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# A Class of Symmetric Graphs With 2-ARC Transitive Quotients

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**Abstract:** Let $\Gamma$ be an $X$-symmetric graph admitting an $X$-invariant partition $\mathcal{B}$ on $V(\Gamma)$ such that $\Gamma_{\mathcal{B}}$ is connected and $(X,2)$-arc transitive. A characterization of $(\Gamma, X, \mathcal{B})$ was given in [S. Zhou Eur J Comb 23 (2002), 741–760] for the case where $|B|>|\Gamma(C)\cap B|=2$ for an arc $(B,C)$ of $\Gamma_{\mathcal{B}}$. We consider in this article the case where $|B|>|\Gamma(C)\cap B|=3$, and prove that $\Gamma$ can be constructed from a 2-arc transitive graph of valency 4 or 7 unless its connected components are isomorphic to $3\mathbf{K}_2$, $\mathbf{C}_6$ or $\mathbf{K}_{3,3}$. As a byproduct, we prove that each connected tetravalent $(X,2)$-transitive graph is either the complete graph $\mathbf{K}_5$ or a near $n$-gonal graph for some $n\geq 4$. © 2009 Wiley Periodicals, Inc. J Graph Theory 00: 1–14, 2009

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## 1. INTRODUCTION

In this article, all graphs are assumed to be finite, nonempty, simple and undirected. The reader is referred to [2, 3, 1], respectively, for notation and terminology on graphs, permutation groups and combinatorial designs.

Let $\Gamma$ be a regular graph with vertex set $V(\Gamma)$, edge set $E(\Gamma)$ and valency $val(\Gamma)$. For an integer $s \geq 1$, an *s-arc* is an ordered $(s+1)$-tuple $(\alpha_0, \alpha_1, \ldots, \alpha_s)$ of vertices in $\Gamma$ such that $\{\alpha_i, \alpha_{i+1}\} \in E(\Gamma)$ for $0 \leq i \leq s-1$, and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leq i \leq s-1$. By $Arc_s(\Gamma)$ we denote the set of $s$-arcs in $\Gamma$. A 1-arc is called an *arc*, and $Arc_1(\Gamma)$ is denoted by $Arc(\Gamma)$.

Let $X$ be a group acting on $V(\Gamma)$. The induced action of $X$ on $V(\Gamma) \times V(\Gamma)$ is given by $(\alpha, \beta)^x = (\alpha^x, \beta^x)$ for $\alpha$, $\beta \in V(\Gamma)$ and $x \in X$. We say that $X$ preserves the adjacency of $\Gamma$ if $Arc(\Gamma)^x = Arc(\Gamma)$ for all $x \in X$. Note that $X$ induces naturally an action on $Arc_s(\Gamma)$ if $X$ preserves the adjacency of $\Gamma$. The graph $\Gamma$ is said to be $(X,s)$-*arc transitive* if $\Gamma$ has at least one $s$-arc, $X$ preserves the adjacency of $\Gamma$ and $X$ acts transitively on both $V(\Gamma)$ and $Arc_s(\Gamma)$; and $\Gamma$ is said to be $(X,s)$-*arc regular* if in addition $X$ acts regularly on $Arc_s(\Gamma)$. Further, $\Gamma$ is said to be $(X,s)$-*transitive* if $\Gamma$ is $(X,s)$-arc transitive but not $(X,s+1)$-arc transitive. An $(X,1)$-arc transitive graph is usually called an $X$-*symmetric* graph.

Let $\Gamma$ be an $X$-symmetric graph admitting a nontrivial $X$-invariant partition $\mathcal{B}$ on $V(\Gamma)$, that is, $1 < |B| < V(\Gamma)$ and $B^x := \{\alpha^x \mid \alpha \in B\} \in \mathcal{B}$ for $B \in \mathcal{B}$ and $x \in X$. Such a graph is said to be an *imprimitive* $X$-symmetric graph. The *quotient graph* $\Gamma_{\mathcal{B}}$ of $\Gamma$ with respect to $\mathcal{B}$ is defined to be the graph with vertex set $\mathcal{B}$ such that $B \in \mathcal{B}$ and $C \in \mathcal{B}$ are adjacent in $\Gamma_{\mathcal{B}}$ if and only if there exist $\alpha \in B$ and $\beta \in C$ adjacent in $\Gamma$. It is easy to see that $\Gamma_{\mathcal{B}}$ is $X$-symmetric. We always assume that $\Gamma_{\mathcal{B}}$ has at least one edge, which implies that each $B \in \mathcal{B}$ is an independent set of $\Gamma$.

For $\alpha \in V(\Gamma)$ and $B \in \mathcal{B}$, set $\Gamma(\alpha) = \{\gamma \mid \{\alpha, \gamma\} \in E(\Gamma)\}$, $\Gamma(B) = \bigcup_{\beta \in B} \Gamma(\beta)$, $\Gamma_{\mathcal{B}}(B) = \{C \in \mathcal{B} \mid \{B, C\} \in E(\Gamma_{\mathcal{B}})\}$ and $\Gamma_{\mathcal{B}}(\alpha) = \{C \in \mathcal{B} \mid \alpha \in \Gamma(C)\}$. Since $\Gamma$ is $X$-symmetric, for $\alpha \in B \in \mathcal{B}$ and $C \in \Gamma_{\mathcal{B}}(B)$, it is easily shown that the parameters $v := |B|$, $k := |\Gamma(C) \cap B|$ and $r := |\Gamma_{\mathcal{B}}(\alpha)|$ are independent of the choices of $B$, $C$ and $\alpha$. The graph $\Gamma$ is said to be a *multicover* of $\Gamma_{\mathcal{B}}$ if $k = v$. Noting that $vr = val(\Gamma_{\mathcal{B}})k$ (see [10], for example), $\Gamma$ is a multicover of $\Gamma_{\mathcal{B}}$ if and only if $r = val(\Gamma_{\mathcal{B}})$. Let $\mathcal{D}(B)$ denote the incidence structure $(B, \Gamma_{\mathcal{B}}(B))$ such that $\beta \in B$ is incident with some $C \in \Gamma_{\mathcal{B}}(B)$ if and only if $C \in \Gamma_{\mathcal{B}}(\beta)$. Then $\mathcal{D}(B)$ is a flag-transitive 1-$(v,k,r)$ design with $val(\Gamma_{\mathcal{B}})$ blocks [12, Lemma 2.1], which is independent of the choice of $B$ up to isomorphism. For $(B,C) \in Arc(\Gamma_{\mathcal{B}})$, denote by $\Gamma[B,C]$ the bipartite subgraph of $\Gamma$ induced by $(\Gamma(C) \cap B) \cup (\Gamma(B) \cap C)$. Then $\Gamma[B,C]$ is independent of the choice of $(B,C) \in Arc(\Gamma_{\mathcal{B}})$ up to isomorphism.

It has been observed in the literature that the quotient graphs of $(X,2)$-arc transitive graphs are usually not $(X,2)$-arc transitive, and that an $X$-symmetric graph with an $(X,2)$-arc transitive quotient itself is not necessarily $(X,2)$-arc transitive. (For example, several examples are given in [4, 5] for the first situation; and for the second situation, it is shown in [12] that every connected $(X,3)$-arc transitive graph is a quotient graph of at least one $X$-symmetric graph which is not $(X,2)$-arc transitive.) This observation gave rise to a series of intensive studies of the following questions [18, 9].

(Q1) When can $\Gamma_{\mathcal{B}}$ be $(X,2)$-arc transitive?

(Q2) What information of the structure of $\Gamma$ can we obtain from an $(X,2)$-arc transitive quotient $\Gamma_{\mathcal{B}}$ of $\Gamma$?

The triple $(\Gamma_{\mathcal{B}}, \Gamma[B,C], \mathcal{D}(B))$ mirrors "global" and "local" information of the structure of $\Gamma$, which allows us to reconstruct $\Gamma$ in some cases. This approach to imprimitive symmetric graphs has received considerable attention in the literature. Gardiner and Praeger [6] first suggested such an approach, and they discussed the case when the stabilizer $X_\alpha$ of a vertex $\alpha \in V(\Gamma)$ in $X$ acts primitively on $\Gamma(\alpha)$; and in [7, 8], they considered the case when $\Gamma_{\mathcal{B}}$ is a complete graph and $X_B$ (the subgroup of $X$ fixing $B$ set-wise) is 2-transitive on $B$. For the case where $k=v-1\geq 2$, Li et al. [10] found an elegant construction (called the *3-arc graph* construction) for constructing certain graphs. Iranmanesh et al. [9], and Lu and Zhou [12] studied the case where $\Gamma_{\mathcal{B}}$ is $(X,2)$-arc transitive and obtained a series of interesting results. In particular, Lu and Zhou [12] found the second type 3-arc graph construction, which led to a classification [19] of a family of symmetric graphs. The reader is referred to [14–18, 11] for further developments in this topic.

In answering the above two questions, a relatively explicit classification of $(\Gamma, X, \mathcal{B})$ has been given in [18], when $\Gamma_{\mathcal{B}}$ is connected and $(X,2)$-arc transitive such that $2=k\leq v-1$. This motivated us to investigate the case where $k=3$. The following is a summary of the main result of this article, and more details will be given in Theorem 4.1.

**Theorem 1.1.** *Let $\Gamma$ be an $X$-symmetric graph which admits an $X$-invariant partition $\mathcal{B}$ on $V(\Gamma)$ such that $val(\Gamma_{\mathcal{B}})\geq 2$, $\Gamma_{\mathcal{B}}$ is connected and $(X,2)$-arc transitive. If $|B|>|B\cap\Gamma(C)|=3$ for $(B,C)\in Arc(\Gamma_{\mathcal{B}})$, then one of the following four cases occurs*: (a) $|B|=4$ *and* $val(\Gamma_{\mathcal{B}})=4$; (b) $|B|=6$ *and* $val(\Gamma_{\mathcal{B}})=4$; (c) $|B|=7$ *and* $val(\Gamma_{\mathcal{B}})=7$; (d) $|B|=3val(\Gamma_{\mathcal{B}})$.

*Notation*: For a group $X$ acting on a set $V$ and $B\subseteq V$, denote by $X^V$ the induced permutation group on $V$, by $X_B$ the set-wise stabilizer of $B$ in $X$, and by $X_{(B)}$ the point-wise stabilizer of $B$ in $X$; for a positive integer $m$ and a graph $\Gamma$, denote by $m\Gamma$ the vertex-disjoint union of $m$ copies of $\Gamma$.

## 2. GRAPHS CONSTRUCTED FROM GIVEN GRAPHS



In this section, we restate several graphs constructed from a given graph, as well as some of their properties, which turn out to be useful in a further characterization of $(\Gamma, X, \mathcal{B})$ stated in Theorem 1.1.

Assume that $\Sigma$ is an $(X,2)$-arc transitive graph with $val(\Sigma)\geq 3$. Let $\Delta$ be a self-paired $X$-orbit on $Arc_3(\Sigma)$, where self-parity means that $(\sigma_3,\sigma_2,\sigma_1,\sigma_0)\in\Delta$ whenever $(\sigma_0,\sigma_1,\sigma_2,\sigma_3)\in\Delta$. Define two kinds of 3-arc graphs [10, 12] as follows:

$\mathcal{I}(\Sigma,\Delta)$, the graph with vertex set $Arc(\Sigma)$ such that two arcs $(\tau,\tau_1)$ and $(\sigma,\sigma_1)$ of $\Sigma$ are adjacent if and only if $(\tau_1,\tau,\sigma,\sigma_1)\in\Delta$; $\mathcal{J}(\Sigma,\Delta)$, the graph with vertices the 2-paths (paths of length 2) in $\Sigma$ such that two distinct paths $\sigma_1\sigma\sigma_2$ and $\tau_1\tau\tau_2$ are adjacent if and only if one of $\sigma=\tau_i$, $\tau=\sigma_j$ and $(\sigma_i,\sigma,\tau,\tau_j)\in\Delta$ for some $i,j\in\{1,2\}$.

Let $H(\Sigma)$ be the set of pairs $(\tau_1\tau\tau_2, \sigma_1\sigma\sigma_2)$ of 2-paths with $\sigma\in\Sigma(\tau)\setminus\{\tau_1,\tau_2\}$, $\tau\in\Sigma(\sigma)\setminus\{\sigma_1,\sigma_2\}$. Let $\Lambda$ be a self-paired $X$-orbit on $H(\Sigma)$, where self-parity means that $(\tau_1\tau\tau_2,\sigma_1\sigma\sigma_2)\in\Lambda$ whenever $(\sigma_1\sigma\sigma_2,\tau_1\tau\tau_2)\in\Lambda$. The *2-path graph* $\mathcal{H}(\Sigma,\Lambda)$ with respect to $\Lambda$ is the graph with vertices the 2-paths in $\Sigma$ such that two 2-paths are adjacent if and only if they give a pair in $\Lambda$.

**Proposition 2.1** (Li et al. [10], Lu and Zhou [12]). *$\mathcal{I}(\Sigma,\Delta)$, $\mathcal{J}(\Sigma,\Delta)$ and $\mathcal{H}(\Sigma,\Lambda)$ are X-symmetric.*

Let $A_\tau=\{(\tau,\sigma)\mid\sigma\in\Sigma(\tau)\}$ for $\tau\in V(\Sigma)$. Set $\mathcal{A}=\{A_\tau\mid\tau\in V(\Sigma)\}$. By [10, Theorem 10], it is easily shown that the following result holds.

**Proposition 2.2.** *Let $\Gamma=\mathcal{I}(\Sigma,\Delta)$. Then $\Sigma\cong\Gamma_{\mathcal{A}}$, $val(\Gamma)=(val(\Sigma)-1)val(\Gamma[A_\tau,A_\sigma])$ for $(\tau,\sigma)\in Arc(\Sigma)$, and each vertex of $\Gamma$ is adjacent to exactly $val(\Sigma)-1$ blocks in $\mathcal{A}$.*

Let $P_\sigma$ denote the set of 2-paths with a given mid vertex $\sigma\in V(\Sigma)$. Set $\mathcal{P}=\{P_\sigma\mid\sigma\in V(\Sigma)\}$. Then, by [12], both $\mathcal{J}(\Sigma,\Delta)$ and $\mathcal{H}(\Sigma,\Lambda)$ admit an $X$-invariant partition $\mathcal{P}$ with quotient graphs isomorphic to $\Sigma$. The following lemma improves [12, Theorem 4.10].

**Lemma 2.3.** *Let $\Gamma$ be an X-symmetric graph admitting an X-invariant partition $\mathcal{B}$ with $val(\Gamma_{\mathcal{B}})\geq 3$ and $|\Gamma_{\mathcal{B}}(\alpha)|=2$ for $\alpha\in V(\Gamma)$. Set*

$$\Delta=\left\{(C,B(\alpha),B(\beta),D)\,\middle|\,\begin{array}{l}(\alpha,\beta)\in Arc(\Gamma)\\ C\in\Gamma_{\mathcal{B}}(\alpha),D\in\Gamma_{\mathcal{B}}(\beta),C\neq B(\beta),D\neq B(\alpha)\end{array}\right\},$$

*where $B(\alpha)$ denotes the block in $\mathcal{B}$ containing $\alpha$. Suppose that $|\Gamma(D)\cap B_0\cap\Gamma(C)|\neq 0$ for any 2-path $DB_0C$ of $\Gamma_{\mathcal{B}}$ with a given mid vertex $B_0\in\mathcal{B}$. Then $\Gamma_{\mathcal{B}}$ is $(X,2)$-arc transitive, $\lambda:=|\Gamma(D)\cap B_0\cap\Gamma(C)|$ is independent of the choice of $DB_0C$, $\Delta$ is a self-paired X-orbit on $Arc_3(\Gamma_{\mathcal{B}})$, and either*

(a) *$\lambda=1$ and $\Gamma\cong\mathcal{J}(\Gamma_{\mathcal{B}},\Delta)$; or*
(b) *$\lambda\geq 2$ and $\Gamma$ admits a second nontrivial X-invariant partition*

$$\mathcal{Q}:=\{\Gamma(D)\cap B\cap\Gamma(C)\,|\,DBC \text{ is a 2-path of } \Gamma_{\mathcal{B}}\}$$

*on $V(\Gamma)$, which is a proper refinement of $\mathcal{B}$ such that $\Gamma_{\mathcal{Q}}\cong\mathcal{J}(\Gamma_{\mathcal{B}},\Delta)$.*

***Proof.*** Note that $val(\Gamma_{\mathcal{B}})\geq 3$. Take three distinct blocks $C,D,D'\in\Gamma_{\mathcal{B}}(B_0)$. Since $|\Gamma(D)\cap B_0\cap\Gamma(C)|\neq 0$ and $|\Gamma(D')\cap B_0\cap\Gamma(C)|\neq 0$, there exist $\alpha,\beta\in\Gamma(C)\cap B_0$ with $\alpha\in\Gamma(D)$ and $\beta\in\Gamma(D')$. Let $\alpha',\beta'\in C$ be such that $(\alpha,\alpha'),(\beta,\beta')\in Arc(\Gamma)$. Then $(\alpha,\alpha')^x=(\beta,\beta')$ for some $x\in X$ as $\Gamma$ is $X$-symmetric. So, $\alpha^x=\beta$ and $\alpha'^x=\beta'$. Then $B_0^x=B_0$ and $C^x=C$, hence $x\in X_{B_0}\cap X_C$. Further $C,D^x,D'\in\Gamma_{\mathcal{B}}(\beta)$, it follows that $D^x=D'$ as $|\Gamma_{\mathcal{B}}(\beta)|=2$. Thus $X_{B_0}\cap X_C$ is transitive on $\Gamma_{\mathcal{B}}(B_0)\setminus\{C\}$, it follows that $X_{B_0}$ is 2-transitive on $\Gamma_{\mathcal{B}}(B_0)$. Therefore, $\Gamma_{\mathcal{B}}$ is $(X,2)$-arc transitive. Then, by [12], $\lambda\geq 1$ is a constant number; and if $\lambda=1$, $\Delta$ is a self-paired $X$-orbit on $Arc_3(\Gamma_{\mathcal{B}})$ and $\Gamma\cong\mathcal{J}(\Gamma_{\mathcal{B}},\Delta)$. In the following we assume $\lambda\geq 2$.

We first show that $\mathcal{Q}$ is an $X$-invariant partition of $V(\Gamma)$. Take two arbitrary 2-paths $D_1B_1C_1$ and $D_2B_2C_2$ of $\Gamma_{\mathcal{B}}$. Suppose that there exists some $\alpha\in V(\Gamma)$ such that $\alpha\in(\Gamma(D_1)\cap B_1\cap\Gamma(C_1))\cap(\Gamma(D_2)\cap B_2\cap\Gamma(C_2))$. Then $B_1=B_2$ and $C_i,D_i\in\Gamma_{\mathcal{B}}(\alpha)$ for $i=1,2$. Since $|\Gamma_{\mathcal{B}}(\alpha)|=2$, we have that $\{C_1,D_1\}=\{C_2,D_2\}$, thus $D_1B_1C_1=D_2B_2C_2$. It follows that $\mathcal{Q}$ is a partition of $V(\Gamma)$. For any 2-path $DBC$ and $x\in X$, we have $(\Gamma(D)\cap B\cap\Gamma(C))^x=\Gamma(D^x)\cap B^x\cap\Gamma(C^x)\in\mathcal{Q}$. Thus $\mathcal{Q}$ is $X$-invariant. Since $\Gamma$ is not a multicover of $\Gamma_{\mathcal{B}}$, we know $|B|>|\Gamma(D)\cap B\cap\Gamma(C)|=\lambda\geq 2$, so $\mathcal{Q}$ is a proper refinement of $\mathcal{B}$. Then $(\mathcal{B},\mathcal{Q})$ gives an $X$-invariant partition $\bar{\mathcal{B}}:=\{\bar{B}\,|\,B\in\mathcal{B}\}$ of $V(\Gamma_{\mathcal{Q}})$, where $\bar{B}=\{\Gamma(D)\cap B\cap\Gamma(C)\,|\,C,D\in\Gamma_{\mathcal{B}}(B),C\neq B\}$.

We denote a vertex $\Gamma(D)\cap B\cap\Gamma(C)$ of $\Gamma_{\mathcal{Q}}$ by $\bar{\alpha}$ if $\alpha\in\Gamma(D)\cap B\cap\Gamma(C)$. Consider the quotient graph $(\Gamma_{\mathcal{Q}})_{\bar{\mathcal{B}}}$ of $\Gamma_{\mathcal{Q}}$ with respect to $\bar{\mathcal{B}}$. For any 2-path $\bar{D}\bar{B}\bar{C}$ of $(\Gamma_{\mathcal{Q}})_{\bar{\mathcal{B}}}$ and any $\bar{\alpha}\in V(\Gamma_{\mathcal{Q}})$, we have $|(\Gamma_{\mathcal{Q}})_{\bar{\mathcal{B}}}(\bar{\alpha})|=2$ and $|\Gamma_{\mathcal{Q}}(\bar{D})\cap\bar{B}\cap\Gamma_{\mathcal{Q}}(\bar{C})|=1$. It follows from (a) that $\Gamma_{\mathcal{Q}}\cong\mathcal{J}((\Gamma_{\mathcal{Q}})_{\bar{\mathcal{B}}},\bar{\Delta})$, where $\bar{\Delta}=\{(\bar{C},\bar{B}(\bar{\alpha}),\bar{B}(\bar{\beta}),\bar{D})\,|\,(C,B(\alpha),B(\beta),D)\in\Delta\}$. Moreover, it is easily shown that $\bar{\mathcal{B}}\to\mathcal{B},\bar{B}\mapsto B$ is an isomorphism from $(\Gamma_{\mathcal{Q}})_{\bar{\mathcal{B}}}$ to $\Gamma_{\mathcal{B}}$. Therefore, $\Gamma_{\mathcal{Q}}\cong\mathcal{J}((\Gamma_{\mathcal{Q}})_{\bar{\mathcal{B}}},\bar{\Delta})\cong\mathcal{J}(\Gamma_{\mathcal{B}},\Delta)$. ■

## 3. DOUBLE STAR GRAPHS

Let $\Gamma$ be an $X$-symmetric graph that admits an $X$-invariant partition $\mathcal{B}$ such that $\Gamma_{\mathcal{B}}$ is $(X,2)$-arc transitive. If $r=1$, 2, $b-2$ or $b-1$ then, by [12], $\Gamma$ or its a quotient is isomorphic to $|E(\Gamma_{\mathcal{B}})|\mathbf{K}_2$, $\mathcal{J}(\Gamma_{\mathcal{B}},\Delta)$, $\mathcal{H}(\Gamma_{\mathcal{B}},\Lambda)$ or $\mathcal{I}(\Gamma_{\mathcal{B}},\Delta)$. This motivates us to consider the general case where $1\le r\le b-1$, and introduce stars and generalized 2-path graphs, called double star graphs.

In this section, we always assume that $\Sigma$ is an $X$-symmetric graph of valency $\mathbb{v}\ge 2$. For $\tau\in V(\Sigma)$ and a $\mathbb{k}$-subset $S$ of $\Sigma(\tau)$, the pair $(\tau,S)$ is called a $\mathbb{k}$-*star* of $\Sigma$. Let $\mathcal{S}t^{\mathbb{k}}(\Sigma)$ denote the set of $\mathbb{k}$-stars of $\Sigma$. An $X$-orbit $\mathcal{S}$ on $\mathcal{S}t^{\mathbb{k}}(\Sigma)$ is *symmetric* if $X_\tau\cap X_S$ acts transitively on $S$ for some $(\tau,S)\in\mathcal{S}$. Let $L$ and $R$ be $\mathbb{k}$-subsets of $\Sigma(\tau)$ and $\Sigma(\sigma)$, respectively, an ordered pair $((\tau,L),(\sigma,R))$ of $\mathbb{k}$-stars is called a *double* $\mathbb{k}$-*star* of $\Sigma$ if $\sigma\in L$ and $\tau\in R$. Denote by $D\mathcal{S}t^{\mathbb{k}}(\Sigma)$ the set of double $\mathbb{k}$-stars of $\Sigma$. Let $\Theta$ be an $X$-orbit on $D\mathcal{S}t^{\mathbb{k}}(\Sigma)$ and set $\mathcal{S}t(\Theta)=\{(\tau,L),(\sigma,R)\,|\,((\tau,L),(\sigma,R))\in\Theta\}$. Then $\Theta$ is said to be *symmetric* if $\mathcal{S}t(\Theta)$ is a symmetric $X$-orbit on $\mathcal{S}t^{\mathbb{k}}(\Sigma)$ and $\Theta$ is self-paired, that is, $((\sigma,R),(\tau,L))\in\Theta$ whenever $((\tau,L),(\sigma,R))\in\Theta$.

Let $\mathcal{S}$ be a symmetric $X$-orbit on $\mathcal{S}t^{\mathbb{k}}(\Sigma)$. For $\tau\in V(\Sigma)$, set $\mathcal{S}_\tau=\{(\tau,S)\,|\,(\tau,S)\in\mathcal{S}\}$. Define an incidence structure $\mathbb{D}(\tau):=(\Sigma(\tau),\mathcal{S}_\tau)$ in which $\sigma\in\Sigma(\tau)$ is incident with $(\tau,S)\in\mathcal{S}_\tau$ if and only if $\sigma\in S$. Then it is easy to see that $\mathbb{D}(\tau)$ is an $X_\tau$-flag-transitive 1-design, and $\mathbb{D}(\tau)$ is independent of the choice of $\tau\in V(\Sigma)$ up to isomorphism.

Let $\tau\in V(\Sigma)$ and $\mathfrak{D}(\tau)$ be an $X_\tau$-flag-transitive 1-$(\mathbb{v},\mathbb{k},\mathbb{r})$ design with vertex set $\Sigma(\tau)$. It may happen that distinct blocks of $\mathfrak{D}(\tau)$ have the same trace. Since $\mathfrak{D}(\tau)$ is flag-transitive, the number of blocks with the same trace is a constant, say $m(\mathfrak{D}(\tau))$, called the *multiplicity* of $\mathfrak{D}(\tau)$. Let $\mathfrak{D}'(\tau)$ be the design with vertex set $\Sigma(\tau)$ and blocks being the traces of blocks of $\mathfrak{D}(\tau)$. Then $\mathfrak{D}'(\tau)$ is an $X_\tau$-flag-transitive 1-$(\mathbb{v},\mathbb{k},\mathbb{r}/m(\mathfrak{D}(\tau)))$ design.

**Theorem 3.1.** *Let $\tau\in V(\Sigma)$. If there exists some $X_\tau$-flag-transitive 1-$(\mathbb{v},\mathbb{k},\mathbb{r})$ design $\mathfrak{D}(\tau)$ on $\Sigma(\tau)$ for $1\le\mathbb{k}\le\mathbb{v}-1$ such that $\mathbb{r}/m(\mathfrak{D}(\tau))$ is odd, then there exists a symmetric $X$-orbit on $D\mathcal{S}t^{\mathbb{k}}(\Sigma)$.*

***Proof.*** Set $\mathcal{S}=\{(\tau^x,S^x)\,|\,x\in X,S\in\mathfrak{D}'(\tau)\}$. It is easily shown that $\mathfrak{D}'(\tau)\cong\mathbb{D}(\tau)$ and $\mathcal{S}$ is a symmetric $X$-orbit. Let $(\tau,\sigma)\in Arc(\Sigma)$. Since $\Sigma$ is $X$-symmetric, $(\tau,\sigma)^y=(\sigma,\tau)$ for some $y\in X$. Set $\mathcal{S}_{(\tau,\sigma)}=\{(\tau,S)\in\mathcal{S}_\tau\,|\,\sigma\in S\}$. Then $\mathbb{r}/m(\mathfrak{D}(\tau))=|\mathcal{S}_{(\tau,\sigma)}|$ is odd, $\mathcal{S}^y_{(\tau,\sigma)}=\mathcal{S}_{(\sigma,\tau)}$ and $\mathcal{S}^{y^2}_{(\tau,\sigma)}=\mathcal{S}_{(\tau,\sigma)}$. Let $\mathcal{O}$ be a $\langle y^2\rangle$-orbit on $\mathcal{S}_{(\tau,\sigma)}$ with odd length $l$. Then, for $(\tau,S)\in\mathcal{O}$, the stabilizer of $(\tau,S)$ in $\langle y^2\rangle$ is $\langle y^{2l}\rangle$. Let $z=y^l$. Then $((\tau,S),(\sigma,S^z))^z=((\sigma,S^z),(\tau,S))$, and hence $\Theta:=\{((\tau,S)^x,(\sigma,S^z)^x)\,|\,x\in X\}$ is a symmetric $X$-orbit on $D\mathcal{S}t^{\mathbb{k}}(\Sigma)$ with $\mathcal{S}t(\Theta)=\mathcal{S}$. ■

Let $1 \le \Bbbk \le \mathbb{v}-1$ and $\Theta$ be a symmetric $X$-orbit on $DSt^{\Bbbk}(\Sigma)$. The *double star graph* $\Pi(\Sigma,\Theta)$ of $\Sigma$ with respect to $\Theta$ is the graph with vertex set $\mathcal{S}t(\Theta)$ such that two $\Bbbk$-stars $(\tau,L)$ and $(\sigma,R)$ are adjacent if and only if they give a pair in $\Theta$.

**Theorem 3.2.** *Let $\Gamma := \Pi(\Sigma,\Theta)$ be as above. Set $\mathcal{S} = \mathcal{S}t(\Theta)$ and $\mathcal{B} = \{\mathcal{S}_\tau \mid \tau \in V(\Sigma)\}$. Then $\Gamma$ is $X$-symmetric, $\mathcal{B}$ is a nontrivial $X$-invariant partition on $V(\Gamma)$ such that $\Gamma_{\mathcal{B}} \cong \Sigma$, $\Gamma$ is not a multicover of $\Gamma_{\mathcal{B}}$, and $\mathcal{D}(\mathcal{S}_\tau) \cong \mathbb{D}^*(\tau)$ for $\tau \in V(\Sigma)$, where $\mathbb{D}^*(\tau)$ is the dual design of $\mathbb{D}(\tau)$.*

***Proof.*** It is easily shown that $\Gamma$ is $X$-symmetric, $\mathcal{B}$ is an $X$-invariant partition of $V(\Gamma)$, and $V(\Sigma) \to V(\Gamma_{\mathcal{B}})$, $\tau \mapsto \mathcal{S}_\tau$ gives an isomorphism from $\Sigma$ to $\Gamma_{\mathcal{B}}$. For any $(\tau,S) \in \mathcal{S}_\tau \in \mathcal{B}$, as $1 \le \Bbbk = |S| \le \mathbb{v}-1$, take $\sigma \in S$ and $\delta \in \Sigma(\tau) \setminus S$. Since $\Sigma$ is $X$-symmetric, there exists $x \in X_\tau$ such that $\delta = \sigma^x$. Then $(\tau,S) \ne (\tau,S^x) \in \mathcal{S}_\tau$, so $v := |\mathcal{S}_\tau| \ge 2$ and $\mathcal{B}$ is nontrivial. Since $(\tau,\delta) \in Arc(\Sigma)$ and $\Theta$ is a symmetric $X$-orbit, there exists $(\delta,R) \in \mathcal{S}_\delta$ with $((\tau,S^x),(\delta,R)) \in \Theta$, hence $\mathcal{S}_\delta \in \Gamma_{\mathcal{B}}(\mathcal{S}_\tau)$. If $((\tau,S),(\delta,R')) \in \Theta$ for some $(\delta,R') \in \mathcal{S}_\delta$, then $\delta \in S$, a contradiction. Thus $(\tau,S) \notin \mathcal{S}_\tau \cap \Gamma(\mathcal{S}_\delta)$, so $|\mathcal{S}_\tau \cap \Gamma(\mathcal{S}_\delta)| < v$ and $\Gamma$ is not a multicover of $\Gamma_{\mathcal{B}}$.

Let $\tau \in V(\Sigma)$. Define $\pi : \mathcal{S}_\tau \cup \Gamma_{\mathcal{B}}(\mathcal{S}_\tau) \to \mathcal{S}_\tau \cup \Sigma(\tau)$; $(\tau,S) \mapsto (\tau,S)$, $\mathcal{S}_\sigma \mapsto \sigma$. If $\mathcal{S}_\sigma \in \Gamma_{\mathcal{B}}(B)$, then there exist $(\tau,L) \in \mathcal{S}_\tau$ and $(\sigma,R) \in \mathcal{S}_\sigma$ such that $((\tau,L),(\sigma,R)) \in \Theta$; in particular, $\sigma \in L \subseteq \Sigma(\tau)$, so $\pi$ is well-defined. It is easily shown that $\pi$ is a bijection. By the definition of $\mathcal{D}(\mathcal{S}_\tau)$, we know that $(\tau,S) \in B$ is incident with $\mathcal{S}_\sigma \in \Gamma_{\mathcal{B}}(B)$ if and only if there is some $(\sigma,T) \in C$ with $((\tau,S),(\sigma,T)) \in \Theta$, that is, $\tau \in T$ and $\sigma \in S$; it follows that $\sigma$ is incident with $(\tau,S)$ in $\mathbb{D}(\tau)$.

Assume that $\sigma' \in \Sigma(\tau)$ is incident with $(\tau,S')$ in $\mathbb{D}(\tau)$. Then $\sigma' \in S'$. Take some $(\tau',T')$ with $((\tau,S'),(\tau',T')) \in \Theta$. Then $\tau' \in S'$. Since $\mathcal{S}$ is a symmetric $X$-orbit, there is some $x \in X_\tau \cap X_{S'}$ with $\tau'^x = \sigma'$. Thus $(\tau,S')^x = (\tau,S')$, $(\tau',T')^x = (\sigma',T'^x) \in \mathcal{S}_{\sigma'}$ and $((\tau,S'),(\tau',T')^x) = ((\tau,S'),(\tau',T'))^x \in \Theta$. Hence $(\tau,S')$ is incident with $\mathcal{S}_{\sigma'}$ in $\mathcal{D}(\mathcal{S}_\tau)$. The above argument says that $\pi$ is an isomorphism from $\mathcal{D}(\mathcal{S}_\tau)$ to $\mathbb{D}^*(\tau)$. So $\mathcal{D}(\mathcal{S}_\tau) \cong \mathbb{D}^*(\tau)$. ∎

In the following, we assume that $\Gamma$ is an $X$-symmetric graph admitting a nontrivial $X$-invariant partition $\mathcal{B}$ such that $val(\Gamma_{\mathcal{B}}) \ge 2$ and $\Gamma$ is not a multicover of $\Gamma_{\mathcal{B}}$. For $\alpha \in B \in \mathcal{B}$, define $B_\alpha = B \cap (\bigcap_{C \in \Gamma_{\mathcal{B}}(\alpha)} \Gamma(C))$. Then $|B_\alpha|$, denoted by $m^*(\Gamma,\mathcal{B})$, is independent of the choices of $B$ and $\alpha$. Since $\Gamma$ is not a multicover of $\Gamma_{\mathcal{B}}$, we have $m^*(\Gamma,\mathcal{B}) \le k := |B \cap \Gamma(C)|$ for $C \in \Gamma_{\mathcal{B}}(B)$. In fact, $m^*(\Gamma,\mathcal{B})$ is the multiplicity of the dual design $\mathcal{D}^*(B)$ of $\mathcal{D}(B)$. Set $\underline{\mathcal{B}} = \{B_\alpha \mid B \in \mathcal{B}, \alpha \in B\}$. Then $\underline{\mathcal{B}}$ is an $X$-invariant partition of $V(\Gamma)$. Let $\bar{B} = \{B_\alpha \mid \alpha \in B\}$. Then $\Gamma_{\underline{\mathcal{B}}}$ is an $X$-symmetric graph with an $X$-invariant partition $\bar{\mathcal{B}} := \{\bar{B} \mid B \in \mathcal{B}\}$ such that $(\Gamma_{\underline{\mathcal{B}}})_{\bar{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$ and $m^*(\Gamma_{\underline{\mathcal{B}}},\bar{\mathcal{B}}) = 1$.

**Theorem 3.3.** *Set $\mathcal{S} = \{(B,\Gamma_{\mathcal{B}}(\alpha)) \mid B \in \mathcal{B}, \alpha \in B\}$. Then $\mathcal{S}$ is a symmetric $X$-orbit on $\mathcal{S}t^r(\Gamma_{\mathcal{B}})$, where $r = |\Gamma_{\mathcal{B}}(\alpha)|$ is a constant. Let $\Theta = \{((B,\Gamma_{\mathcal{B}}(\alpha)),(C,\Gamma_{\mathcal{B}}(\beta))) \mid \alpha \in B \in \mathcal{B}, \beta \in C \in \mathcal{B}, (\alpha,\beta) \in Arc(\Gamma)\}$. Then $\Theta$ is a symmetric $X$-orbit on $DSt^r(\Gamma_{\mathcal{B}})$ with $\mathcal{S}t(\Theta) = \mathcal{S}$ and $\Gamma_{\underline{\mathcal{B}}} \cong \Pi(\Gamma_{\mathcal{B}},\Theta)$, and $X$ acts faithfully on $\mathcal{B}$ if and only $X$ acts faithfully on $\underline{\mathcal{B}}$.*

***Proof.*** It is easily shown that $\Theta$ is a symmetric $X$-orbit on $DSt^r(\Gamma_{\mathcal{B}})$ with $\mathcal{S}t(\Theta) = \mathcal{S}$. Assume $m^*(\Gamma,\mathcal{B}) = 1$. Then $B_\alpha = \{\alpha\}$ and $C_\beta = \{\beta\}$ for two distinct vertices $\alpha \in B \in \mathcal{B}$ and $\beta \in C \in \mathcal{B}$, it implies that $\Gamma_{\mathcal{B}}(\alpha) \ne \Gamma_{\mathcal{B}}(\beta)$, hence $(B,\Gamma_{\mathcal{B}}(\alpha)) \ne (C,\Gamma_{\mathcal{B}}(\beta))$. Thus $V(\Gamma) \to V(\Pi(\Gamma_{\mathcal{B}}))$, $\alpha \mapsto (B,\Gamma_{\mathcal{B}}(\alpha))$ is a bijection, which gives an isomorphism between $\Gamma$ and $\Pi(\Gamma_{\mathcal{B}},\Theta)$.

Now assume $m^*(\Gamma,\mathcal{B})>1$. Recall that $m^*(\Gamma,\mathcal{B})\le k:=|B\cap\Gamma(C)|$ for $C\in\Gamma_{\mathcal{B}}(B)$. Then $\underline{\mathcal{B}}$ is a proper refinement of $\mathcal{B}$. Consider the pair $(\Gamma_{\underline{\mathcal{B}}},\bar{\mathcal{B}})$. Then $m^*(\Gamma_{\underline{\mathcal{B}}},\bar{\mathcal{B}})=1$. A similar argument as above leads to $\Gamma_{\underline{\mathcal{B}}}\cong\Pi(\Sigma,\bar{\Theta})$, where $\Sigma=(\Gamma_{\underline{\mathcal{B}}})_{\bar{\mathcal{B}}}$ and $\bar{\Theta}=\{((\bar{B},\Sigma(B_\alpha)),(\bar{C},\Sigma(C_\beta)))\,|\,B_\alpha\in\bar{B}\in\bar{\mathcal{B}},C_\beta\in\bar{C}\in\bar{\mathcal{B}},(B_\alpha,C_\beta)\in Arc(\Gamma_{\underline{\mathcal{B}}})\}$. Noting that $B_\alpha=B_{\alpha'}$ for any $\alpha'\in B_\alpha$, it follows that $(\bar{B},\Sigma(B_\alpha))\mapsto(B,\Gamma_{\mathcal{B}}(\alpha))$ gives a bijection between $V(\Pi(\Sigma,\bar{\Theta}))$ and $V(\Pi(\Gamma_{\mathcal{B}},\Theta))$, which is in fact an isomorphism between $\Pi(\Sigma,\bar{\Theta})$ and $\Pi(\Gamma_{\mathcal{B}},\Theta)$. Hence $\Gamma_{\underline{\mathcal{B}}}\cong\Pi(\Gamma_{\mathcal{B}},\Theta)$.

Let $K$ and $H$ be the kernels of $X$ acting on $\mathcal{B}$ and on $\underline{\mathcal{B}}$, respectively. Noting that $\underline{\mathcal{B}}$ is a refinement of $\mathcal{B}$, we have $H\le K$. Let $x\in K$ and $B_\alpha\in\bar{B}\in\underline{\mathcal{B}}$. Since $m^*(\Gamma_{\underline{\mathcal{B}}},\bar{\mathcal{B}})=1$, we have $\{B_\alpha\}=\bar{B}\cap(\bigcap_{\bar{C}\in(\Gamma_{\underline{\mathcal{B}}})_{\bar{\mathcal{B}}}(B_\alpha)}\Gamma_{\underline{\mathcal{B}}}(\bar{C}))=\bar{B}\cap(\bigcap_{C\in\Gamma_{\mathcal{B}}(\alpha)}\Gamma_{\underline{\mathcal{B}}}(\bar{C}))$, yielding $B_\alpha^x=B_\alpha$. The above argument gives $x\in H$. Hence $K\le H$, and so $H=K$. Therefore, $X$ acts faithfully on $\mathcal{B}$ (that is, $K=1$) if and only if $X$ acts faithfully on $\underline{\mathcal{B}}$ (that is, $H=1$). ∎

Finally, we list a simple fact which will be used in the following sections.

**Theorem 3.4.** *If $m^*(\Gamma,\mathcal{B})=1=m(\mathcal{D}(B))$, then $X_B^B\cong X_B^{\Gamma_{\mathcal{B}}(B)}$ for $B\in\mathcal{B}$.*

***Proof.*** If $x\in X$ fixes $B$ set-wise, then it also fixes the neighborhood $\Gamma_{\mathcal{B}}(B)$ of $B$ in $\Gamma_{\mathcal{B}}$. Now consider the action of $X_B$ on $\Gamma_{\mathcal{B}}(B)$, and let $K$ be the kernel of this action. For any $\alpha\in B$, since $m^*(\Gamma,\mathcal{B})=1$, we have $\{\alpha\}=B\cap(\bigcap_{C\in\Gamma_{\mathcal{B}}(\alpha)}\Gamma(C))$. It follows that $K$ fixes $\alpha$. Thus $K\le X_{(B)}$. On the other hand, $x$ fixes $B\cap\Gamma(C)$ point-wise for any $x\in X_{(B)}$ and any $C\in\Gamma_{\mathcal{B}}(B)$; in particular, $B\cap\Gamma(C^x)=(B\cap\Gamma(C))^x=B\cap\Gamma(C)$. It follows from $m(\mathcal{D}(B))=1$ that $C=C^x$. Therefore, $x\in K$. Thus $X_{(B)}\le K$, and so $X_{(B)}=K$. Then $X_B^B\cong X_B/X_{(B)}=X_B/K\cong X_B^{\Gamma_{\mathcal{B}}(B)}$. ∎



## 4. THE MAIN RESULT

A *near n-gonal graph* [13] is a connected graph $\Sigma$ of girth at least 4 together with a set $\mathcal{E}$ of $n$-cycles of $\Sigma$ such that each 2-arc of $\Sigma$ is contained in a unique member of $\mathcal{E}$. Let $Arc_3(\mathcal{E})$ be the set of 3-arcs appearing on cycles in $\mathcal{E}$. For a cycle $\mathbf{C}$ in an $X$-symmetric graph, denote by $X_{\mathbf{C}}$ the subgroup of $X$ which preserves the adjacency of $\mathbf{C}$, and set $X_{\mathbf{C}}^{\mathbf{C}}=X_{\mathbf{C}}/X_{(V(\mathbf{C}))}$.



**Theorem 4.1.** *Let $\Gamma$ be an $X$-symmetric graph admitting a nontrivial $X$-invariant partition $\mathcal{B}$ such that $val(\Gamma_{\mathcal{B}})\ge 2$, $\Gamma_{\mathcal{B}}$ is connected and $X$ is faithful on $V(\Gamma)$. Assume that $|B|>|\Gamma(C)\cap B|=3$ for $(B,C)\in Arc(\Gamma_{\mathcal{B}})$. Set $e=|E(\Gamma_{\mathcal{B}})|$. If further $\Gamma_{\mathcal{B}}$ is $(X,2)$-arc transitive, then*

(a) $|B|=4$, *$val(\Gamma_{\mathcal{B}})=4$ and $X_B^B\cong A_4$ or $S_4$; or*
(b) $|B|=6$, *$val(\Gamma_{\mathcal{B}})=4$ and $X_B^B\cong A_4$ or $S_4$; or*
(c) $|B|=7$, *$val(\Gamma_{\mathcal{B}})=7$ and $X_B^B\cong PSL(3,2)$; or*
(d) $|B|=3val(\Gamma_{\mathcal{B}})$ *and $\Gamma\cong 3e\mathbf{K}_2$, $e\mathbf{C}_6$ or $e\mathbf{K}_{3,3}$.*

*Further, each of* (a), (b) *and* (c) *implies that $\Gamma_{\mathcal{B}}$ is $(X,2)$-arc transitive with $X$ faithful on $\mathcal{B}$, $\Gamma$ is connected provided $\Gamma[B,C]\not\cong 3\mathbf{K}_2$, and $\Gamma$ is isomorphic to one of $\mathcal{I}(\Gamma_{\mathcal{B}},\Delta)$, $\mathcal{J}(\Gamma_{\mathcal{B}},\Delta)$ and $\Pi(\Gamma_{\mathcal{B}},\Theta)$, respectively, where $\Delta$ is a self-paired $X$-orbit on $Arc_3(\Gamma_{\mathcal{B}})$*

*and* $\Theta$ *is a symmetric X-orbit on* $D\mathcal{S}t^3(\Gamma_{\mathcal{B}})$; *moreover, one of* (a) *and* (b) *yields* (1) *or* (2), *and* (c) *yields* (3).

(1) *Either* $\Gamma_{\mathcal{B}} \cong \mathbf{K}_5$ *or* $\Gamma_{\mathcal{B}}$ *is near n-gonal with respect to an X-orbit* $\mathcal{E}$ *of n-cycles of* $\Gamma_{\mathcal{B}}$ *such that* $|\mathcal{E}| \geq 6$, $n \geq 4$, $n|\mathcal{E}| = 3e = 6|\mathcal{B}|$ *and* $X_{\mathbf{C}}^{\mathbf{C}} \cong \mathrm{D}_{2n}$ (*the dihedral group of order* $2n$) *for* $\mathbf{C} \in \mathcal{E}$; *and either*

(1.1) $\Gamma[B,C] \cong 3\mathbf{K}_2$, $X_B \cong \mathrm{A}_4$ *or* $\mathrm{S}_4$, $\Delta = Arc_3(\mathcal{E})$, $val(\Gamma) = 3$ *if* (a) *holds or* $\Gamma \cong |\mathcal{E}|\mathbf{C}_n$ *if* (b) *holds*; *or*

(1.2) $\Gamma[B,C] \cong \mathbf{C}_6$, $X_B \cong \mathrm{S}_4$, $\Gamma$ *is* $(X,1)$*-arc regular*, $val(\Gamma) = 6$ *if* (a) *holds or* $val(\Gamma) = 4$ *if* (b) *holds, and* $Arc_3(\Gamma_{\mathcal{B}}) \setminus \Delta = Arc_3(\mathcal{E})$ *is a self-paired X-orbit on* $Arc_3(\Gamma_{\mathcal{B}})$.

(2) $\Gamma[B,C] \cong \mathbf{K}_{3,3}$, $\Gamma_{\mathcal{B}}$ *is* $(X,3)$*-arc transitive, and* $val(\Gamma) = 9$ *or* 6 *for* (a) *or* (b) *respectively.*

(3) $val(\Gamma) = 3, 6$ *or* 9 *depending on* $\Gamma[B,C] \cong 3\mathbf{K}_2$, $\mathbf{C}_6$ *or* $\mathbf{K}_{3,3}$, *respectively*; *and if* $val(\Gamma) = 3$ *then* $\Gamma$ *is* $(X,2)$*-arc transitive.*

## 5. SELF-PAIRED ORBITS OF 3-ARCS

The following lemma is formulated from [10, Remark 4(c)(ii)] by noting that it is available to symmetric graphs.

**Lemma 5.1.** *Every X-symmetric graph* $\Sigma$ *with even valency contains a self-paired X-orbit on* $Arc_3(\Sigma)$.

Let $\Sigma$ be an $X$-symmetric graph with valency $\mathbb{v} \geq 2$ and $\Delta$ be a self-paired $X$-orbit on $Arc_3(\Sigma)$. For $(\tau_1,\tau,\sigma,\sigma_1) \in \Delta$, consider the action of $X_{(\tau_1,\tau,\sigma)}$ on $\Sigma(\sigma)\setminus\{\tau\}$, and use $\ell(\Delta)$ to denote the length of the orbit containing $\sigma_1$. Then $\ell(\Delta)$ is independent of the choice of $(\tau_1,\tau,\sigma,\sigma_1) \in \Delta$.

**Theorem 5.2.** *Let* $\Sigma$ *be a connected* $(X,2)$*-arc transitive graph with valency* $\mathbb{v} \geq 3$ *and* $\Delta$ *be a self-paired X-orbit on* $Arc_3(\Sigma)$ *such that* $\ell(\Delta) = 1$. *If X is faithful on* $V(\Sigma)$, *then* $X_\tau$ *is faithful on* $\Sigma(\tau)$ *for* $\tau \in V(\Sigma)$. *Set* $f = |V(\Sigma)|$ *and* $e = |E(\Sigma)|$. *Then* $\mathcal{J}(\Sigma,\Delta) \cong m\mathbf{C}_n$ *such that*

(1) $m \geq \mathbb{v}(\mathbb{v}-1)/2$, $n \geq girth(\Sigma)$ *and* $mn = f\mathbb{v}(\mathbb{v}-1)/2 = e(\mathbb{v}-1)$;

(2) $\Delta = Arc_3(\mathcal{E})$ *for an X-orbit* $\mathcal{E}$ *of n-cycles of* $\Sigma$ *with* $|\mathcal{E}| = m$ *and* $X_{\mathbf{C}}^{\mathbf{C}} \cong \mathrm{D}_{2n}$ *for* $\mathbf{C} \in \mathcal{E}$, *where* $\mathrm{D}_{2n}$ *is the dihedral group of order* $2n$;

(3) *each* 2*-path of* $\Sigma$ *is contained in a unique member of* $\mathcal{E}$, *and either* $\Sigma \cong \mathbf{K}_{\mathbb{v}+1}$ *or* $n \geq 4$ *and* $\Sigma$ *is a near n-gonal graph with respect to* $\mathcal{E}$.

***Proof.*** Since $\Sigma$ is $(X,2)$-arc transitive, each 2-arc of $\Sigma$ lies in a member of $\Delta$. Let $(\tau,\sigma)$ be an arbitrary arc of $\Sigma$. Since $\ell(\Delta) = 1$ and $\Delta$ is a self-paired $X$-orbit, we conclude that, for any $\tau_1 \in \Sigma(\tau)\setminus\{\sigma\}$, there is a unique $\sigma_1 \in \Sigma(\sigma)\setminus\{\tau\}$ such that $(\tau_1,\tau,\sigma,\sigma_1) \in \Delta$, $X_{(\tau_1,\tau,\sigma)} = X_{(\tau,\sigma,\sigma_1)}$ and $(\tau_1',\tau,\sigma,\sigma_1) \in \Delta$ yielding $\tau_1' = \tau_1$. Then $(X_\tau)_{(\Sigma(\tau))} = \bigcap_{\tau_1 \in \Sigma(\tau)\setminus\{\sigma\}} X_{(\tau_1,\tau,\sigma)} = \bigcap_{\sigma_1 \in \Sigma(\sigma)\setminus\{\tau\}} X_{(\tau,\sigma,\sigma_1)} = (X_\sigma)_{(\Sigma(\sigma))}$. It follows from the connectedness of $\Sigma$ that $(X_\tau)_{(\Sigma(\tau))}$ fixes every vertex of $\Sigma$. Thus, if $X$ is faithful on $V(\Sigma)$, then $(X_\tau)_{(\Sigma(\tau))} = 1$ and $X_\tau$ is faithful on $\Sigma(\tau)$.

Let $\Gamma = \mathcal{J}(\Sigma, \Delta)$. By [12, Theorem 4.4], $\Gamma$ is $X$-symmetric and admits an $X$-invariant partition $\mathcal{P} := \{P_\sigma \mid \sigma \in V(\Sigma)\}$ such that $\Sigma \cong \Gamma_{\mathcal{P}}$, where $P_\sigma$ is the set of 2-paths of $\Sigma$ with mid vertex $\sigma$. It follows from [12] that $r := |\Gamma_{\mathcal{P}}(\alpha)| = 2$ and $\lambda := |P_\delta \cap \Gamma(P_\tau) \cap \Gamma(P_\sigma)| = 1$ for any vertex $\alpha$ (a 2-path of $\Sigma$) in $V(\Gamma)$ and $P_\delta$ with $\alpha \in P_\delta$ and $\Gamma_{\mathcal{P}}(\alpha) = \{P_\tau, P_\sigma\}$. Since $\ell(\Delta) = 1$ and $\Delta$ is self-paired, for any 2-path $\tau_1\tau\sigma$ of $\Sigma$, there exist exactly two 2-paths $\tau\sigma\sigma_1$ and $\tau_2\tau_1\tau$ such that $(\tau_1, \tau, \sigma, \sigma_1) \in \Delta$ and $(\tau_2, \tau_1, \tau, \sigma) \in \Delta$. It follows that $val(\Gamma) = 2$, so $\Gamma \cong m\mathbf{C}_n$ for some $m$ and $n$. Then $mn$ is the number of 2-paths of $\Sigma$, hence $mn = f\mathbb{v}(\mathbb{v}-1)/2 = e(\mathbb{v}-1)$. Noting that $val(\Gamma) = 2$ and each $P_\sigma$ is an independent set of $\Gamma$, it follows that different vertices in $P_\sigma$ appear in different $n$-cycles of $\Gamma$. Thus $m \geq |P_\sigma| = \mathbb{v}(\mathbb{v}-1)/2$.

Let $\bar{\mathrm{C}} = \alpha_1\alpha_2 \ldots \alpha_n\alpha_1$ be an arbitrary $n$-cycle of $\Gamma$, where $\alpha_i = \tau_i\sigma_i\delta_i$ are $n$ distinct 2-paths of $\Gamma$ with mid vertices $\sigma_i$, respectively. Without loss of generality, we assume $\delta_i = \sigma_{i+1} = \tau_{i+2}$ for $1 \leq i \leq n$, where the subscripts are reduced modulo $n$. Since $\alpha_i$ is a 2-path of $\Sigma$, $\sigma_i \neq \delta_i$, hence $\sigma_i \neq \sigma_{i+1}$. Then $(\sigma_i, \sigma_{i+1}) \in Arc(\Sigma)$. Since $\{\alpha_i, \alpha_{i+1}\}$ is an edge of $\Gamma$, we have $(\sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2}) = (\tau_i, \sigma_i, \delta_i, \delta_{i+1}) \in \Delta$.

Now we show that $\mathbf{C} := \sigma_1\sigma_2 \ldots \sigma_n\sigma_1$ is an $n$-cycle of $\Sigma$; in particular, $n \geq girth(\Sigma)$. Note that $\bar{\mathrm{C}}$ is a component of $\Gamma$. Then $\bar{\mathrm{C}}$ is $X_{\bar{\mathrm{C}}}$-symmetric; in particular, $X_{\bar{\mathrm{C}}}^{\bar{\mathrm{C}}} \cong \mathrm{D}_{2n}$. Thus there exist $x, y \in X_{\bar{\mathrm{C}}}$ such that $\alpha_i^x = \alpha_{i+1}$ and $\alpha_i^y = \alpha_{n-i+1}$, hence $\sigma_i^x = \sigma_{i+1}$ and $\sigma_i^y = \sigma_{n-i+1}$ for $1 \leq i \leq n$ with the subscripts modulo $n$. Assume that $\sigma_i = \sigma_j$ for some $i$ and $j$. Then $\sigma_{i+1} = \sigma_i^x = \sigma_j^x = \sigma_{j+1}$ and $\sigma_{i+2} = \sigma_{i+1}^x = \sigma_{j+1}^x = \sigma_{j+2}$. Thus $P_{\sigma_i} = P_{\sigma_j}$, $P_{\sigma_{i+1}} = P_{\sigma_{j+1}}$ and $P_{\sigma_{i+2}} = P_{\sigma_{j+2}}$. It yields $(\alpha_i, \alpha_{i+1}), (\alpha_j, \alpha_{j+1}) \in Arc(\Gamma[P_{\sigma_i}, P_{\sigma_{i+1}}])$ and $(\alpha_{i+1}, \alpha_{i+2}), (\alpha_{j+1}, \alpha_{j+2}) \in Arc(\Gamma[P_{\sigma_{i+1}}, P_{\sigma_{i+2}}])$. It follows that $\alpha_{i+1}$, $\alpha_{j+1} \in P_{\sigma_{i+1}} \cap \Gamma(P_{\sigma_i}) \cap \Gamma(P_{\sigma_{i+2}})$. Since $1 = \lambda = |P_{\sigma_{i+1}} \cap \Gamma(P_{\sigma_i}) \cap \Gamma(P_{\sigma_{i+2}})|$, we have $\alpha_{i+1} = \alpha_{j+1}$. Thus $i = j$. Then all $\sigma_i$ are distinct, and so $\mathbf{C}$ is an $\langle x, y\rangle$-symmetric $n$-cycle. Hence $X_{\mathbf{C}}^{\mathbf{C}} \cong \mathrm{D}_{2n}$.

Set $\mathcal{E} = \{\mathbf{C}^x \mid x \in X\}$. Then $\mathcal{E}$ is an $X$-orbit of $n$-cycles of $\Sigma$. Since $\mathbf{C}$ is $X_{\mathbf{C}}$-symmetric, $\mathbf{C}$ is $(X_{\mathbf{C}}, 3)$-arc transitive. Recall that the 3-arc $(\sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2})$ of $\mathbf{C}$ is contained in $\Delta$. It follows that $\Delta = Arc_3(\mathcal{E})$.

It is easily shown that $X_{\bar{\mathrm{C}}}$ is a subgroup of $X_{\mathbf{C}}$. Suppose that $X_{\bar{\mathrm{C}}}$ is a proper subgroup of $X_{\mathbf{C}}$. Then there is some $z \in X_{\mathbf{C}}$ with $\mathbf{C}^z = \mathbf{C}$ but $\bar{\mathrm{C}}^z \neq \bar{\mathrm{C}}$, so $V(\bar{\mathrm{C}}) \cap V(\bar{\mathrm{C}}^z) = \emptyset$ as $\bar{\mathrm{C}}$ and $\bar{\mathrm{C}}^z$ are distinct connected components of $\Gamma$. Since $\mathbf{C}^z = \mathbf{C}$, there exist $i$, $j$ and $l$ with $\sigma_1 = \sigma_i^z$, $\sigma_2 = \sigma_j^z$ and $\sigma_3 = \sigma_l^z$. Then $\alpha_i^z = \tau_i^z\sigma_1\delta_i^z \in P_{\sigma_1}$, $\alpha_j^z = \tau_j^z\sigma_2\delta_j^z \in P_{\sigma_2}$ and $\alpha_l^z = \tau_l^z\sigma_3\delta_l^z \in P_{\sigma_3}$. Since $(\sigma_1, \sigma_2, \sigma_3)$ is a 2-arc of $\mathbf{C}$, we know that $(\sigma_i, \sigma_j, \sigma_l)$ is also a 2-arc of $\mathbf{C}$. It follows that $i - j \equiv j - l \equiv \pm 1 \pmod{n}$. Then $\alpha_i\alpha_j\alpha_l$ is a 2-path of $\bar{\mathrm{C}}$, and so $\alpha_i^z\alpha_j^z\alpha_l^z$ is a 2-path of $\bar{\mathrm{C}}^z$. Thus $\alpha_2, \alpha_j^z \in P_{\sigma_2} \cap \Gamma(P_{\sigma_1}) \cap \Gamma(P_{\sigma_3})$. Since $V(\bar{\mathrm{C}}) \cap V(\bar{\mathrm{C}}^z) = \emptyset$, we have $\alpha_2 \neq \alpha_j^z$, which contradicts $\lambda = 1$. Then $X_{\bar{\mathrm{C}}} = X_{\mathbf{C}}$ and so $|\mathcal{E}| = |X : X_{\mathbf{C}}| = |X : X_{\bar{\mathrm{C}}}| = m$.

Recall that the number of 2-paths of $\Sigma$ is equal to $mn$. Since $\Sigma$ is $(X, 2)$-arc transitive, every 2-path is contained in some $n$-cycle in $\mathcal{E}$. Noting each of the $m$ cycles in $\mathcal{E}$ has exactly $n$ paths of length 2, it follows that each 2-path of $\Sigma$ is contained in a unique member of $\mathcal{E}$. Thus either $\Sigma \cong \mathrm{K}_{\mathbb{v}+1}$, or $n \geq girth(\Sigma) \geq 4$ and $\Sigma$ is a near $n$-gonal graph with respect to $\mathcal{E}$. ∎

The following result follows from Lemmas 5.1 and 5.2.

**Corollary 5.3.** *Every connected $(X, 2)$-arc regular graph with even valency and girth no less than 4 is a near $n$-gonal graph for some integer $n \geq 4$.*

**Remark.** We would like to mention a recent result on near polygonal graphs of odd valency. Zhou [20] gave a necessary and sufficient condition for a trivalent 2-arc transitive to be near polygonal.

## 6. TETRAVALENT 2-ARC TRANSITIVE GRAPHS

The main aim of this section is to give a characterization of tetravalent 2-arc transitive graphs. The following simple lemma is useful.

**Lemma 6.1.** *Let $\Gamma$ be an $X$-symmetric graph admitting an $X$-invariant partition $\mathcal{B}$ with connected $(X,2)$-arc transitive quotient $\Gamma_{\mathcal{B}}$. Assume that $|\Gamma_{\mathcal{B}}(\gamma)|>1$ and $\Gamma[B,C]$ are connected for $\gamma\in V(\Gamma)$ and $(B,C)\in Arc(\Gamma_{\mathcal{B}}(B))$. Then $\Gamma$ is connected.*

***Proof.*** It suffices to show that any two distinct vertices $\alpha$ and $\beta$ are joined by a path in $\Gamma$. Since $|\Gamma_{\mathcal{B}}(\gamma)|>1$ and $\Gamma_{\mathcal{B}}$ is $(X,2)$-arc transitive, $\lambda:=|\Gamma(C)\cap B\cap\Gamma(D)|\neq 0$ is a constant for $B\in\mathcal{B}$ and distinct $C,D\in\Gamma_{\mathcal{B}}(B)$.

Assume that $\alpha,\beta\in B$. Without loss of generality, we assume $\alpha\in\Gamma(C)\cap B\cap\Gamma(D)$. If $\beta\in\Gamma(C)\cap B$, then there is a path between $\alpha$ and $\beta$ as $\Gamma[B,C]$ is connected. Assume $\beta\notin\Gamma(C)\cap B$. Take $D'\in\Gamma_{\mathcal{B}}(\beta)$. Then $D'\in\Gamma_{\mathcal{B}}(B)$, $\beta\in B\cap\Gamma(D')$ and $|\Gamma(C)\cap B\cap\Gamma(D')|=\lambda>0$. Let $\gamma\in\Gamma(C)\cap B\cap\Gamma(D')$. Then either $\alpha=\gamma$ or there is a path between $\alpha$ and $\gamma$, and there is a path between $\gamma$ and $\beta$. Thus there is a path between $\alpha$ and $\beta$.

Now let $\alpha\in B$ and $\beta\in B'$ with $B\neq B'$. Since $\Gamma_{\mathcal{B}}$ is connected, there is a path $B=B_1B_2\ldots B_l=B'$. Let $\beta_l'\in B_l$ and $\beta_{l-1}\in B_{l-1}$ such that $\{\beta_{l-1},\beta_l'\}\in E(\Gamma)$. Thus there is a path between $\beta_{l-1}$ and $\beta$. Then induction on $l$ implies that there is a path between $\alpha$ and $\beta$. ■

Let $\Sigma$ be an $(X,2)$-arc transitive graph with $val(\Sigma)=4$. Recall that $H(\Sigma)$ is the set of pairs $(\tau'\tau\tau'',\sigma'\sigma\sigma'')$ of 2-paths in $\Sigma$ such that $\sigma\in\Sigma(\tau)\setminus\{\tau',\tau''\}$, $\tau\in\Sigma(\sigma)\setminus\{\sigma',\sigma''\}$. For $\Delta\subseteq Arc_3(\Sigma)$, define $H(\Delta)=\{(\tau_2\tau\tau_3,\sigma_2\sigma\sigma_3)\,|\,(\tau_1,\tau,\sigma,\sigma_1)\in Arc_3(\Sigma),\{\sigma,\tau_1,\tau_2,\tau_3\}=\Sigma(\tau),\{\tau,\sigma_1,\sigma_2,\sigma_3\}=\Sigma(\sigma)\}$. Then $H(\Delta)\subseteq H(\Sigma)$. It is easily shown that $\Delta$ is a self-paired $X$-orbit on $Arc_3(\Sigma)$ if and only if $H(\Delta)$ is a symmetric $X$-orbit on $H(\Sigma)$.

**Lemma 6.2.** *Let $\Sigma$ be a connected $(X,2)$-arc transitive graph of valency 4. If $\Delta$ is a self-paired X-orbit on $Arc_3(\Sigma)$, then $\mathcal{J}(\Sigma,\Delta)\cong\mathcal{H}(\Sigma,H(\Delta))$.*

***Proof.*** Define $\phi:[\tau_1,\tau,\tau_2]\mapsto[\tau_3,\tau,\tau_4]$, where $\{\tau_3,\tau_4\}=\Sigma(\tau)\setminus\{\tau_1,\tau_2\}$. It is easy to check that $\phi$ is an isomorphism from $\mathcal{J}(\Sigma,\Delta)$ to $\mathcal{H}(\Sigma,H(\Delta))$. ■

**Theorem 6.3.** *Let $\Sigma$ be a connected $(X,2)$-arc transitive graph with valency 4 and $X$ acting faithfully on $V(\Sigma)$. Then $\Sigma$ has a self-paired $X$-orbit $\Delta$ of 3-arcs. Let $\Gamma=\mathcal{J}(\Sigma,\Delta)$ and $\Gamma'=\mathcal{I}(\Sigma,\Delta)$. Then $\Gamma[P_\tau,P_\sigma]\cong\Gamma'[A_\tau,A_\sigma]$ for $(\tau,\sigma)\in Arc(\Sigma)$, and one of the following cases occurs.*

(1) *Either $\Sigma\cong\mathrm{K}_5$ or $\Sigma$ is a near $n$-gonal graph with respect to an $X$-orbit $\mathcal{E}$ of $n$-cycles of $\Sigma$ with $|\mathcal{E}|\geq 6$, $n\geq girth(\Sigma)$, $n|\mathcal{E}|=3|E(\Sigma)|=6|V(\Sigma)|$ and $X_{\mathbf{C}}^{\mathbf{C}}\cong\mathrm{D}_{2n}$ for $\mathbf{C}\in\mathcal{E}$; and either.*

(1.1) $\Gamma[P_\tau,P_\sigma]\cong 3\mathbf{K}_2$, $\Gamma\cong m\mathbf{C}_n$, $val(\Gamma')=3$, $\Delta=Arc_3(\mathcal{E})$, $X_{P_\tau}=X_{A_\tau}=X_\tau\cong\mathrm{A}_4$ *or* $\mathrm{S}_4$; *or*

(1.2) $\Gamma[P_\tau, P_\sigma] \cong \mathbf{C}_6$, $val(\Gamma) = 4$, $val(\Gamma') = 6$, $X_{P_\tau} = X_{A_\tau} = X_\tau \cong \mathrm{S}_4$, *both* $\Gamma$ *and* $\Gamma'$ *are connected and* $(X,1)$*-arc regular, and* $Arc_3(\mathcal{E}) = Arc_3(\Sigma) \setminus \Delta$ *is a self-paired* $X$*-orbit on* $Arc_3(\Sigma)$.

(2) $\Gamma[P_\tau, P_\sigma] \cong \mathbf{K}_{3,3}$, $val(\Gamma) = 6$, $val(\Gamma') = 9$, *both* $\Gamma$ *and* $\Gamma'$ *are connected, and* $\Sigma$ *is* $(X,3)$*-arc transitive.*

***Proof.*** By Lemma 5.1, $\Sigma$ has a self-paired $X$-orbit $\Delta$ on $Arc_3(\Sigma)$. Let $\ell(\Delta)$ be defined as in Section 5. Then $\ell(\Delta) \le 3$ as $val(\Sigma) = 4$. By [12, Theorem 4.4], $\Gamma = \mathcal{J}(\Sigma, \Delta)$ is $X$-symmetric and admits an $X$-invariant partition $\mathcal{P} = \{P_\sigma \mid \sigma \in V(\Sigma)\}$. By Proposition 2.2, $\Gamma' = \mathcal{I}(\Sigma, \Delta)$ is $X$-symmetric and admits an $X$-invariant partition $\mathcal{A} = \{A_\sigma \mid \sigma \in V(\Sigma)\}$.

Let $(\tau, \sigma) \in Arc(\Sigma)$. Then there is a 3-arc $(\tau_1, \tau, \sigma, \sigma_1) \in \Delta$ as $\Sigma$ is $X$-symmetric. It follows that $\{\tau_1\tau\sigma, \tau\sigma\sigma_1\}$ is an edge of $\Gamma[P_\tau, P_\sigma]$, and that $\{(\tau,\tau_1),(\sigma,\sigma_1)\}$ is an edge of $\Gamma'[A_\tau, A_\sigma]$. It is easily shown that $X_{(\tau,\sigma)} = X_\tau \cap X_\sigma = X_{P_\tau} \cap X_{P_\sigma}$ acts transitively on the edges of $\Gamma[P_\tau, P_\sigma]$. It implies that $X_{(\tau_1,\tau,\sigma)}$ acts transitively on the neighborhood of $\tau_1\tau\sigma$ in $\Gamma[P_\tau, P_\sigma]$. Then $val(\Gamma[P_\tau, P_\sigma]) = |X_{(\tau_1,\tau,\sigma)} : X_{(\tau_1,\tau,\sigma,\sigma_1)}| = \ell(\Delta)$. Since $\Sigma$ is $(X,2)$-arc transitive, $X_{(\tau,\sigma)}$ is transitive on both $\Sigma(\tau) \setminus \{\sigma\} := \{\tau_1, \tau_2, \tau_3\}$ and $\Sigma(\sigma) \setminus \{\tau\} := \{\sigma_1, \sigma_2, \sigma_3\}$. Thus $V(\Gamma[P_\tau, P_\sigma]) = \{\tau_i\tau\sigma \mid i = 1,2,3\} \cup \{\tau\sigma\sigma_i \mid i = 1,2,3\}$. A similar argument leads to $V(\Gamma'[A_\tau, A_\sigma]) = \{(\tau,\tau_i) \mid i = 1,2,3\} \cup \{(\sigma,\sigma_i) \mid i = 1,2,3\}$. It is easy to check that $\tau_i\tau\sigma \mapsto (\tau,\tau_i)$, $\tau\sigma\sigma_i \mapsto (\sigma,\sigma_i)$ gives an isomorphism from $\Gamma[P_\tau, P_\sigma]$ to $\Gamma'[A_\tau, A_\sigma]$. Further, $\Gamma[P_\tau, P_\sigma] \cong 3\mathbf{K}_2$, $\mathbf{C}_6$ or $\mathbf{K}_{3,3}$ according to $\ell(\Delta) = 1$, 2 or 3, respectively. By [12, Theorem 4.3], $2 = |\Gamma_{\mathcal{P}}(\tau_1\tau\sigma)|$ for $\tau_1\tau\sigma \in V(\Gamma)$. Then $val(\Gamma) = \ell(\Delta)|\Gamma_{\mathcal{P}}(\tau_1\tau\sigma)| = 2\ell(\Delta)$. By Lemma 2.2, $val(\Gamma') = 3\ell(\Delta)$. Further, by Lemma 6.1, both $\Gamma$ and $\Gamma'$ are connected provided $\Gamma[P_\tau, P_\sigma] \ncong 3\mathbf{K}_2$.

If $\ell(\Delta) = 3$, then $val(\Gamma) = 2\ell(\Delta) = 6$, $val(\Gamma') = 3\ell(\Delta) = 9$, $\Gamma[P_\tau, P_\sigma] \cong \mathbf{K}_{3,3}$, and (2) follows from [10, Theorem 2]. Thus we assume that $\ell(\Delta) \le 2$ in the following.

It is easy to see $X_\tau = X_{P_\tau} = X_{A_\tau}$, $(X_\tau)_{(\Sigma(\tau))} = X_{(P_\tau)} = X_{(A_\tau)}$ and hence $X_\tau^{\Sigma(\tau)} \cong X_{P_\tau}^{P_\tau} = X_{A_\tau}^{A_\tau}$. Since $\Sigma$ is $(X,2)$-arc transitive, $X_\tau^{\Sigma(\tau)} \cong \mathrm{A}_4$ or $\mathrm{S}_4$. Further, if $\ell(\Delta) = 2$ then $|X_\tau^{\Sigma(\tau)}| > 12$ as $\Sigma$ is not $(X,2)$-arc regular in this case. Let $\Delta' = \Delta$ or $Arc_3(\Sigma) \setminus \Delta$ depending on $\ell(\Delta) = 1$ or 2, respectively. It is easily shown that $\ell(\Delta') = 1$ and $\Delta'$ is a self-paired $X$-orbit on $Arc_3(\Sigma)$. Then (1) follows from Theorem 5.2 and the above argument. ■



**Corollary 6.4.** *Let* $\Sigma$ *be a connected tetravalent* $(X,2)$*-transitive graph. Then either* $\Sigma \cong \mathbf{K}_5$, *or* $\Sigma$ *is a near* $n$*-gonal graph for some integer* $n \ge 4$.

## 7. HEPTAVALENT GRAPHS WITH $X_\tau^{\Sigma(\tau)} \cong \mathrm{PSL}(3,2)$

**Theorem 7.1.** *Let* $\Sigma$ *be an* $(X,2)$*-arc transitive graph of valency 7 with* $X_\tau^{\Sigma(\tau)} \cong \mathrm{PSL}(3,2)$ *for* $\tau \in V(\Sigma)$. *Then there exists a symmetric* $X$*-orbit* $\Theta$ *on* $DSt^3(\Sigma)$. *Let* $\Gamma = \Pi(\Sigma, \Theta)$ *and* $\mathcal{S} = St(\Theta)$. *Then, for* $\sigma \in \Sigma(\tau)$, *one of the following cases occurs.*

(1) $\Gamma[\mathcal{S}_\tau, \mathcal{S}_\sigma] \cong 3\mathbf{K}_2$, *and* $\Gamma$ *is a trivalent* $(X,2)$*-arc transitive graph*;
(2) $\Gamma[\mathcal{S}_\tau, \mathcal{S}_\sigma] \cong \mathbf{C}_6$, $val(\Gamma) = 6$ *and* $\Gamma$ *is connected*;
(3) $\Gamma[\mathcal{S}_\tau, \mathcal{S}_\sigma] \cong \mathbf{K}_{3,3}$, $val(\Gamma) = 9$ *and* $\Gamma$ *is connected.*

***Proof.*** Let $\tau \in V(\Sigma)$. Since $X_\tau^{\Sigma(\tau)} \cong \mathrm{PSL}(3,2)$, we may identify $\Sigma(\tau)$ with the point set of the seven-point plane $\mathrm{PG}(2,2)$, which is an $X_\tau$-flag-transitive 1-(7,3,3) design with multiplicity 1. By Theorem 3.1, there exists a symmetric $X$-orbit $\Theta$ on $\mathcal{D}\mathcal{S}t^3(\Sigma)$. Let $\mathcal{S}=\mathcal{S}t(\Theta)$ and $\Gamma=\Pi(\Sigma,\Theta)$. Then, by Theorem 3.2, $\Gamma$ is $X$-symmetric and $\Gamma_{\mathcal{B}}\cong\Sigma$, where $\mathcal{B}=\{\mathcal{S}_\tau \mid \tau\in V(\Sigma)\}$ and $\mathcal{S}_\tau=\{(\tau,S)\mid(\tau,S)\in\mathcal{S}\}$. Further, for $\mathcal{S}_\tau\in\mathcal{B}$, we have $X_\tau=X_{\mathcal{S}_\tau}$ and $\mathcal{D}(\mathcal{S}_\tau)\cong\mathbb{D}^*(\tau)\cong\mathrm{PG}(2,2)$. In particular, $|\mathcal{S}_\tau\cap\Gamma(\mathcal{S}_\sigma)|=3$ for $\sigma\in\Sigma(\tau)$; thus $\Gamma[\mathcal{S}_\tau,\mathcal{S}_\sigma]\cong 3\mathbf{K}_2$, $\mathbf{C}_6$ or $\mathbf{K}_{3,3}$. Noting that two distinct lines of PG(2,2) intersect a unique point and two distinct points determine a unique line, it follows that $\lambda:=|\Gamma(\mathcal{S}_\sigma)\cap\mathcal{S}_\tau\cap\Gamma(\mathcal{S}_\delta)|=1$ for $\sigma,\delta\in\Sigma(\tau)$ with $\sigma\neq\delta$. By Lemma 6.1, $\Gamma$ is connected if $\Gamma[\mathcal{S}_\tau,\mathcal{S}_\sigma]\ncong 3\mathrm{K}_2$. Note that each point of $\mathcal{D}(\mathcal{S}_\tau)$ is incident with three blocks. Then $val(\Gamma)=3val(\Gamma[\mathcal{S}_\tau,\mathcal{S}_\sigma])$. Thus (2) or (3) holds if $\Gamma[\mathcal{S}_\tau,\mathcal{S}_\sigma]\ncong 3\mathbf{K}_2$.

Assume that $\Gamma[\mathcal{S}_\tau,\mathcal{S}_\sigma]\cong 3\mathbf{K}_2$. Then $val(\Gamma)=3$. Let $\alpha\in\mathcal{S}_\tau$, and $\Gamma(\alpha)=\{\alpha_1,\alpha_2,\alpha_3\}$ with $\alpha_i\in\mathcal{S}_{\tau_i}$ for $i=1,2,3$. Then $\tau_1$, $\tau_2$ and $\tau_3$ are distinct vertices of $\Sigma$. Recall $\mathcal{D}(\mathcal{S}_\tau)\cong\mathbb{D}^*(\tau)\cong\mathrm{PG}(2,2)$. Then we may identify $\alpha$ with a line of PG(2,2), and $\mathcal{S}_{\tau_i}$ with the points on this line. Then $(X_\tau^{\Sigma(\tau)})_\alpha\cong\mathrm{S}_4$ acts 2-transitively on $\{\mathcal{S}_{\tau_i}\mid i=1,2,3\}$. It implies that $(X_\tau)_\alpha=X_\alpha$ acts 2-transitively (and unfaithfully) on $\{\alpha_1,\alpha_2,\alpha_3\}$. Thus $\Gamma$ is $(X,2)$-arc transitive, and (1) holds. ■

## 8. PROOF OF THEOREM 4.1

Let $\Gamma$ be an $X$-symmetric graph admitting an $X$-invariant partition $\mathcal{B}$ such that $\Gamma_{\mathcal{B}}$ is connected and $X$ is faithful on $V(\Gamma)$. Set $b=val(\Gamma_{\mathcal{B}})$, $v=|B|$, $r=|\Gamma_{\mathcal{B}}(\alpha)|$ and $k=|B\cap\Gamma(C)|$ for $\alpha\in V(\Gamma)$ and $(B,C)\in Arc(\Gamma_{\mathcal{B}})$. Assume that $b\geq 2$ and $v>k=3$. Recall that $\mathcal{D}(B)$ is a 1-$(v,b,r)$-design.

We first show that each of Theorem 4.1(a)–(c) implies that $\Gamma_{\mathcal{B}}$ is $(X,2)$-arc transitive. Assume that one of (a), (b) and (c) occurs. Since $vr=bk$, we have $(v,b,r)$ is one of $(4,4,3)$, $(6,4,2)$ and $(7,7,3)$.

Consider the multiplicity $m(\mathcal{D}(B))$ of $\mathcal{D}(B)$. Suppose that $m(\mathcal{D}(B))\neq 1$. Then $\Gamma_{\mathcal{B}}(B)$ admits an $X_B$-invariant partition $\mathcal{M}:=\{\mathcal{M}_C\mid C\in\Gamma_{\mathcal{B}}(B)\}$, where $\mathcal{M}_C$ is a set of blocks of $\mathcal{D}(B)$ with the same trace $B\cap\Gamma(C)$ of $C$. Thus $m(\mathcal{D}(B))=|\mathcal{M}_C|$ is a divisor of $b$. For $\alpha\in B$, it is easy to see that $C\in\Gamma_{\mathcal{B}}(\alpha)$ yields $D\in\Gamma_{\mathcal{B}}(\alpha)$ for any $D\in\mathcal{M}_C$. This observation says that $m(\mathcal{D}(B))$ is also a divisor of $r$. It follows that $(v,b,r)=(6,4,2)$, $m(\mathcal{D}(B))=2=r$ and $|\mathcal{M}|=2$. Set $\mathcal{M}=\{\mathcal{M}_C,\mathcal{M}_D\}$. Then $\mathcal{T}:=\{B\cap\Gamma(C),B\cap\Gamma(D)\}$ is an $X_B$-invariant partition of $B$. Let $K$ be the kernel of $X_B$ acting on $\mathcal{T}$. Then $|X_B:K|=2$ and $X_{(B)}\leq K$. It follows that $X_B^B\cong\mathrm{S}_4$ and $K/X_{(B)}\cong\mathrm{A}_4$. Note that $K$ is in fact the set-wise stabilizer of $B\cap\Gamma(C)$, and also of $B\cap\Gamma(D)$, in $X_B$. Then $K$ is transitive on both $B\cap\Gamma(C)$ and $B\cap\Gamma(D)$. Let $H$ and $H_1$ be the kernels of $K$ acting on $B\cap\Gamma(C)$ and on $B\cap\Gamma(D)$, respectively. Then $K/H$ and $K/H_1$ are permutation groups of degree 3. Noting that $X_{(B)}\leq H$ and $X_{(B)}\leq H_1$, it follows that $H/X_{(B)}$ and $H_1/X_{(B)}$ are normal subgroups of $K/X_{(B)}$ with index 3 in $K/X_{(B)}$. Hence $H_1/X_{(B)}=H/X_{(B)}$ as $\mathrm{A}_4$ has only one normal subgroup of order 4. Thus $H_1=H$ fixes $B$ point-wise, and so $H\leq X_{(B)}$, which contradicts $|H/X_{(B)}|=4$. Thus $m(\mathcal{D}(B))=1$.

Recall that $m^*(\Gamma,\mathcal{B})$ is the multiplicity of the dual design $\mathcal{D}^*(B)$ of $\mathcal{D}(B)$ and $m^*(\Gamma,\mathcal{B})=|B_\alpha|$ for $\alpha\in B\in\mathcal{B}$ and $B_\alpha=B\cap(\bigcap_{C\in\Gamma_{\mathcal{B}}(\alpha)}\Gamma(C))$. It is easily shown that $\{B_\alpha\mid\alpha\in B\}$ is an $X_B$-invariant partition of $B$; in particular, $m^*(\Gamma,\mathcal{B})=|B_\alpha|$ is a divisor of $|B|=v$. Noting that $B_\alpha\subseteq B\cap\Gamma(C)$ for $\alpha\in B$ and $C\in\Gamma_{\mathcal{B}}(\alpha)$, it follows that $m^*(\Gamma,\mathcal{B})$ is also a divisor of $k=|B\cap\Gamma(C)|$. If $m^*(\Gamma,\mathcal{B})\neq 1$, then $(v,k,r)=(6,3,2)$ and $m^*(\Gamma,\mathcal{B})=k$, so $m(\mathcal{D}(B))\geq|\Gamma_{\mathcal{B}}(\alpha)|=2$, a contradiction. Thus $m^*(\Gamma,\mathcal{B})=1$.

Therefore, $m(\mathcal{D}(B)) = 1 = m^*(\Gamma, \mathcal{B})$, and $X_B^{\Gamma_{\mathcal{B}}(B)} \cong X_B^B$ by Theorem 3.4. Thus, if one of cases (a), (b) and (c) occurs then $X_B^{\Gamma_{\mathcal{B}}(B)}$ is 2-transitive on $\Gamma_{\mathcal{B}}(B)$, and hence $\Gamma_{\mathcal{B}}$ is $(X, 2)$-arc transitive.

Now assume that $\Gamma_{\mathcal{B}}$ is $(X, 2)$-arc transitive. Then $\lambda := |\Gamma(C) \cap B \cap \Gamma(D)|$ is independent of the choice of 2-path $CBD$ of $\Gamma_{\mathcal{B}}$, and $m(\mathcal{D}(B)) = 1$ by [12, Lemma 2.4]. By [12, Corollary 3.3], $vr = 3b$ and $\lambda(b-1) = 3(r-1)$, thus $(9 - \lambda v)r = 3(3-\lambda)$. Since $v > k = 3$, we have $\lambda \le k - 1 = 2$. If $\lambda = 0$, then $r = 1$ and $v = 3b$. Let $\lambda \ge 1$. Then, by [12, Theorem 3.2], the dual design $\mathcal{D}^*(B)$ of $\mathcal{D}(B)$ is a 2-$(b, r, \lambda)$ design with $v$ blocks. The well-known Fisher's Inequality applied to $\mathcal{D}^*(B)$ gives $b \le v$, and so $r \le k = 3$. If $\lambda = 2$, then $\lambda(b-1) = 3(r-1)$, $(9 - 2v)r = 3$ yields $(v, b, r) = (4, 4, 3)$. If $\lambda = 1$, then $r \le k$, $vr = 3b$ and $(9 - v)r = 6$ yield $(v, b, r) = (6, 4, 2)$ or $(7, 7, 3)$.

Note that $m^*(\Gamma, \mathcal{B}) \le \lambda$ if $\lambda \ne 0$. Suppose that $m^*(\Gamma, \mathcal{B}) \ne 1$ for some $\lambda \ne 0$. Then $\lambda = 2 = m^*(\Gamma, \mathcal{B})$. Since $r = 3$, there are $C, D \in \Gamma_{\mathcal{B}}(\alpha)$ with $C \ne D$ and $B \cap \Gamma(C) = B \cap \Gamma(D)$. Thus $C$ and $D$ has the same trace, so $m(\mathcal{D}(B)) \ge 2$, a contradiction. Therefore, if $\lambda \ne 0$ then $m^*(\Gamma, \mathcal{B}) = 1$ and, by Theorem 3.3 and 3.4, $X_B^{\Gamma_{\mathcal{B}}(B)} \cong X_B^B$ and $X$ is faithful on $\mathcal{B}$.

Assume that $(v, b, r, \lambda) = (4, 4, 3, 2)$ or $(6, 4, 2, 1)$. Then $val(\Gamma_{\mathcal{B}}) = 4$, and $X_B^B \cong \mathrm{A}_4$ or $\mathrm{S}_4$ as $X_B$ acts 2-transitively on $\Gamma_{\mathcal{B}}(B)$. Thus (a) or (b) holds, so either $\Gamma \cong \mathcal{I}(\Gamma_{\mathcal{B}}, \Delta)$ by [10, Theorem 2] or $\Gamma \cong \mathcal{J}(\Gamma_{\mathcal{B}}, \Delta)$ by Lemma 2.3, where $\Delta$ is a self-paired $X$-orbit on $Arc_3(\Gamma_{\mathcal{B}})$. Then, by Theorem 6.3, one of Theorem 4.1 (1) and (2) occurs.

Assume that $(v, b, r, \lambda) = (7, 7, 3, 1)$. Then $\mathcal{D}(B) \cong \mathrm{PG}(2, 2)$ is $X_B$-flag-transitive, and so $X_B^{\Gamma_{\mathcal{B}}(B)}$ is isomorphic to a subgroup of $\mathrm{PSL}(3, 2)$, the automorphism group of $\mathrm{PG}(2, 2)$. Since $\Gamma_{\mathcal{B}}$ is $(X, 2)$-arc transitive, $X_B^{\Gamma_{\mathcal{B}}(B)}$ is 2-transitive on $\Gamma_{\mathcal{B}}(B)$, and hence $|X_B^{\Gamma_{\mathcal{B}}(B)}| \ge 42$. It follows that $X_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(3, 2)$. Thus $X_B^B \cong X_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(3, 2)$ by Theorem 3.4. Hence (c) holds. Since $m^*(\Gamma, \mathcal{B}) = 1$, by Theorem 3.3, $\Gamma \cong \Pi(\Gamma_{\mathcal{B}}, \Theta)$ for a symmetric $X$-orbit $\Theta$ on $DSt^3(\Gamma_{\mathcal{B}})$. Then, by Theorem 7.1, Theorem 4.1(3) holds.

Assume that $\lambda = 0$, $r = 1$ and $v = 3b$. Then $\Gamma \cong e\Gamma[B, C]$ for $\{B, C\} \in E(\Gamma_{\mathcal{B}})$. Since $|B \cap \Gamma(C)| = 3$, we have $\Gamma[B, C] \cong 3\mathbf{K}_2, \mathbf{C}_6$ or $\mathbf{K}_{3,3}$. Thus (d) occurs.

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