Schelling Games with Continuous Types (short paper)*

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Abstract
In most major cities and urban areas, residents form homogeneous neighborhoods along ethnic or socio-economic lines. This phenomenon is widely known as residential segregation and has been studied extensively. Fifty years ago, Schelling proposed a landmark model that explains residential segregation in an elegant agent-based way. A recent stream of papers analyzed Schelling's model using game-theoretic approaches. However, all these works considered models with a given number of discrete types modeling different ethnic groups. We focus on segregation caused by non-categorical attributes, such as household income or position in a political left-right spectrum. For this, we consider agent types that can be represented as real numbers. This opens up a great variety of reasonable models and, as a proof of concept, we focus on several natural candidates. In particular, we consider agents that evaluate their location by the average type-difference or the maximum type-difference to their neighbors, or by having a certain tolerance range for type-values of neighboring agents. We study the existence and computation of equilibria and provide bounds on the Price of Anarchy and Stability.

1. Introduction

"Birds of a feather flock together" is an often used proverb to describe *homophily* [1], i.e., the phenomenon that homogeneous groups are prevalent in society. The group members might be similar in terms of, for example, their ethnic group, their socioeconomic status, or their political orientation. Within a city, such groups typically cluster together, which then leads to segregated neighborhoods, called *residential segregation* [2].

Segregated neighborhoods have a strong impact on the socioeconomic prospects [3] and on the health of its inhabitants [4, 5]. This explains why residential segregation is widely studied. Typical models are agent-based and they assume that the agents are partitioned into a given fixed set of *types*, which can be understood as an ethnic group, a trait, or an affiliation. The landmark model of this kind was proposed by Schelling [6, 7] roughly fifty years ago. There,

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agents of two types are placed on the line or a grid and it is assumed that an agent is content with her current location, if at least a τ-fraction of all neighbors are of her type, for some $\tau \in [0, 1]$. Discontent agents try to relocate. As a result, large homogeneous neighborhoods eventually form, even if all the agents are tolerant, i.e., if $\tau \leq 1/2$.

However, real-world agents show more complex behavior than predicted by Schelling's two-type model. For example, people might not care about the ethnic group, but they might compare themselves with their neighbors along non-categorical aspects like age, household income, or position in a political left-right spectrum. Given these more complex preferences, the agents cannot be assumed to simply classify their neighbors into friends and enemies. This is in line with recent economics research which reveals that individuals' happiness is relative to a particular peer group. E.g., the reference income hypothesis [8, 9] states that people compare their income with a reference value, e.g., the mean or median income of their neighborhood [10, 11].

With this paper, we initiate the study of agent-based models for residential segregation that use non-categorical type-values for the agents. This allows for modeling more realistic agent preferences. Using arbitrary type-values in $[0, 1]$ unlocks an entirely new class of game-theoretic models, that we call *Schelling Games with Continuous Types*. As the first steps, we consider three natural behavioral models: agents compare their type-value with the most different or the average type-value in their neighborhood, or they have a tolerance range for the accepted type-value difference. All three variants of the cost function are motivated by plausible real-world behavior. Comparing with the maximum-difference type-value is motivated by considering types as positions in a political left-to-right spectrum, comparing with the average type-value is suggested by the setting where types are household incomes, and the model with a tolerance range is inspired by types being the age of the agents, where agents consider other agents as similar if they are roughly their age.

1.1. Model

A *Schelling Game with Continuous Types* is defined by an undirected connected graph $G = (V, E)$, a set $\mathcal{N}$ of n strategic agents and a type function $t : \mathcal{N} \to [0, 1]$ mapping agent $i \in \mathcal{N}$ to her *type* $t(i) \in [0, 1]$. Unless stated otherwise, we assume without loss of generality that $t(i) \leq t(j)$ for $i, j \in \mathcal{N}$ and $i \leq j$. The *type-distance* $d : \mathcal{N}^2 \to [0, 1]$ between two agents i and j is defined as $d(i, j) = |t(i) - t(j)|$.

An agent's strategy is her location on the graph, i.e., a node of G. A *strategy profile* σ is an n-dimensional vector whose i-th entry corresponds to the strategy of the i-th agent and where all strategies are pairwise disjoint. For an agent $i \in \mathcal{N}$, let the *neighborhood* of i be the set $N_i(\sigma) = \{j \in \mathcal{N} : \{\sigma(i), \sigma(j)\} \in E\}$ of agents living in the neighborhood of $\sigma(i)$ in G. If $N_i(\sigma) = \emptyset$, we say that i is *isolated*.

In *Swap Schelling Games with Continuous Types*, every node is occupied by exactly one agent, so $n = |V|$. Agents can change their strategies only by swapping their location with another agent. As agents are rational, we only consider *profitable swaps* that strictly decrease the individual cost of both involved agents. A strategy profile is a *swap equilibrium (SE)* if it does not admit any profitable swaps.

In *Jump Schelling Games with Continuous Types*, empty nodes exist, i.e., $n < |V|$ with

$e := |V| - n$. An agent can change her strategy by jumping to any empty node. Agents only perform *profitable jumps* that strictly decrease their cost. A strategy profile is a *jump equilibrium (JE)* if it does not admit any profitable jumps.

We consider the following three cost models. In *Average Type-Distance Games (ADGs)*, the cost of agent i in σ is defined as the average distance towards her neighbors, i.e., $\text{cost}_i(\sigma) = \frac{\sum_{j \in N_i(\sigma)} d(i,j)}{|N_i(\sigma)|}$. In *Maximum Type-Distance Games (MDGs)*, the cost of agent i in σ is defined as the maximum distance towards her neighbors, i.e., $\text{cost}_i(\sigma) = \max_{j \in N_i(\sigma)} d(i,j)$. In *Cutoff Games (CGs)*, given a cutoff parameter $\lambda \in [0,1]$, let $N_i^+(\sigma) = \{j \in N_i(\sigma) : d(i,j) \leq \lambda\}$ be the set of *friends* of agent i in σ and $N_i^-(\sigma) = N_i(\sigma) \setminus N_i^+(\sigma)$ be the set of *enemies*. The cost i in σ is the fraction of enemies in the neighborhood of i, i.e., $\text{cost}_i(\sigma) = \frac{|N_i^-(\sigma)|}{|N_i(\sigma)|}$. The model of CGs is closer to the original Schelling model, i.e., neighbors whose type difference is within the cutoff are considered as *friends*; however, in contrast to previous models, friendship is not transitive.

For all cost models, we consider two possible variants depending on how we define the cost of an isolated agent. Under the *unhappy-in-isolation* (UIS) variant, this cost is set to 1; under the *happy-in-isolation* (HIS) variant, it is set to 0. Since we are considering connected graphs, an agent can never be isolated in swap games, so the two variants create a different model only for jump games. In summary, we obtain nine different games that we denote as X-Y-Z, where X $\in$ {J,S} stands for the deviation model, either jump (J) or swap (S), Z $\in$ {ADG,MDG,CG} stands for cost model and, whenever X = J, Y $\in$ {UIS,HIS} states which cost is paid in isolation.

We measure the quality of a strategy profile σ by its *social cost* $\text{cost}(\sigma) = \sum_{i \in \mathcal{N}} \text{cost}_i(\sigma)$ and denote by σ^* a *social optimum*, i.e., a strategy profile minimizing the social cost[1]. The quality of equilibria is measured by the price of anarchy (PoA) and the price of stability (PoS). The PoA of a game $\mathcal{G}$ is obtained by comparing the equilibrium with the largest social cost with the social optimum, while the PoS refers to the equilibrium with the lowest social cost. The PoA (resp. PoS) of a class of games $\mathcal{C}$ is obtained by taking the worst-case PoA (resp. PoS) over all games in the class. Formally, $\mathsf{PoA}(\mathcal{C}) = \sup_{\mathcal{G} \in \mathcal{C}} \mathsf{PoA}(\mathcal{G})$ and $\mathsf{PoS}(\mathcal{C}) = \sup_{\mathcal{G} \in \mathcal{C}} \mathsf{PoS}(\mathcal{G})$.

1.2. Related Work

Schelling's seminal residential segregation model was formulated as a strategic game by Chauhan et al. [12]. In their model, agents of two types have a threshold-based utility function and an agent gets maximum utility if for this agent the fraction of same-type neighbors is at least τ. Later, Echzell et al. [13] extended this model to more than two types and showed that the convergence behavior of improving response dynamics strongly depends on τ. Agarwal et al. [14] focused on the case with $\tau = 1$ and proved that equilibrium existence on trees is not guaranteed and that computing socially optimal strategy profiles or equilibria with high social welfare is NP-hard. Also, the authors introduced a new welfare measure that counts the number of agents that have an other-type neighbor, called the degree of integration. For $\tau = 1$ also the influence of the underlying graph and of locality was studied [15] and welfare guarantees have been investigated [16].

[1]The social cost can be considered as a segregation measure, since a low social cost in our models means that many agent neighborhoods are very homogeneous, i.e., are strongly segregated.

A variant where the agent itself is counted in the fraction of same-type neighbors has been introduced in [17]. Moreover, recently also agents with non-monotone utility functions, in particular, with single-peaked utilities, have been considered in [18].

Closest to our work is another very recent variant, called Tolerance Schelling Games, introduced by Kanellopoulos et al. [19]. In this model agents have a discrete type and all k types are ordered according to a given total ordering $\succ$, i.e., $T_1 \succ T_2 \succ \cdots \succ T_k$. Agents have tolerance values to agents of other types depending on the number of types in between the two in the given ordering. Specifically, the model uses a tolerance vector $\mathbf{t} = (t_0, \ldots, t_{k-1})$ and the tolerance between agents of type T_i and type T_j is equal to $t_{|i-j|}$. In their work, they specifically analyze balanced tolerance Schelling games in which every type has the same number of agents and only consider the jump variant of the game. The authors show that for every tolerance vector with $t_1 < 1$ there are graphs that do not admit equilibria. Furthermore, they look at α-*binary* Tolerance Schelling Games where agents tolerate all other agents of types with at most $\alpha - 1$ other types in between in the ordering. For specific values of α and k this game admits at least one equilibrium on trees and grid graphs and they provide algorithms to find such states. Also, they prove high tight asymptotic bounds on the PoA and the PoS.

1.3. Our Contribution

We introduce very general strategic residential segregation models with the decisive new feature that non-categorical types are possible. This allows for modeling more complex and arguably also more realistic agent behavior. Moreover, the power of our models can be seen by noting that they generalize several existing variants. For example, the k-type model by Agarwal et al. [14] with $k = 2$ can be captured by both ADGs and CGs, by setting one type to value 0 and the other to value 1 (and $\lambda < 1$). Also, the k-type model by [13] for $\tau = 1$ can be modeled via a CG with suitable type-values and low enough λ. Moreover, also the α-binary Tolerance Schelling Game by Kanellopoulos et al. [19] is captured by a CG, with equally spaced type-values and a suitably chosen cutoff λ[2].

Besides generalizing several known models, our results go beyond what was known for the special cases. In particular, we demonstrate with the MDG that our model allows for drastically different games that behave very differently, compared to previously considered variants. For the S-MDG, not only do equilibria always exist, independently of the underlying graph, but we are also able to construct these states very efficiently. The same holds for the J-HIS-MDG, the ADG, and the CG on specific graph classes. Also, the HIS-assumption has not been studied before.

More precisely, we obtain the following results.

Existence of Equilibria. First of all, we observe that whenever the type function t is such that $t(0) = 1$, $t(1) = 1$ and $t(i) \in \{0, 1\}$ for each $i \in \mathcal{N}$, ADGs and CGs boil down to classical Swap or Jump Schelling games with two types considered in [14] for which non-existence of equilibria is known in general graphs. Hence, we immediately derive that S-ADGs, S-CGs, J-UIS-ADGs, and J-UIS-CGs may not have equilibria when played on general graphs. However,

[2]In the other direction, the J-UIS-ADG can be represented as a Tolerance Schelling Game by using enough types, with the tolerance of two types $x, y \in [0, 1]$ being $1 - |r - s|$. However, irrational type-values cannot be translated.

we show that a SE always exists in S-MDGs and it can be computed in $O(|E|)$ time by exploiting the properties of a breadth-first search tree of a graph. For S-ADGs, we prove existence of SEa on regular graphs, while, for S-CGs, existence is extended to even almost Δ-regular graphs[3]. Both results are obtained by resorting to a potential function technique. For the latter, in particular, an $O(\Delta n)$ algorithm can be derived. For jump games, stability is much harder to achieve and we can prove positive results mostly only under the HIS variant. For J-UIS-MDGs, in fact, existence, provable again via a potential function argument, is guaranteed as long as e is smaller than the minimum degree of G. To complement this result, we exhibit a game on a Δ-regular graph, with $e = \Delta$, without any JE. For J-HIS-MDGs, instead, JEa are always guaranteed to exist, still via a potential function argument. Interestingly, we can compute a JE in $O(|V|^{1+\min\{2,e\}})$, as long as G admits $K_{2,e}$ as a subgraph. In fact, a JE can be constructed by assigning agents 1 and n to the two nodes of the left bipartition and the remaining agents arbitrarily to nodes not belonging to the right bipartition. Two direct consequences of this result is an $O(|V|^2)$ algorithm for the case of $e = 1$ and an $O(|V|^3)$ algorithm for graphs with $\omega(|V|^{3/2})$ edges. We also design an $O(|V|^3)$ algorithm for the case of $e = 2$. Finally, for J-HIS-MDGs, J-HIS-ADGs and J-HIS-CGs played on a path, we give an $O(n \log n)$ algorithm for computing a JE.

Efficiency of Equilibria. We also provide extensive results on the PoA and the PoS. In particular, under a worst-case perspective, we can prove that there can be equilibria of positive social cost, while a social optimum with social cost zero exists, yielding an unbounded PoA, which often holds also for games played on paths. The only exceptions are S-MDGs and S-ADGs, having a PoA in $\Theta(n)$ and in $\Theta(n\Delta)$, respectively, where Δ is the maximum degree of the underlying graph. For characterizing the PoS, usually, the existence of either potential functions or algorithms computing equilibria with provable approximation guarantees are required. As we have seen, this may be either impossible or require quite an effort; nevertheless, these difficulties are common also in previous models of Schelling games. For games played on paths, we derive a bound of 1 in S-ADGs and an upper bound of 2 in S-MDGs, S-CGs and J-HIS-MDGs. On regular graphs, a bound of 1 holds for both S-ADGs and S-CGs. Finally, for games played on unrestricted topologies, we show that the PoS is in $\Theta(n)$ for both S-MDGs and J-HIS-MDGs and even unbounded for J-UIS-MDGs.

For all missing details, we refer the reader to the extended version of this paper [20].

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[3] A Δ-regular graph is a graph in which all nodes have degree Δ; an almost Δ-regular graph is a graph in which all nodes have degree in $\{\Delta, \Delta + 1\}$.

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