Generation of Skew Convex Polyominoes

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Abstract

We study the family of skew polyominoes, an intermediate class between L-convex and 4-stack polyominoes, defined by geometrical constraints satisfied by pairs of rows/columns. The problem of enumerating this family according to the semi-perimeter (size) is open, and can lead to a simplified enumeration of Z-convex polyominoes. We define a recursive method for the exhaustive generation of these objects of given size, based on generating trees. In practice, we introduce a set of operations on skew polyominoes, which perform local expansions on the objects, and such that every skew polyomino of size $n + 1$ is uniquely generated from one of size n. This leads to a simple algorithm for the exhaustive generation of skew polyominoes of size n in constant amortized time.

Keywords

Convex Polyomino, Convexity Degree, Generating Tree, Enumeration.

1. Introduction

We assume the reader is confident with the concept of *polyomino* and with the most important classes of polyominoes, such as the *stack*, the *row/column convex*, and the *convex* polyominoes. For the main definitions concerning these objects we refer to [4, 13].

In [11], the authors propose a hierarchy on convex polyominoes in terms of internal paths. A convex polyomino is said to be k-convex if every pair of its cells can be connected by a monotone path, i.e., a path made of (unit) steps in two directions only among the four North, East, West and South, that is internal to the polyomino and has at most k changes of direction. For $k = 1$, we have L-convex polyominoes (see Fig. 1 (b)), which have been studied by several points of view (see [8, 9, 11]). For $k = 2$, we have Z-convex polyominoes (Fig. 1 (d)), studied and enumerated in [14, 16]. The methods applied for the enumeration of L-convex and Z-convex polyominoes are not easily extendable to the general case, and, as a matter of fact, the enumeration of k-convex polyominoes is still an open problem for $k > 2$. On the other side, k-convex polyominoes recently became a fruitful subject of investigation, from the enumeration of several subfamilies defined by imposing geometric constraints, such as k-parallelograms and k-directed convex in [5], to the definition of efficient algorithms for their detection and exhaustive generation [6, 7, 10].

A convex polyomino is said to be *h-centered* (resp., *v-centered*) if it contains at least one row (resp., column) touching both the left and the right (resp., top and bottom) side of its minimal

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bounding rectangle (see Fig. 1 (c)).

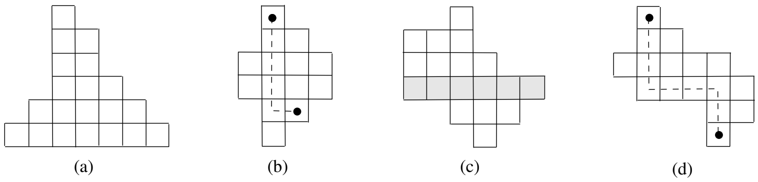

Figure 1: (a) Stack polyomino, (b) L-convex polyomino, (c) h-centered polyomino (not L-convex), (d) Z-convex polyomino (not 4-stack).

A polyomino is said to be *4-stack* if it can be decomposed into a central rectangle (*supporting rectangle*) supporting four stack polyominoes, one on each side of the rectangle, as graphically shown in Fig. 2. The four stacks are called, respectively, the *up*, *right*, *down*, and *left* stack of the polyomino. We denote by F the class of 4-stack polyominoes. In general, a 4-stack polyomino does not have a unique decomposition in terms of supporting rectangle and four stacks (see Fig. 2). Moreover, when dealing with a certain decomposition of a 4-stack polyomino, we will always assume that the supporting rectangle is not properly included in any other supporting rectangle.

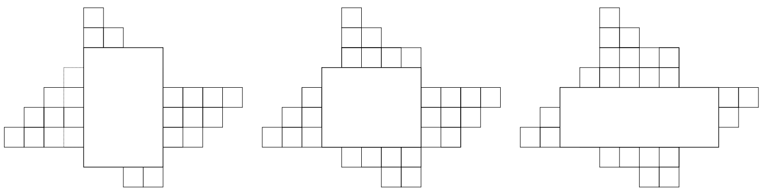

Figure 2: Three possible decompositions of a 4-stack polyomino.

Let L_n (resp., C_n, F_n, Z_n) be the class of L-convex (resp., h-centered convex, 4-stack, Z-convex) polyominoes of semi-perimeter (size) equal to $n \geq 2$. Clearly, $L_n \subseteq C_n \subseteq F_n \subseteq Z_n$ (see Figures 1 and 2). The main results concerning the nature of the generating functions and the asymptotic behaviors of these families of polyominoes (according to the size) are reported in the table below.

Class of polyominoes	**Type of g.f.**	**Asymptotic growth**
L-convex	rational	$\left(\frac{1+\sqrt{2}}{4}\right)\left(2+\sqrt{2}\right)^n$
h-centered	rational	$\frac{3}{8} \cdot 4^n$
4-stack	algebraic	$\frac{1}{64\sqrt{\pi}} \cdot \sqrt{n} \cdot 4^n$
Z-convex	algebraic	$\frac{n}{24} \cdot 4^n$
convex	algebraic	$\frac{n}{8} \cdot 4^n$

From these observations, Marc Noy, as reported in [14], pointed out that there should be a

subclass of Z-convex polyominoes, strictly including the class of h-centered polyominoes, growing as $O(n4^n)$ and having a rational generating function.
So, in this paper we introduce and study the family of *skew convex* (briefly, *skew*) polyominoes, given by 4-stacks where every pair of cells can be connected by means of a monotone internal path using only North and East (briefly N and E) unit steps, and having at most two changes of direction. Skew polyominoes are a subclass of Z-convex polyominoes which includes h-centered and v-centered convex polyominoes (Fig. 4). We determine a recursive algorithm for generating skew polyominoes of size n based on ECO method and generating trees [2, 3]. This allows us firstly to delve the problem of enumerating skew polyominoes, by translating their generating tree into a functional equation satisfied by the generating function of skew polyominoes according to the size. Moreover, applying the ideas carried out in some recent works [1, 13], we show that from the generating tree of skew polyominoes we can easily develop a CAT (Constant Amortize Time) algorithm for their exhaustive generation.

2. Skew convex polyominoes

Generating and enumerating the classes Z_n [14] and F_n [15] is a hard task. To get around this problem, we will focus our attention on a subclass of F_n which can be seen as a natural extension of L-convex polyominoes. Some further definitions are required: for a generic row r (resp., column c), we denote x_r, x'_r (resp., y_c, y'_c) the abscissas of the left and right side of r (resp., the ordinate of the lowest and highest side of c). We define the following relations on the rows (resp., columns) of convex polyominoes (see Figure 3):

i) I_r: given two rows s, t, we say that $(s, t) \in I_r$ if ($x_s \leq x_t$ and $x'_t \leq x'_s$) or ($x_s \geq x_t$ and $x'_t \geq x'_s$). The relation I_c, *column inclusion*, is defined analogously.

ii) L: given two rows s, t, we say that $(s, t) \in L$ if $x_s < x_t$ and $x'_s < x'_t$. In words, the relation L indicates that row s lies on the Left of row t.

iii) U: given two columns s, t, we say that $(s, t) \in U$ if $y_s < y_t$ and $y'_s < y'_t$. Again in words, the relation U indicates that column t lies above (Up) column t.

Definition 2.1. *An NE-skew polyomino is a 4-stack such that:*

i) for every pair of rows r_1, r_2 such that r_1 is below r_2, $(r_1, r_2) \in I_r$ or $(r_1, r_2) \in L$ holds, and

ii) for every pair of columns c_1, c_2 such that c_1 is on the left of c_2, $(c_1, c_2) \in I_c$ or $(c_1, c_2) \in U$ holds.

Using the terminology of [6], Definition 2.1 can be rephrased saying that NE-skew polyominoes are precisely convex polyominoes which are 4-stacks, and having degree of NE-convexity 2 and degree of NW-convexity 1 (Fig. 5). The family of NE-skew polyominoes includes those of h-centered and of v-centered polyominoes, and is strictly included in 4-stacks (see Fig. 4). The definition of NW-skew polyomino is dual. Observe that the intersection of NE-skew and NW-skew polyominoes gives the class of L-convex polyominoes. Moreover, NE-skew and NW-skew polyominoes are trivially bijective, therefore, from now on, we will only study NE-skew polyominoes, and we will call them simply *skew polyominoes*, denoted by $\mathcal{S}_K$.

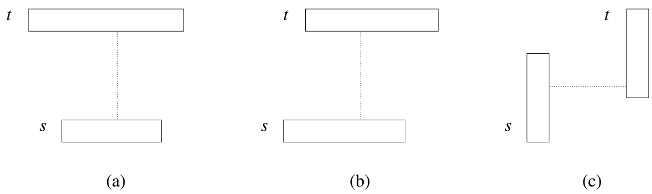

Figure 3: (a) Two rows s, t in relation I_r. (b) Two rows $(s, t) \in L$. (c) Two columns $(s, t) \in U$.

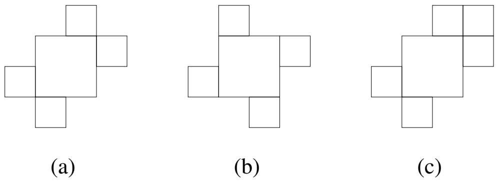

Figure 4: (a) NE-skew polyomino (not h-centered); (b) 4-stack polyomino, but not NE-skew; (c) Z-convex polyomino, but not 4-stack.

Generating trees and succession rules. Consider a combinatorial class $\mathcal{C}$, i.e., a set of discrete objects equipped with a notion of size such that the number of objects of size n is finite, for any n, and such that there is exactly one object of size 1. A *generating tree* for $\mathcal{C}$ is an infinite rooted tree whose vertices are the objects of $\mathcal{C}$, each appearing exactly once in the tree, and such that objects of size n are at level n (with the convention that the root is at level 1). The children of some object $c \in \mathcal{C}$ are obtained by adding an *atom* (*i.e.*, a piece of object that increases its size by 1) to c. Since every object appears only once in the generating tree, not all possible additions are usually acceptable. We enforce the unique appearance property by considering only those additions that follow some prescribed rules, and call *growth* of $\mathcal{C}$ the process of adding atoms according to these rules. When the growth of $\mathcal{C}$ is particularly regular, we encapsulate it in a *succession rule*. This applies more precisely when there exist statistics whose evaluations control the number of objects produced in the generating tree. A succession rule consists of one starting label (*axiom*), corresponding to the value of the statistics on the root object, and of a *set of productions* encoding the way in which these evaluations spread in the generating tree. Obviously, the sequence enumerating the class $\mathcal{C}$ can be recovered from the succession rule itself, without reference to the specifics of the objects in $\mathcal{C}$: indeed, the n-th term of the sequence is the number of nodes at level n in the generating tree. The reader interested in the concepts of generating tree and rule of succession may refer to [2, 3, 13].

3. A generating tree for skew polyominoes

We partition the class $\mathcal{S}_K$ of *skew polyominoes* in two disjoint subfamilies: those ones which are h-centered, $\mathcal{S}_C$, and those ones which are not, $\mathcal{S}_R$. In this section, we introduce operations to generate skew polyominoes of size $n+1$ in a unique way from those ones of size n.

3.1. More definitions on h-centered skew polyominoes

Given an h-centered polyomino P in $\mathcal{S}_C$, the set of rows running from the left to the right side of the minimal bounding rectangle of P form a supporting rectangle, precisely the supporting rectangle of minimal height, which is called the *basis* $\mathcal{B}(P)$, briefly $\mathcal{B}$, of P. On the other hand, the supporting rectangle having maximal height is called the *canonical rectangle*, and it is denoted by $\mathcal{R}(P)$, briefly $\mathcal{R}$, see Fig. 5 (a). Observe that $\mathcal{B}$ and $\mathcal{R}$ always exist, and that they may coincide. Let us define (see Fig. 5):

- $b(P) > 0$, briefly b, is the height of the basis.

- $r(P) \geq 0$, briefly r, is the minimum between the number of cells of the last column of P and above $\mathcal{B}$ and the number $y_R - y_B$, with y_R (resp., y_B) the ordinate of the top side of $\mathcal{R}$ (resp., $\mathcal{B}$). In practice, r takes into account each cell J on the last column of P such that gluing a cell to the right side of J makes the polyomino pass to the class $\mathcal{S}_R$.

- $h(P) \geq 0$, briefly h, is the number of cells in the last column of P above $\mathcal{R}$. In practice, gluing a cell to the right side of one of these cells makes the polyomino be no more a 4-stack.

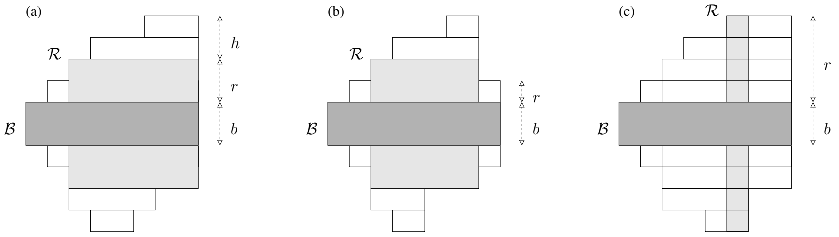

Figure 5: (a) A skew polyomino (which is not v-centered) where the right side of $\mathcal{R}$ lies on the minimal bounding rectangle of P. (b) The right side of $\mathcal{R}$ does not lie on the minimal bounding rectangle of P. (c) A v-centered polyomino.

Figure 5 (a) shows the three parameters on a skew polyomino in $\mathcal{S}_C$. Moreover, we introduce the following boolean parameters:

- $c(P)$, briefly c, is equal to 0 if and only if the first column of P contains exactly one cell (otherwise $c = 1$). Observe that $c = 0$ implies $b = 1$.

- $t(P)$, briefly t, is equal to 1 if there is a cell of P in the topmost right corner of the minimal bounding rectangle (otherwise $t = 0$). In Figure 5 (a) and (c) we have $t = 1$, in (b) we have $t = 0$.

- $s(P)$, briefly s, is equal to 1 if and only if there is no cell of P above the basis $\mathcal{B}$ (otherwise $s = 0$). If $s = 1$, then P is a down stack and $t = 1$.

Observe that if P is v-centered, then $h = 0$ (see Fig. 5 (c)), while the converse does not hold (see Fig. 5 (b)). On the other hand, if $h = 0$ and $t = 1$, then P is v-centered.

3.2. Operations on skew polyominoes

To a generic $P \in \mathcal{S}_C$, we assign a *label* which contains all the combinatorial information necessary to understand how P grows in the generating tree:

$$\begin{pmatrix} h & r & b \\ s & c & t \end{pmatrix}, \quad b > 0,\ h, r \geq 0,\ s, c, t \in \{0, 1\}.$$

The unit cell polyomino has label $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Then, to a generic $P \in \mathcal{S}_R$, we assign the *label* (r), r being the number of cells in the rightmost column.
We will first define operations applied on a polyomino $P \in \mathcal{S}_C$ of size n, producing skew polyominoes of size $n+1$.

Left Cell (LC). It can be applied to all polyominoes in $\mathcal{S}_C$, and consists in gluing a cell on the left of the basis of P, in all possible positions (see Fig. 6). It produces b polyominoes, still in $\mathcal{S}_C$, all having $b = 1$ and $c = 0$. The labels of the produced objects are:

- when we add the cell in the topmost position, the values of s and t remain how they were, so we have the label $\begin{pmatrix} h & r & 1 \\ s & 0 & t \end{pmatrix}$.
- when we add the cell in a different position, at distance $1, \dots, b-1$ from the topmost position, the value of s always becomes 0, while t remains how it was before, so we have the labels
$$\begin{pmatrix} h & r+1 & 1 \\ 0 & 0 & t \end{pmatrix}, \quad \dots \quad, \begin{pmatrix} h & r+b-1 & 1 \\ 0 & 0 & t \end{pmatrix}.$$

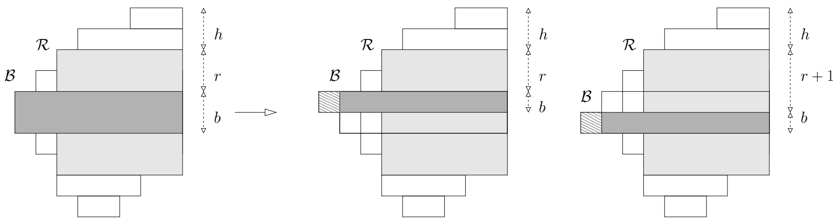

Figure 6: The application of the operation *Left Cell* to a skew polyomino with $b = 2$.

Right Cell (RC). It can be applied to all polyominoes in $\mathcal{S}_C$, provided $c = 1$ to avoid ambiguity. It consists in gluing a cell on the right of the basis of P, in all possible positions, and produces b polyominoes, still in $\mathcal{S}_C$, all having $b = c = 1$ and $h = r = 0$ (see Fig. 7). The labels of the produced objects are:

- when we add the cell in the topmost position of the basis, the obtained polyomino P' has $s(P') = s(P)$, while $t(P') = 1$ if and only if $s(P) = 1$, so we set $t(P') = s(P)$, getting the label $\begin{pmatrix} 0 & 0 & 1 \\ s & 1 & s \end{pmatrix}$.

- when we add the cell in a different position, at distance $1, \ldots, b-1$ from the topmost position of the basis, s and t become 0, so we have $b-1$ labels as $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

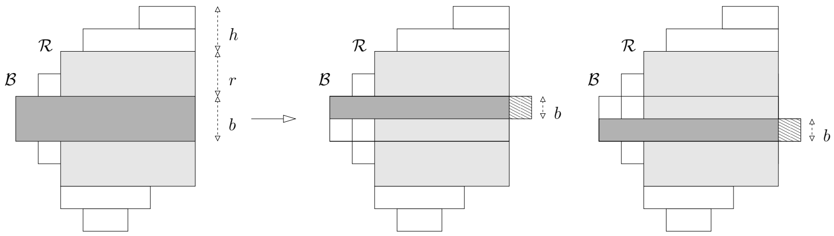

Figure 7: The application of the operation *Right Cell* to a skew polyomino with $b = 2$.

Row. It consists in adding a new row to the basis of P (see Fig. 8). It produces one polyomino, still in $\mathcal{S}_C$, with label $\begin{pmatrix} h & r & b+1 \\ s & 1 & t \end{pmatrix}$.

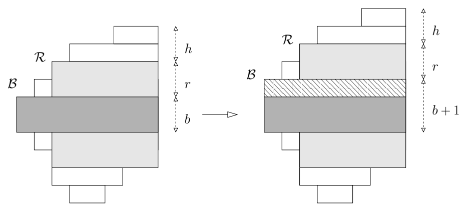

Figure 8: The application of the operation *Row* to a skew polyomino with $b = 2$.

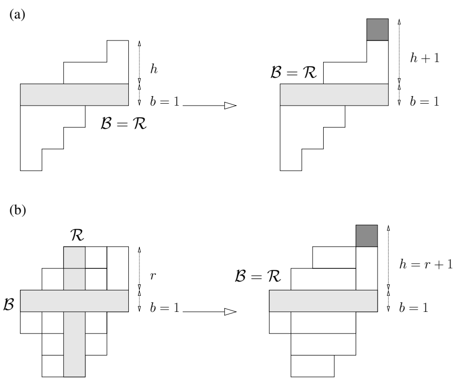

Figure 9: The *Top* operation: (a) on a non v-centered; (b) on a v-centered polyomino.

Top. It consists in adding a cell onto the top of the rightmost column of P (see Fig. 9). This operation can be applied only to polyominoes such that: *i)* $t = 1$, so that the addition of the new cell does not violate the convexity of the polyomino; *ii)* $c = b = 1$, to ensure that the polyomino has not been yet obtained by means of the operation *LC* (i.e., $c = 0$) or *Row* (i.e., $b > 1$). Summing up, provided $t = c = b = 1$, there are two cases where the operation *Top* can be applied:

(T1) P is not v-centered (see Fig. 9 (a)). In this case, $\mathcal{R}$ coincides with $\mathcal{B}$, so $h > 0$ and $r = 0$. The application of *Top* to a polyomino of this kind just increases by one the value of h, while the values of all the other parameters remain the same, giving the production

$$\begin{pmatrix} h & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} h+1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(T2) P is v-centered (see Fig. 9 (b)). In this case, $r \geq 0$ and $h = 0$. Observe that, since $b = c = 1$, the vertical basis does not include the last column of P, then applying *Top* we obtain a polyomino P' which fits in case (T1), $r(P') = 0$ and $h(P') = r(P) + 1$. So, the production for this case can be described by

$$\begin{pmatrix} 0 & r & 1 \\ s & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} r+1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The productions for both cases (T1) and (T2) can be written as

$$\begin{pmatrix} h & r & 1 \\ s & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} h+r+1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

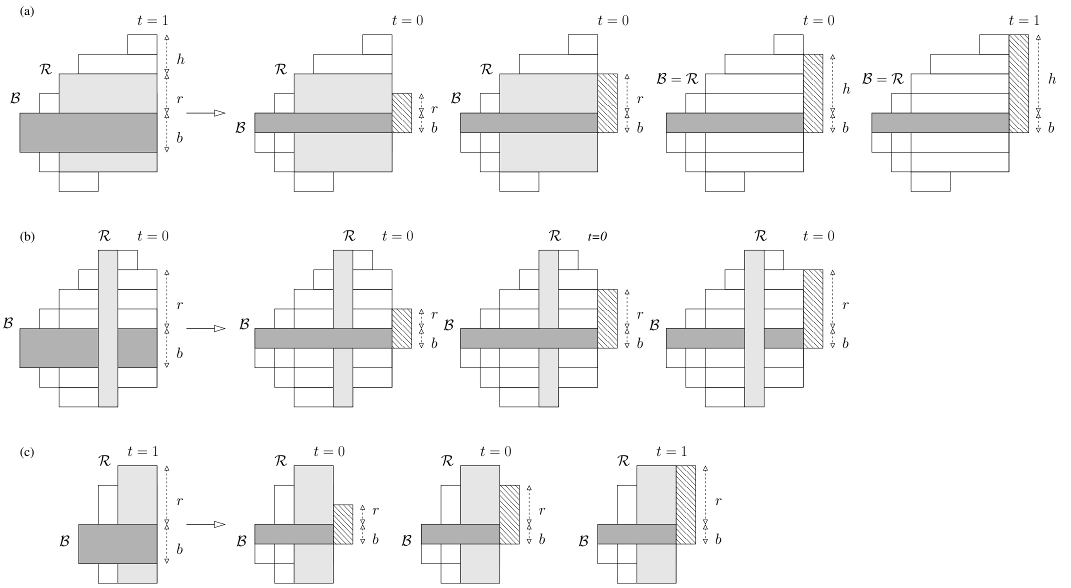

Figure 10: The *Parall* operation on: (a) a non v-centered polyomino, with $h, r > 0, t = 1$; (b) a v-centered polyomino, with $h = 0, t = 0$; (c) a v-centered polyomino, with $t = 1$.

Parall. This operation can be applied to all polyominoes in $\mathcal{S}_C$, provided $h \geq 1$ or $r \geq 1$, and $c = 1$ (for ambiguity reasons). It consists in gluing a column of length q, with $2 \leq q \leq h+r+1$, on the right of the basis of P, with the lowest cell being glued to the topmost cell of the right side of the basis. It produces $r+h$ polyominoes still in $\mathcal{S}_C$, with $b=1$. Two cases may occur:

(P1) $h>0$**.** The polyomino P is not v-centered and has label $\begin{pmatrix} h & r & b \\ 0 & 1 & t \end{pmatrix}$ (see Fig. 10 (a)). Let us consider separately the cases:

- if $2 \leq q \leq r+1$ (*r-Par*), the addition of the new column does not change the canonical rectangle $\mathcal{R}$. The labels are

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \dots \quad \begin{pmatrix} 0 & r & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

- if $r+2 \leq q \leq r+h+1$ (*h-Par*), then, for each of the obtained polyominoes, the canonical rectangle coincides with its basis and has height equal to 1. So, $r=0$, and the labels are

$$\begin{pmatrix} r+1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \dots \quad \begin{pmatrix} r+h-1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r+h & 0 & 1 \\ 0 & 1 & t \end{pmatrix}.$$

  We observe that, if $t(P)=1$, then adding a column of length $r+h+1$ keeps $t=1$.

(P2) $h=0$**.** Here, P is v-centered, and necessarily $r>0$, so its label is

$$\begin{pmatrix} 0 & r & b \\ 0 & 1 & t \end{pmatrix}.$$

In this case, only columns of length $2 \leq q \leq r+1$ can be added, and the addition of the new column does not change the canonical rectangle $\mathcal{R}$ (Fig. 10 (b), (c)). It gives the same labels of the production *r-Par* of the previous case, except for the last production, corresponding to adding a column of length $r+1$. In this case, the obtained polyomino P' has $t(P')=t(P)$, so we have labels:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \dots \quad \begin{pmatrix} 0 & r-1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & r & 1 \\ 0 & 1 & t \end{pmatrix}.$$

Right Stack (RS). It consists in gluing a column of length f, with $1 \leq f \leq r$, to the right side of P, in the area belonging to $\mathcal{R}$ and above $\mathcal{B}$ (taken into account by the parameter r), in all possible ways. This operation can be applied to all polyominoes in $\mathcal{S}_C$, provided $r \geq 1$. It produces $\binom{r+1}{2}$ polyominoes in $\mathcal{S}_R$, with labels $(1)^r\,(2)^{r-1}\,\dots\,(r-1)^2\,(r)$, where the power notation stands for repetitions, i.e., with $1 \leq k \leq r$ the label (k) is produced $r-k+1$ times.

A polyomino $P \in \mathcal{S}_R$ has label (r) given by the length of the rightmost column of P. To a polyomino of this kind we can only apply the operation

Right Stack*, analogous to *Right Stack* defined for $\mathcal{S}_C$ polyominoes. It consists in gluing a column of length f, $1 \leq f \leq r$, to the right side of the last column of P, in all possible ways. This operation produces $\binom{r+1}{2}$ polyominoes in $\mathcal{S}_R$, with labels $(1)^r\,(2)^{r-1}\,\dots\,(r-1)^2\,(r)$.

Theorem 3.1. *Every skew polyomino of size $n+1$, with $n \geq 2$, is uniquely generated from a skew polyomino of size n through the application of one among the operations* LC, RC, Row, Top, Parall, Right Stack, Right Stack*.

Proof. To prove the statement, we just need to show that, given a skew polyomino P of size $n+1$, $n \geq 2$, there is a unique skew polyomino $\phi(P)$ of size n such that P is generated from $\phi(P)$ applying exactly one of the operations defined above. Let us consider all the possible, mutually disjoint, cases:
1) $P \in \mathcal{S}_R$ (not h-centered). Then, P is obtained by adding the rightmost column to the polyomino $\phi(P)$: if $\phi(P) \in \mathcal{S}_C$, by performing *Right stack*, while if $\phi(P) \in \mathcal{S}_R$, by performing *Right stack** (Fig. 11 (a)).

2) $P \in \mathcal{S}_C$ (h-centered). Then, we have the following disjoint possibilities:

$b > 1$: Then, the last applied operation is *Row* (Fig. 11 (b)).

$b = 1$ and $c = 0$: An exhaustive check reveals that there is only one operation that produces polyominoes with $c = 0$ and $b = 1$, and it is precisely *LC*. Then, P is produced by applying *LC* to the polyomino $\phi(P)$, obtained from P by removing the first column (see Fig. 11 (c)).

$b = 1$ and $c = 1$: We can consider two possibilities: the last column of P has exactly one or more than one cell.
In the first case, we observe that the last applied operation needs to be one among *LC*, *RC*, *Right stack* and *Right stack**, whereas the last two are not possible, since P is not in $\mathcal{S}_R$. Moreover, since $c = 1$, also *LC* cannot be the last applied operation. Then, the last applied operation is necessarily *RC*, and it is applied to the polyomino $\phi(P)$ obtained from P by removing the cell in the last column (see Fig. 11 (d)).
In the second case, the last column of P has more than one cell. The only possibility is that the first column q' of P contains no cell above the basis, and at least one cell below it; the last column q'' of P contains at least one cell above the basis, and no cell below it. So, the columns q', q'' are in relation U, precisely $(q', q'') \in U$, and $\mathcal{R} = \mathcal{B}$. Again two cases arise: on one hand, the top side of q'' has ordinate greater than all other cells in P. Then, the last applied operation is *Top* (Fig. 11 (e)). On the other, the top side of q'' has ordinate smaller than or equal to the top side of the column on its left. Removing q'' from P, we have a (h-centered) polyomino $\phi(P)$ of size n. Moreover, since $b(P) = 1$, P can only be obtained by gluing q'' on the right of the topmost right cell of the basis of $\phi(P)$ (Fig. 11 (f)). So, the last applied operation is *Parall.*
We conclude by noticing that no other possible cases arise. $\square$
The growth of the generating tree can be formally expressed by the following two succession

rules. The first, Ω_C, describes the growth of polyominoes in $\mathcal{S}_C$,

$$\Omega_C : \begin{cases} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} & & \\ \begin{pmatrix} h & r & b \\ s & c & t \end{pmatrix} & \rightarrow \begin{pmatrix} h & r & 1 \\ s & 0 & t \end{pmatrix} \begin{pmatrix} h & r+1 & 1 \\ 0 & 0 & t \end{pmatrix} \cdots \begin{pmatrix} h & r+b-1 & 1 \\ 0 & 0 & t \end{pmatrix} & \mathit{LC} \\ & \rightarrow \begin{pmatrix} h & r & b+1 \\ s & 1 & t \end{pmatrix} & \mathit{Row} \\ (\text{if } b=c=t=1) & \rightarrow \begin{pmatrix} h+r+1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} & \mathit{Top} \\ (\text{if } c=1) & \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{(b-1)} \begin{pmatrix} 0 & 0 & 1 \\ s & 1 & s \end{pmatrix} & \mathit{RC} \\ (\text{if } r>0, h=0) & \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & r-1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & r & 1 \\ 0 & 1 & t \end{pmatrix} & \mathit{r\text{-}Par} \\ (\text{if } r>0, h>0) & \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & r-1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & r & 1 \\ 0 & 1 & 0 \end{pmatrix} & \mathit{r\text{-}Par} \\ & \rightarrow \begin{pmatrix} r+1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} r+h-1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r+h & 0 & 1 \\ 0 & 1 & t \end{pmatrix} & \mathit{h\text{-}Par} \\ (\text{if } r>1) & \rightarrow (1)^r \, (2)^{r-1} \, \ldots \, (r-1)^2 \, (r) & \mathit{RS} \end{cases}$$

The second rule, Ω_R, describes the production of the labels (r) of elements in $\mathcal{S}_R$,

$$\Omega_R : \quad (r) \rightarrow \quad (1)^r \, (2)^{r-1} \, \ldots \, (r-1)^2 \, (r) \quad \mathit{RS}^*.$$

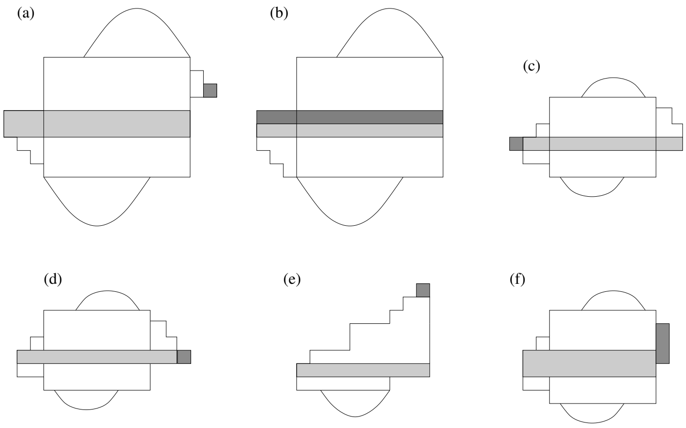

Figure 11: The last operation applied on a skew polyomino of size $n+1$, $n \geq 2$.

The first few levels of the generating tree, with the corresponding labels, are depicted in Figure 12. From the generating tree we can generate the first terms of the sequence s_n of skew polyominoes of size $n \geq 2$:

$$1, 2, 7, 26, 101, 403, 1636, 6720, 27838, 116068, 486446, 2047558 \ldots$$

We point out that this sequence is not yet included in the Online Encyclopedia of Integer Sequences [17]. Using standard techniques ([2, 4, 13]), the generating tree can be translated into a system of functional equations satisfied by the generating function of skew polyominoes

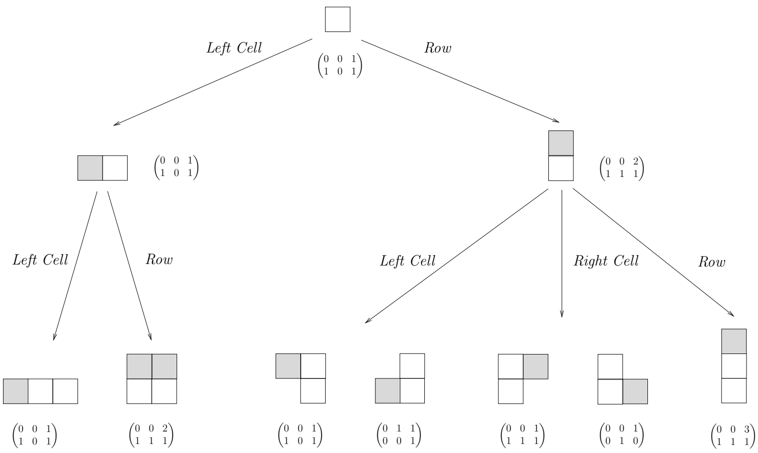

Figure 12: The first three levels of the generating tree.

according to the size. In some further research we plan to solve this system, by applying a generalization of the classical *kernel method*, known under the name of *obstinate kernel method* ([4]). Moreover, applying the techniques in [1, 13], the recursive construction given by the generating tree for skew polyominoes can be used to write down a Constant Amortized Time (CAT) algorithm for generating all skew polyominoes with given size.

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