variables being the $x_{v,t}$ variables. The significance of these MIPs is revealed by the following theorem.

Theorem 4.1: The combinatorial relaxations of the IP formulations of Section III that are obtained by replacing the integrality requirements of the variables $u_{v,v',t}$, $y_{v,t}$, s_t, and $\psi_{v,t,\theta}$ with Constraints 17–20, are strong; i.e., the optimal objective values of these relaxations are equal to the optimal objective values of the corresponding IPs.

Furthermore, if the MIPs that correspond to these relaxations are solved by a Branch-&-Bound (B&B) method [23] where the generated linear programming (LP) formulations are solved by a method that provides extreme-point solutions to these LPs,[5] any obtained optimal solution for these MIPs is also a feasible optimal solution for the corresponding IP.

Proof: First we notice that even though the variables $x_{v,t},\ v \neq o$, are defined as nonnegative integer variables in Section III, they act as binary variables in the IP formulations that are developed in that section and in the relaxations that are considered in Theorem 4.1, because of Constraint 7. This result, when combined with Constraints 6 and 17, further implies that the variables $u_{v,v',t}$ are upper-bounded by 1.0.

Constraints 10–14, 18–20 when combined with the structure of the objective functions of Equations 15 and 16, also imply that for any valid pricing of the variables $x_{v,t},\ v \in$

[4] We remind the reader that $[x]^+ = \max\{x, 0\}$.

[5] e.g., the Simplex algorithm [16] is such a method.

$V, \ t \in \{0, \ldots, \bar{T}\}$, all variables $\psi_{v,t,\theta}, \ y_{v,t}$ and s_t will take binary values that evaluate correctly the solution that is implied by the considered pricing of the variables $x_{v,t}$, with respect to the objective functions of Equations 15 and 16.

Next we show that for any valid pricing of the variables $x_{v,t}, \ v \in V, \ t \in \{0, \ldots, \bar{T}\}$, we have a valid pricing of the variables $u_{v,v',t}, \ v \in V, \ v' \in \mathcal{N}(v), t \in \{1, \ldots, \bar{T}\}$ that is integer. Hence, consider such a valid pricing of $x_{v,t}$ and some given period $\hat{t} \in \{1, \ldots, \bar{T}\}$. Constraints 5 and 6 imply that the variables $u_{v,v',\hat{t}}, \ v \in V, \ v' \in \mathcal{N}(v)$, constitute a static flow $\mathcal{F}(\hat{t})$, supported by the guidepath network $\mathcal{G}$. The demand and the supply of this static flow are defined by the differences $x_{v,\hat{t}} - x_{v,\hat{t}-1}, \ v \in V$, and the edge capacities for this flow are defined by the variables $x_{v,\hat{t}-1}, \ v \in V$, via Constraint 6. Hence, all the defining parameters for static flow $\mathcal{F}(\hat{t})$ are integer (actually, binary). But then, according the corresponding theory of network flows [16], the flows $\widehat{\mathcal{F}}(\hat{t})$ that constitute extreme points of the feasibility space for the variables $u_{v,v',\hat{t}}, \ v \in V, \ v' \in \mathcal{N}(v)$ that is defined by the considered pricing of the variables $x_{v,t}, \ v \in V, \ t \in \{0, \ldots, \bar{T}\}$ and the Constraints 5 and 6 corresponding to period $\hat{t}$, are integer. Since the considered period $\hat{t}$ was chosen arbitrarily, it follows that there exists an integer flow $\widehat{\mathcal{F}}(t)$ for every period $t \in \{1, \ldots, \bar{T}\}$, and our claim has been proven.

From the above developments it follows that (i) for any valid pricing of the variables $x_{v,t}, \ v \in V, \ t \in \{0, \ldots, \bar{T}\}$, there is an integral sequence of flows $\widehat{\mathcal{F}}(t), t \in \{1, \ldots, \bar{T}\}$, that supports this pricing of $x_{v,t}$ by pricing the variables $u_{v,v't}$ as discussed in the previous paragraph, and (ii) the plan P that is defined by the variables $x_{v,t}$ and $u_{v,v',t}$, is assessed correctly regarding its feasibility with respect to the imposed connectivity constraints 10–14 and the employed objective function 15 or 16, via the resulting pricing of the variables $\psi_{v,t,\theta}, y_{v,t}$ and s_t. The results of Theorem 4.1 follow immediately from this remark. □

The practical implication of Theorem 4.1 is that we can solve the IP formulations of Section III by solving their combinatorial relaxations that were defined at the beginning of this section. This capability can be very significant from a computational standpoint, since the number of integer variables in the IP formulations of Section III is of the order $O(|V|^2 \cdot \bar{T}) = O(|V|^3)$, while the number of integer variables involved in the considered relaxations is only $|V| \cdot \bar{T}$, which is $O(|V|^2)$. Furthermore, it is worth-mentioning that the most extensively used MIP solvers, like CPLEX©, satisfy the conditions that are posed in the second part of Theorem 4.1. Finally, we also notice, for completeness, that the strong combinatorial relaxations developed in [18] for the IP formulations of the M- and TVT-problems on dendritic topologies involve only $|L| \cdot \bar{T}$ binary variables, instead of $|V| \cdot \bar{T}$, but the derivation of these relaxations relies strongly on the tree structure of the underlying guidepath network $\mathcal{G}$.

V. Some experimental results

In this section, we report the results of a numerical experiment that analyzes the efficacy and the tractability of the relaxed MIP formulations of Section IV. More specifically, we have generated a number of instances of the M-problem by varying a set of parameters that affect the difficulty of this problem, and we consider the quality of the solutions that are provided by the corresponding combinatorial relaxations of Section IV under a fixed computational time budget.

The complete experimental design and the obtained results are tabulated in Table I. More specifically, in Table I, column $|V|$ reports the number of nodes of the guidepath network $\mathcal{G}$ of the generated problem instances. For each $|V|$ value, we consider three levels of $|\mathcal{R}|$: (a) *low (L)*, where $|\mathcal{R}|$ is set equal to the *eccentricity* Θ of node o in the guidepath network $\mathcal{G}$, (b) *moderate M*, where $|\mathcal{R}|$ is set equal to the average of Θ and $|V|$, and (c) *high (H)*, where $|\mathcal{R}|$ is set equal to $|V|$. Also, for each pair of $|V|$ and $|\mathcal{R}|$, we define three levels for the number of target nodes, $|L|$: (a) 10%, (b) 30%, and (c) 50%, where $|L|$ is respectively set equal to 10%, 30%, and 50% of $|V|$. Furthermore, we consider some structural elements which may affect the difficulty of the problem by defining the following three types of planar graphs: Type-I graphs are connected planar graphs where, for any pair of nodes, there exists at least one cycle containing these nodes; Type-II graphs are planar graphs consisting of a type-I subgraph and a set of subtrees rooted at some nodes of the type-I subgraph; Type-III graphs are planar graphs consisting of a set of type-II subgraphs which are connected to each other through unique paths (also known as bridges). In order to check the impact of the additional structural elements of type-II and type-III graphs on the difficulty of the problem, we ensure that some target nodes are located in the subtrees that are contained in these graphs. Columns "Type-I", "Type-II", and "Type-III" report the corresponding experimental results for problem instances, respectively, with type-I, type-II, and type-III guidepath networks, obtained by the relaxed MIP formulations.

For each triplet ($|V|$, $|\mathcal{R}|$-level, $|L|$-level), we randomly generated five instances for each type of guidepath network and, for each generated problem instance, we tried to solve the MIP formulation of Section IV for the M-problem with a time budget of one hour (3600 seconds). Furthermore, in order to interpret the results more easily, we redefined the objective function of these MIP formulations as follows:

$$\min\left(\bar{T} + 1 - \sum_{t \in \{1, \ldots, \bar{T}\}} s_t\right)$$

We solved the resulting MIPs using CPLEX© in Python, on a Mac OS-laptop with i7-8850H 2.6GHz CPU, and 16 GB RAM. Columns entitled "Computation time", for each of the three guidepath-network types, in Table I reports the average computational time for the corresponding five replications, to obtain an optimal solution or to reach the specified time limit of 3600 secs. In the latter case, CPLEX© also returned a suboptimal solution together with a suboptimality (MIP) gap which is defined by the following equation:

TABLE I: Some experimental results for the solution of the M-problem through the corresponding combinatorial relaxation of Section IV.

$\lvert V\rvert$	$\lvert\mathcal{R}\rvert$	$\lvert L\rvert$	**Type-I**		**Type-II**		**Type-III**	
			Computation time (sec)	**Inflation ratio**	**Computation time (sec)**	**Inflation ratio**	**Computation time (sec)**	**Inflation ratio**
		10%	0.83	1.00	2.00	1.00	1.37	1.00
	L	30%	2.09	1.00	9.24	1.00	723.62	1.02
		50%	6.74	1.00	31.96	1.00	2163.35	1.16
		10%	0.49	1.00	1.25	1.00	2.04	1.00
20	M	30%	2.33	1.00	7.40	1.00	2.08	1.00
		50%	1.07	1.00	15.86	1.00	2.61	1.00
		10%	0.36	1.00	0.76	1.00	0.60	1.00
	H	30%	0.71	1.00	14.67	1.00	1.19	1.00
		50%	1.03	1.00	3.40	1.00	3.10	1.00
		10%	669.53	1.00	1211.51	1.16	779.08	1.18
	L	30%	2063.78	1.17	3600	1.44	3280.21	1.61
		50%	3134.02	2.13	3600	2.25	3600	4.47
		10%	350.35	1.00	673.96	1.00	796.33	1.03
40	M	30%	742.58	1.08	1955.19	1.16	1435.84	1.17
		50%	882.38	1.06	2671.91	1.16	1632.57	1.18
		10%	22.63	1.00	418.63	1.00	734.35	1.02
	H	30%	423.44	1.00	1444.03	1.10	1202.87	1.06
		50%	783.49	1.05	2883.10	1.19	1207.22	1.16
		10%	2964.76	1.15	2277.31	1.38	3600	4.17
	L	30%	3600	3.36	3600	5.64	3600	9.04
		50%	3600	5.30	3600	5.59	3600	8.80
		10%	762.53	1.00	777.20	1.20	2170.12	1.94
60	M	30%	2884.00	1.17	2542.03	1.37	3600	7.80
		50%	2885.22	2.21	3600	3.10	3600	7.02
		10%	287.44	1.00	1482.23	1.00	1986.57	1.07
	H	30%	2965.50	1.43	3600	1.57	2760.85	1.46
		50%	2620.86	1.64	3600	2.03	3600	2.39
		10%	2629.96	1.14	2973.27	2.17	3600	2.46
	L	30%	3017.39	2.56	3600	6.75	3600	7.15
		50%	3600	3.73	3600	6.78	3600	6.48
		10%	921.36	1.21	1531.72	1.29	2688.14	1.40
80	M	30%	2707.56	2.01	3600	4.62	3600	6.84
		50%	3600	2.60	3600	6.58	3600	6.77
		10%	1877.20	1.24	2251.87	1.19	1733.37	1.50
	H	30%	3424.37	1.46	2888.28	2.35	3515.64	2.37
		50%	3600	2.41	3436.78	3.23	3600	6.70
		10%	3600	3.19	3600	7.24	3600	9.57
	L	30%	3600	9.45	3600	7.83	3600	16.09
		50%	3600	9.34	3600	8.28	3600	13.82
		10%	1684.63	1.29	2488.93	1.61	3600	2.91
100	M	30%	3187.38	2.44	3600	3.89	3600	7.53
		50%	3419.69	2.43	3600	4.10	3600	7.43
		10%	1907.28	1.31	2609.79	1.68	3600	3.61
	H	30%	3491.24	2.46	3286.87	3.42	3600	4.72
		50%	3380.24	2.23	3600	3.90	3600	7.33

$$\text{MIP gap} = 100 \times \left(1.0 - \frac{\text{best lower bound obtained by CPLEX}}{\text{best objective value obtained by CPLEX}}\right) \tag{21}$$

But for an easier understanding of the solution suboptimality that is implied by the MIP-gap values, in Table I, we translate these values into an "inflation ratio" which is obtained from the corresponding MIP gaps by rearranging Equation 21 as follows:

$$\frac{1}{1 - \left(\dfrac{\text{MIP gap}}{100}\right)} = \left(\frac{\text{best objective value obtained by CPLEX}}{\text{best lower bound obtained by CPLEX}}\right) \tag{22}$$

The right-hand-side of Equation 22 is the "inflation ratio" of the attained objective value with respect to the best lower bound attained by CPLEX© within the provided time budget. For instance, the inflation ratio of a returned solution with a MIP gap of 50% is 2.00; i.e., the attained objective value is at most two times higher than the optimal value. Similarly, the inflation ratio of a solution with a MIP gap of 90% is 10.00. Obviously, the inflation ratio for a solution with a MIP gap of 0% – i.e., an optimal solution – is 1.00, while a high value of the inflation ratio implies that CPLEX© could not return a certified good-quality solution within the provided time budget. In Table I, column "Inflation ratio" for each type of graph reports the average inflation ratio for the corresponding five repetitions.

We can see in Table I that, as $|V|$ increases, the difficulty of solving the corresponding MIPs increases significantly.

Hence, for many problem instances with the larger values of $|V|$, CPLEX© can generate only a suboptimal solution within the provided time budget. Furthermore, we notice that the levels of $|\mathcal{R}|$ and $|L|$, and the type of instances, can affect significantly the difficulty of the formulated MIPs. For some pairs ($|\mathcal{R}|$, $|L|$-level) and types of instances, CPLEX© can attain an optimal solution even for a large value of $|V|$ in a reasonable computation time, but for some other pairs of those parameters and types, it cannot attain an optimal solution in the specified time budget, even for moderate levels of $|V|$.

More specifically, for each pair ($|V|$, $|\mathcal{R}|$-level) with some type of graph, a higher portion of the target zones relative to $|V|$ increases the difficulty of the M-problem. The intuitive explanation for this fact is that, as the number of target zones increases, the number of the permutations that characterize the visitation sequence of the target nodes increases as well, and therefore, the difficulty of the problem increases.

Furthermore, for each pair ($|V|$, $|L|$-level) with some type of graph, a low level of $|\mathcal{R}|$ also increases the difficulty of the M-problem significantly, especially when $|V|$ obtains some larger values. This result is consistent with the experimental results of [18], and it is intuitively explained by the fact that the scarcity of available robots requires a more careful allocation of this resource in each period.

Also, for each triplet ($|V|$, $|\mathcal{R}|$-level, $|L|$-level), the type of graphs significantly affects the difficulty of the M-problem, with guidepath networks of types-II and -III corresponding to harder problem instances. This fact is due to the limited concurrency and routing flexibility that are enabled by the additional structural elements of these guidepath networks, even in the presence of a very large number of robots, $|\mathcal{R}|$.

Finally, we note that the aforementioned remarks also apply to the TVT-problem instances that are defined by configurations considered in Table I. But we have not included explicitly the corresponding results in this section due to the page limit that is imposed for this paper.

VI. Conclusions

In this paper, we have extended the investigation of the M- and TVT-problems on dendritic topologies, originally introduced in [15], by considering a more arbitrary topology of the underlying guidepath network. Our developments parallel, in spirit, the corresponding developments of [15], [18] for the more restricted case of dendritic topologies, but the analytical details of the derived results are substantially different.

A juxtaposition of the numerical results that are reported in those previous works with the corresponding results that are reported in this paper, reveals that the routing flexibility that is inherent in the new problem instances, can ease up considerably the computational effort and time that is required for the solution of these problems through their (relaxed) MIP formulations.

On the other hand, the new problem versions remain NP-hard, and the overall tractability of the corresponding MIPs eventually is limited by this fact. Hence, in our future work, we shall also seek the development of pertinent heuristics that can provide good suboptimal solutions for harder problem instances within (more) reasonable computational times. The corresponding developments of [19] can be a good starting point in this endeavor.

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