
A proof-checking experiment on representing graphs as membership digraphs^{*}

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Abstract. We developed, and computer-checked by means of the Ref verifier, a formal proof that every weakly extensional, acyclic (finite) digraph can be decorated injectively *à la* Mostowski by finite sets so that its arcs mimic membership. We managed to have one sink decorated with \emptyset by this injection.

We likewise proved that a graph whatsoever admits a weakly extensional and acyclic orientation; consequently, and in view of what precedes, one can regard its edges as membership arcs, each deprived of the direction assigned to it by the orientation.

These results will be enhanced in a forthcoming scenario, where every connected claw-free graph G will receive an extensional acyclic orientation and will, through such an orientation, be represented as a transitive set T so that the membership arcs between members of T will correspond to the edges of G .

Key words: Theory-based automated reasoning; proof checking; Referee aka *ÆtnaNova*; graphs and digraphs; Mostowski's decoration.

1 Can graphs be represented as membership digraphs?

One usually views the edges of a graph as vertex doubletons;⁴ but various ways of representing graphs can be devised (as quickly surveyed in [5, Sec. 2]). Thanks to a convenient choice on how to represent connected claw-free graphs, Milanič and Tomescu [2] proved with relative ease two classical results on graphs of that kind, namely that any such graph owns a near-perfect matching and has a Hamiltonian cycle in its square. Those results are, in fact, legitimately transferred to a special class of digraphs, whose vertices are hereditarily finite sets and whose

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⁴ We call undirected graphs simply *graphs*, and directed graphs, *digraphs*.

arcs reflect the membership relation. Under this change of perspective, a fully formal reconstruction of those results became affordable and, once carried out, was certified correct with the Ref proof-checker [3,4].

Can we, with equal ease, formalize in Ref the Milanič-Tomescu representation result *per se*? That result is the claim that for every connected claw-free graph G there exist a set $\nu_G \supseteq \bigcup \nu_G$ and an injection f from the vertices of G onto ν_G such that $\{x, y\}$ is an edge of G if and only if either $fx \in fy$ or $fy \in fx$ holds. The proof articulates as follows:

1. One shows that for any graph $G = (V, E)$ as said, there is a $D \subseteq V \times V$ such that $E = \{\{x, y\} : [x, y] \in D\}$ and (V, D) is an acyclic digraph which is EXTENSIONAL: i.e., no two vertices in V have the same out-neighbors.
2. One DECORATES vertices by putting $fv = \{fw : [v, w] \in D\}$ à la Mostowski, for all $v \in V$. Acyclicity ensures that this recursion makes sense; extensionality ensures the injectivity of f .

As a preparatory, simpler formalization task, we have proved with Ref that a graph G whatsoever admits a set ν_G and an injection f from the vertices of G onto ν_G such that $\{x, y\}$ is an edge of G if and only if either $fx \in fy$ or $fy \in fx$ holds. We *could not* have insisted on the transitivity condition $\nu_G \supseteq \bigcup \nu_G$ here, because we have nohow restrained G . The proof now articulates as follows:

- 1'. For any $G = (V, E)$, there is a $D \subseteq V \times V$ s.t. $E = \{\{x, y\} : [x, y] \in D\}$ and (V, D) is an acyclic digraph which is WEAKLY EXTENSIONAL: i.e., any two vertices that share the same out-neighbors have *no* out-neighbors.
- 2'. We decorate vertices by putting: $fv = \{fw : [v, w] \in D\}$ for each $v \in V$ endowed with out-neighbors, $fz = \emptyset$ for one sink z , and $fu = \{\{V\} \cup V \setminus \{u\}\}$ for each sink $u \neq z$.

(Notice that 2'. subsumes 2. altogether, because an extensional digraph has exactly one sink.)

2 Ingredients of a Ref's scenario

What one submits to the Ref checker, to have its correctness verified, is a SCENARIO: namely, a script file consisting of definitions and of theorems endowed with their proofs; a construct, named THEORY, enables one to package definitions and theorems into reusable proofware components. A variant of the Zermelo-Fraenkel set theory, postulating global choice, regularity, and infinity, underlies the logical armory of Ref: this is apparent from the fifteen or so inference rules available in the proof-specification language (see [5, Sec. 3]), of which only a few sprout directly from first-order predicate calculus, while most embody some form of set-theoretic reasoning. Multi-level syllogistic [1] acts as a ubiquitous inference mechanism, while THEORIES add a touch of second-order reasoning ability to Ref's overall machinery.

Our figures offer a glimpse of the Ref's language. Fig. 1 shows the definitions of graph-theoretic notions relevant to the proof-checking experiment on which

DEF acyclic :	[Acyclicity]	$\text{Acyclic}(V, D) \leftrightarrow_{\text{Def}}$
	$\langle \forall w \subseteq V \mid w \neq \emptyset \rightarrow \langle \exists t \in w \mid \emptyset = \{y \in w \mid [t, y] \in D\} \rangle \rangle$	
DEF xtens₀ :	[Extensionality]	$\text{Extensional}(V, D) \leftrightarrow_{\text{Def}}$
	$\langle \forall x \in V, y \in V, \exists z \mid ([x, z] \in D \leftrightarrow [y, z] \in D) \rightarrow x = y \rangle$	
DEF xtens₁ :	[Weak extensionality]	$\text{WExtensional}(V, D) \leftrightarrow_{\text{Def}}$
	$\text{Extensional}(V \cap \text{domain}(D \cap (V \times V)), D \cap (V \times V))$	
DEF orien :	[Orientation of a graph]	$\text{Orientates}(D, V, E) \leftrightarrow_{\text{Def}}$
	$E \cap \{\{x, y\} : x \in V, y \in V \setminus \{x\}\} = \{\{p^{[1]}, p^{[2]}\} : p \in D \mid p = [p^{[1]}, p^{[2]}\}\}$	
DEF Finite :	[Finitude]	$\text{Finite}(F) \leftrightarrow_{\text{Def}}$
	$\langle \forall g \in \mathcal{P}(\mathcal{P}F) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle$	
DEF maps₅ :	[Map predicate]	$\text{Is_map}(F) \leftrightarrow_{\text{Def}}$
	$\langle \forall p \in F \mid p = [p^{[1]}, p^{[2]}\rangle$	
DEF maps₆ :	[Single-valued map]	$\text{Svm}(F) \leftrightarrow_{\text{Def}}$
	$\text{Is_map}(F) \ \& \ \langle \forall p \in F, q \in F \mid p^{[1]} = q^{[1]} \rightarrow p = q \rangle$	

Fig. 1. Four properties refer to digraphs, the other three to generic sets

THEOREM part_whole₀ .	$\text{Svm}(F) \rightarrow (\text{Finite}(F) \leftrightarrow \text{Finite}(\text{domain}(F)))$.	PROOF:
Suppose_not(f_1)	\Rightarrow AUTO	
Suppose	\Rightarrow Finite(f_1)	
APPLY $\langle \rangle$	$\text{finitelImage}(s_0 \mapsto f_1, f(X) \mapsto X^{[1]}) \Rightarrow$	Finite($\{\{x^{[1]} : x \in f_1\}\}$)
Use_def(domain)	\Rightarrow false	
Discharge	\Rightarrow AUTO	
$\langle f_1 \rangle \leftrightarrow T\text{svm}_2$	$\Rightarrow f_1 = \{[x, f_1 \upharpoonright x] : x \in \text{domain}(f_1)\}$	
APPLY $\langle \rangle$	$\text{finitelImage}(s_0 \mapsto \text{domain}(f_1), f(X) \mapsto [x, f_1 \upharpoonright x]) \Rightarrow$	Finite($\{[x, f_1 \upharpoonright x] : x \in \text{domain}(f_1)\}$)
EQUAL	\Rightarrow false	
Discharge	\Rightarrow QED	

Fig. 2. Example of a theorem proved in the Ref language

we report, and introduces finitude and the notion of mapping ('Svm').⁵ Fig. 2

⁵ To enforce a useful distinction, we denote by $G(x)$ the application of a *global* function G to an argument x ('global' meaning that the domain of G is the class of all sets), while denoting by $f \upharpoonright x$ the application to x of a *map* f (typically single-valued), viewed as set of pairs.

shows the formal development of a proof, consisting of nine steps, each indicating which inference rule is employed to get the corresponding statement. This proof invokes twice a **THEORY** named `finiteImage`, whose interface is displayed in Fig. 3. While `finiteImage` does not return any symbol, the other, subtler **THEORY** displayed in the same figure, namely `finiteInduction`, returns a symbol, `fin θ` , representing an \subseteq -minimal set which meets P —given that at least one finite set satisfying property P exists. Likewise, the **THEORY** `finiteAcycLabeling` shown in Fig. 4 returns a labeling of a given acyclic digraph, thereby furnishing the technique for decorating the graph *à la* Mostowski.

THEORY <code>finiteImage</code> ($s_0, f(X)$) <code>Finite</code> (s_0) \Rightarrow <code>Finite</code> ($\{f(x) : x \in s_0\}$) END <code>finiteImage</code>	THEORY <code>finiteInduction</code> ($s_0, P(S)$) <code>Finite</code> (s_0) & $P(s_0)$ \Rightarrow (<code>finθ</code>) $\langle \forall S \mid S \subseteq \text{fin}\theta \rightarrow \text{Finite}(S) \ \& \ (P(S) \leftrightarrow S = \text{fin}\theta) \rangle$ END <code>finiteInduction</code>
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Fig. 3. Interfaces of two **THEORYS** regarding finitude

THEORY <code>finAcycLabeling</code> ($v_0, d_0, h(s, x)$) <code>Acyclic</code> (v_0, d_0) & <code>Finite</code> (v_0) \Rightarrow (<code>labθ</code>) <code>Svm</code> (<code>labθ</code>) & <code>domain</code> (<code>labθ</code>) = v_0 $\langle \forall x \in v_0 \mid \text{lab}\theta \upharpoonright x = h \left(\left\{ \text{lab}\theta \upharpoonright p^{[2]} : p \in d_0 \upharpoonright \{x\} \mid p^{[2]} \in v_0 \right\}, x \right) \rangle$ END <code>finAcycLabeling</code>

Fig. 4. Interface of a **THEORY** usable to label an acyclic digraph

3 Our experiment in a nutshell

The two **THEORYS** in which our experiment culminates are shown in Fig. 5; the key theorem which makes the second of them derivable from the first was stated in Ref as follows:

THEOREM `weaXtensionalization $_0$` . $\text{Finite}(V) \ \& \ S \in V \rightarrow$
 $\langle \exists d \mid \text{Orientates}(d, V, E) \ \& \ \text{Acyclic}(V, d) \ \& \ \text{WExtensional}(V, d) \ \& \ S \notin \text{range}(d) \rangle$.

Due to its centrality in our scenario, we wish to briefly sketch the proof of the orientability theorem just cited. Arguing by contradiction, suppose that there

<p>THEORY finMostowskiDecoration(v_0, d_0) $v_0 \times v_0 \supseteq d_0$ & $v_0 \neq \emptyset$ & Finite(v_0) & Acyclic(v_0, d_0) & WExtensional(v_0, d_0) \Rightarrow (mski$_{\emptyset}$) Svm(mski$_{\emptyset}$) & domain(mski$_{\emptyset}$) = v_0 $\langle \forall w \mid w \in \text{domain}(d_0) \rightarrow \text{mski}_{\emptyset} w = \{ \text{mski}_{\emptyset} p^{[2]} : p \in d_{0 \{w\}} \} \text{ \& } \text{mski}_{\emptyset} w \neq \emptyset \rangle$ $\emptyset \in \text{range}(\text{mski}_{\emptyset})$ & $\langle \forall y \mid y \in \text{range}(\text{mski}_{\emptyset}) \rightarrow \text{Finite}(y) \rangle$ $\langle \forall x, y \mid \{x, y\} \subseteq v_0 \text{ \& } \text{mski}_{\emptyset} x = \text{mski}_{\emptyset} y \rightarrow x = y \rangle$ $\langle \forall y \mid y \in v_0 \rightarrow (\text{mski}_{\emptyset} y \in \text{mski}_{\emptyset} x \leftrightarrow [x, y] \in d_0) \rangle$ END finMostowskiDecoration</p> <p>THEORY finGraphRepr(v_0, e_0) $e_0 \subseteq \{ \{x, y\} : x \in v_0, y \in v_0 \setminus \{x\} \}$ & $v_0 \neq \emptyset$ & Finite(v_0) \Rightarrow (wski$_{\emptyset}$) Svm(wski$_{\emptyset}$) & domain(wski$_{\emptyset}$) = v_0 & $\emptyset \in \text{range}(\text{wski}_{\emptyset})$ $\langle \forall y \mid y \in \text{range}(\text{wski}_{\emptyset}) \rightarrow \text{Finite}(y) \rangle$ $\langle \forall x, y \mid \{x, y\} \subseteq v_0 \text{ \& } \text{wski}_{\emptyset} x = \text{wski}_{\emptyset} y \rightarrow x = y \rangle$ $\langle \forall x, y \mid \{x, y\} \subseteq v_0 \rightarrow ((\text{wski}_{\emptyset} y \in \text{wski}_{\emptyset} x \vee \text{wski}_{\emptyset} x \in \text{wski}_{\emptyset} y) \leftrightarrow \{x, y\} \in e_0) \rangle$ $\langle \forall x \mid \text{wski}_{\emptyset} x \cap \text{range}(\text{wski}_{\emptyset}) \neq \emptyset \rightarrow \text{wski}_{\emptyset} x \subseteq \text{range}(\text{wski}_{\emptyset}) \rangle$ END finGraphRepr</p>
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Fig. 5. THEORIES on Mostowski's decoration and on graph representation

is a counterexample; then, exploiting the finiteness hypothesis, take a *minimal* counterexample v_1, s_1, e_0 . We are supposing that there is no acyclic, weakly extensional orientation of the graph $(v_1, e_0 \cap \{ \{x, y\} : x \in v_1, y \in v_1 \setminus \{x\} \})$ having s_1 as a source; whereas, for every $v_0 \subsetneq v_1$, one can orient $(v_0, e_0 \cap \{ \{x, y\} : x \in v_0, y \in v_0 \setminus \{x\} \})$ in an acyclic and weakly extensional way, for any vertex $t \in v_0$, so that t plays the role of a source. Let, in particular, $v_0 = v_1 \setminus \{s_0\}$. Unless s_1 is an isolated vertex, an acyclic and weakly extensional orientation of v_0 exists that has as a source a chosen neighbor t_1 of s_1 . However, that orientation could trivially be extended to the graph with vertices v_1 so that s_1 becomes a source; this contradiction shows that s_1 cannot have neighbors in v_1 , which is also untenable: if so, any orientation for v_0 would in fact work also as an orientation for v_1 and, as such, would have s_0 as a source.

The full Ref scenario can be seen at <http://www2.units.it/eomodeo/wERS.pdf> (cf. also <http://www2.units.it/eomodeo/ClawFreeness.html>).

4 Planned work on representing claw-free graphs

The larger experiment we have in mind will associate with each connected claw-free graph $G = (V, E)$ an injection f from V onto a transitive, hereditarily finite set ν_G so that $\{x, y\} \in E$ if and only if either $f x \in f y$ or $f y \in f x$.

The new notions entering into play can be rendered formally as follows:

$$\begin{aligned}
\text{ClawFreeG}(V, E) &\leftrightarrow_{\text{Def}} \langle \forall w, x, y, z \mid \{w, x, y, z\} \subseteq V \ \& \ \{\{w, y\}, \{y, x\}, \{y, z\}\} \subseteq E \rightarrow \\
&\quad (x = z \vee w \in \{z, x\} \vee \{x, z\} \in E \vee \{z, w\} \in E \vee \{w, x\} \in E) \rangle, \\
\text{Connected}(E) &\leftrightarrow_{\text{Def}} \emptyset \notin E \wedge \\
&\quad \langle \forall p \mid \left(\bigcup p = E \wedge \langle \forall b \in p, c \in p \mid \bigcup b \cap \bigcup c \neq \emptyset \leftrightarrow b = c \rangle \right) \rightarrow p = \{E\} \rangle, \\
\text{HerFin}(S) &\leftrightarrow_{\text{Def}} \text{Finite}(S) \ \& \ \langle \forall x \in S \mid \text{HerFin}(x) \rangle.
\end{aligned}$$

Here, the first *definiens* requires that no subgraph of (V, E) induced by four vertices has the shape of a ‘Y’. The second *definiens* requires that the set E of edges can nohow be split into multiple disjoint blocks so that no edge acts as a ‘bridge’ by sharing endpoints with edges in distinct blocks. Hereditary finitude is a recursive notion.

We aim at getting the analogue, shown in Fig. 6, of **THEORY** `finGraphRepr` (cf. Fig. 5). For that, we must again exploit **THEORY** `finMostowskiDecoration`; in addition, a key theorem will ensure the acyclic extensional orientability of a connected and claw-free graph:

$$\begin{aligned}
\text{THEOREM } \text{cClawFreeG}_2. &\text{ Finite}(V) \ \& \ \text{Connected}(V, E) \ \& \\
&\text{ClawFreeG}(V, E) \ \& \ E \subseteq \{\{x, y\} : x \in V, y \in V \setminus \{x\}\} \rightarrow \\
&\langle \exists d \subseteq V \times V \mid \text{Orientates}(d, V, E) \ \& \ \text{Acyclic}(V, d) \ \& \ \text{Extensional}(V, d) \rangle.
\end{aligned}$$

THEORY `herfinCCFGraphRepr`(v_0, e_0)
 $e_0 \subseteq \{\{x, y\} : x \in v_0, y \in v_0 \setminus \{x\}\} \ \& \ \text{Finite}(v_0)$
 $\text{Connected}(v_0, e_0) \ \& \ \text{ClawFreeG}(v_0, e_0)$
 $\Rightarrow (\text{trans}_\emptyset)$
 $\text{Svm}(\text{trans}_\emptyset) \ \& \ \text{domain}(\text{trans}_\emptyset) = v_0$
 $\langle \forall x, y \mid \{X, Y\} \subseteq v_0 \ \& \ \text{trans}_\emptyset \upharpoonright X = \text{trans}_\emptyset \upharpoonright Y \rightarrow X = Y \rangle$
 $\langle \forall x, y \mid \{X, Y\} \subseteq v_0 \rightarrow$
 $\quad (\text{trans}_\emptyset \upharpoonright Y \in \text{trans}_\emptyset \upharpoonright X \vee \text{trans}_\emptyset \upharpoonright X \in \text{trans}_\emptyset \upharpoonright Y \leftrightarrow \{X, Y\} \in e_0) \rangle$
 $\{y \in \text{range}(\text{trans}_\emptyset) \mid y \not\subseteq \text{range}(\text{trans}_\emptyset)\} = \emptyset$
 $\text{range}(\text{trans}_\emptyset) \neq \emptyset \ \& \ \text{HerFin}(\text{range}(\text{trans}_\emptyset))$
END `herfinCCFGraphRepr`

Fig. 6. **THEORY** on representing a connected claw-free graph via membership

Another fact we must exploit is that every connected graph has a vertex whose removal (along with all edges incident to it) does not disrupt connectivity. The existence of such a NON-CUT VERTEX is easily proved for a tree. So, in order to cheaply achieve our goal, we will define

$$\begin{aligned}
\text{HankFree}(T) &\leftrightarrow_{\text{Def}} \langle \forall e \subseteq T \mid e = \emptyset \vee \langle \exists a \in e \mid a \not\subseteq \bigcup (e \setminus \{a\}) \rangle \rangle, \\
\text{Is_tree}(T) &\leftrightarrow_{\text{Def}} \text{Connected}(T) \ \wedge \ \text{HankFree}(T)
\end{aligned}$$

and—if only provisionally—recast the connectivity assumption, $\text{Connected}(v_0, e_0)$, of **THEORY** `herfinCCFGraphRepr` as the assumption that (v_0, e_0) has a ‘spanning tree’:

$$\langle \exists t \mid \text{Is_tree}(t) \ \& \ \bigcup t = v_0 \ \& \ (v_0 = \{\text{arb}(v_0)\} \vee t \subseteq e_0) \rangle .$$

This eases things: for, any vertex with fewer than 2 incident edges in the spanning tree of a connected graph will be a non-cut vertex of the graph.

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