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On a Certain Extension of the Hurwitz-Lerch Zeta Function

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Abstract. Our purpose in this paper is to consider a generalized form of the extended Hurwitz-Lerch Zeta function. For this extended Hurwitz-Lerch Zeta function, we obtain some classical properties which includes various integral representations, a differential formula, Mellin transforms and certain generating relations. We further consider an application to probability distributions and also point out some important special cases of the main results.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}^-$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ denote the sets of positive integers, negative integers, real numbers, positive real numbers and complex numbers, respectively, and also, let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$.

The Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ is defined by (see, *e.g.*, [5, p.

27, Eq. 1.11(1)]; see also [11, p. 121] and [12, p. 194]):

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.1}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s)>1 \quad \text{when} \quad |z|=1\right).$$

Also, the integral representation of $\Phi(z,s,a)$ (see, *e.g.*, [5, p. 27, Eq. 1.11(3)]; see also [12, p. 194, Eq. 2.5(4)]) is given by

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-(a-1)t}}{e^t - z}dt \tag{1.2}$$

$$\left(\Re(a)>0;\ \Re(s)>0 \quad \text{when} \quad |z| \leqq 1\ (z \neq 1);\ \Re(s)>1 \quad \text{when} \quad z=1\right).$$

Certain forms of generalizations of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ have been considered by various authors, see, *e.g.*, [3–7]). Goyal and Laddha [8, p. 100, Eq. (1.5)] and Garg *et al.* [7, p. 313, Eq. (1.7)] studied certain functions which are, respectively, defined by

$$\Phi_{\mu}^{*}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$$\left(\mu \in \mathbb{C};\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z|<1; \Re(s-\mu)>1 \quad \text{when} \quad |z|=1\right)$$

and

$$\Phi_{\lambda,\mu;\nu}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu)_n}{(\nu)_n n!} \frac{z^n}{(n+a)^s} \tag{1.4}$$

$\big(\lambda, \mu \in \mathbb{C};\ \nu,\, a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C}$ when $|z|<1;\ \Re(s+\nu-\lambda-\mu)>1$ when $|z|=1\big)$, where $(\lambda)_\nu$ denotes the usual Pochhammer symbol (for $\lambda \in \mathbb{C}$) defined by (see [12, p. 2 and p. 5]):

$$\begin{aligned}(\lambda)_\nu :&= \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+\nu-1) & (\nu \in \mathbb{N}) \end{cases} \\ &= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-),\end{aligned} \tag{1.5}$$

where $\Gamma(\lambda)$ is the familiar (Euler's) Gamma function (see, *e.g.*, [12, Chapter 1]).

The corresponding integral representations of (1.3) and (1.4) are given, respectively, by

$$\Phi_{\mu}^{*}(z, s, a) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1} e^{-at}}{(1 - ze^{-t})^{\mu}} \, dt = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1} e^{-(a-1)t}}{(e^{t} - z)^{\mu}} \, dt \tag{1.6}$$

$$\big(\Re(a) > 0;\ \Re(s) > 0 \quad \text{when} \quad |z| \leqq 1\ (z \neq 1);\ \Re(s) > 1 \quad \text{when} \quad z = 1\big)$$

and

$$\Phi_{\lambda,\mu;\nu}(z, s, a) := \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-at} \ {}_2F_1(\lambda, \mu; \nu; ze^{-t}) \, dt \tag{1.7}$$

$$\big(\Re(a) > 0;\ \Re(s) > 0 \quad \text{when} \quad |z| \leqq 1\ (z \neq 1);\ \Re(s) > 1 \quad \text{when} \quad z = 1\big).$$

In their paper, Chaudhry *et al.* [1] presented the following extension of the Beta function:

$$B(x, y\,; p) = \int_{0}^{1} t^{x-1} (1-t)^{y-1} \exp\left(-\frac{p}{t(1-t)}\right) dt \quad (\Re(p) > 0), \tag{1.8}$$

and showed that this extension has certain connections with the Macdonald, Error, and Whittaker functions.

Subsequently, Chaudhry *et al.* [2] used the above function $B(x, y\,; p)$ to provide an extension of the hypergeometric function which is given by

$$F_p\,(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \, \frac{B(b+n, c-b\,; p)}{B(b, c-b)} \, \frac{z^n}{n!} \tag{1.9}$$

$$(p \geqq 0,\ |z| < 1\,; \Re(c) > \Re(b) > 0)\,.$$

Evidently, the special cases of (1.8) and (1.9) when $p = 0$ reduce immediately to the classical Eulerian Beta and Gauss's hypergeometric function, respectively.

Motivated by these aforecited cosiderations, we first give an extension of the generalized Hurwitz-Lerch Zeta function (1.4). We then investigate certain properties of the extended generalized Hurwitz-Lerch Zeta function and derive for this class of functions various integral representations, a derivative formula, Mellin transform and certain generating relations. An application to probability distributions is also considered and we also point out some special cases of the main results.

2 Extended Generalized Hurwitz-Lerch Zeta Function

In terms of the extended Beta function $B(x,\, y\,;\, p)$ defined by (1.8), we propose a new extension of generalized Hurwitz-Lerch Zeta function by changing

$$\frac{(\mu)_n}{(\nu)_n} = \frac{B(\mu+n,\, \nu-\mu)}{B(\mu,\, \nu-\mu)} \to \frac{B(\mu+n,\, \nu-\mu\,;\, p)}{B(\mu,\, \nu-\mu)}$$

in (1.4) and thus, define this new extended form as follows:

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) := \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B(\mu+n,\, \nu-\mu\,;\, p)}{B(\mu,\, \nu-\mu)} \frac{z^n}{(n+a)^s} \tag{2.1}$$

$\big(p \geqq 0;\ \lambda, \mu \in \mathbb{C};\ \nu,\ a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ s \in \mathbb{C}$ when $|z| < 1;\ \Re(s+\nu-\lambda-\mu) > 1$ when $|z| = 1\big)$.

Remark 2.1. The following special or limit cases of the generalized Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}(z,s,a;p)$ defined by (2.1) are worth noting.

Case 1. If we set $\lambda = 1$, then (2.1) yields the following special case of another extended form of the generalized Hurwitz-Lerch Zeta function $\Phi^{1,1}_{\mu;\nu}(z,s,a;p)$ studied earlier by Lin and Srivastava [9, p. 727, Eq. (8)] (with $\rho = \sigma = 1$):

$$\Phi^{1,1}_{\mu;\nu}(z,s,a;p) := \Phi_{1,\mu;\nu}(z,s,a;p) := \sum_{n=0}^{\infty} \frac{B(\mu+n,\, \nu-\mu\,;\, p)}{B(\mu,\, \nu-\mu)} \frac{z^n}{(n+a)^s} \tag{2.2}$$

$\big(p \geqq 0;\ \mu \in \mathbb{C};\ \nu,\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C}$ when $|z| < 1;\ \Re(s+\nu-\mu) > 1$ when $|z| = 1\big)$.

Case 2. Upon setting $\lambda = \nu = 1$ in (2.1), we have the extended generalized Hurwitz-Lerch Zeta function $\Phi^*_\mu(z,s,a;p)$ of Goyal and Laddha [8] in the following form:

$$\Phi^*_\mu(z,s,a;p) := \Phi_{1,\mu;1}(z,s,a;p) := \sum_{n=0}^{\infty} \frac{B(\mu+n,\, 1-\mu\,;\, p)}{B(\mu,\, 1-\mu)} \frac{z^n}{(n+a)^s} \tag{2.3}$$

$\big(p \geqq 0;\ \mu \in \mathbb{C};\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C}$ when $|z| < 1;\ \Re(s+1-\mu) > 1$ when $|z| = 1\big)$.

Case 3. A limit case of the generalized Hurwitz-Lerch function $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$ defined by (2.1) (which is particularly of interest in our present investigation) is given by

$$\Phi^*_{\mu;\nu}(z, s, a; p) := \lim_{|\lambda|\to\infty} \{\Phi_{\lambda,\mu;\nu}\left(\frac{z}{\lambda}, s, a; p\right)\} := \sum_{n=0}^{\infty} \frac{B(\mu+n,\, \nu-\mu\,;\, p)}{B(\mu,\, \nu-\mu)} \frac{z^n}{n!(n+a)^s} \tag{2.4}$$

$\big(p \geqq 0;\ \mu \in \mathbb{C};\ \nu,\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C}$ when $|z| < 1;\ \Re(s+\nu-\mu) > 1$ when $|z| = 1\big)$.

Remark 2.2. The special cases of (2.1) and (2.3) when $p = 0$ are easily seen to reduce to (1.4) and (1.3), respectively, in view of the following relationships of the functions (defined above):

$$\Phi_{\lambda,\mu;\nu}(z, s, a; 0) = \Phi_{\lambda,\mu;\nu}(z, s, a) \quad \text{and} \quad \Phi_{1,\mu;1}(z, s, a; 0) = \Phi^*_{\mu}(z, s, a),$$

Also, for $p = 0$ in (2.2), we have the following special case of the generalized Hurwitz-Lerch function of Lin and Srivastava [9, p. 727, Eq. (8)] (with $\rho = \sigma = 1$):

$$\Phi_{1,\mu;\mu}(z, s, a; 0) = \Phi^{1,1}_{\mu;\nu}(z, s, a).$$

3 Integral Representations and Derivative Formula

In this section, we present certain integral representations and a derivative formula for the extended Hurwitz-Lerch Zeta function defined by (2.1).

Theorem 3.1. *The following integral representation holds true:*

$$\Phi_{\lambda,\mu;\nu}(z, s, a; p) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at}\, F_p(\lambda, \mu; \nu; ze^{-t}) dt \tag{3.1}$$

$\big(p \geqq 0;\ \Re(a) > 0;\ \Re(s) > 0$ *when* $|z| \leqq 1\ (z \neq 1);\ \Re(s) > 1$ *when* $z = 1\big)$.

Proof. Using the following Eulerian integral:

$$\frac{1}{(n+a)^s} := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(n+a)t} dt \quad (\min\{\Re(s),\, \Re(a)\} > 0;\ n \in \mathbb{N}_0) \tag{3.2}$$

in (2.1), we have

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \frac{B(\mu+n,\, \nu-\mu\,;\, p)}{B(\mu,\, \nu-\mu)} \left(\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+a)t} dt \right).$$

Interchanging the order of summation and integration under the condition stated in Theorem 3.1, we get

$$\Phi_{\lambda,\mu;\nu,}(z,s,a;p) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} \left(\sum_{n=0}^{\infty} (\lambda)_n \frac{B(\mu+n,\, \nu-\mu\,;\, p)}{B(\mu,\, \nu-\mu)} \frac{(ze^{-t})^n}{n!} \right) dt.$$

Now using the definition (1.9) to express the involved sum as an extended hypergeometric function, we are led to the desired result. □

Theorem 3.2. *The following integral representations hold true:*

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) = \frac{e^{-2p}\,\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^{\infty} \frac{x^{\mu-1}}{(1+x)^{\nu}} \exp\left(-p\left(x+\frac{1}{x}\right)\right) \Phi_{\lambda}^{*} \left(\frac{zx}{1+x}, s, a\right) dx \tag{3.3}$$

$$\big(p \geqq 0;\ \Re(\nu) > \Re(\mu) > 0\big)$$

and

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) = \frac{e^{-2p}\,\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)} \times \int_0^{\infty}\int_0^{\infty} \frac{t^{s-1}e^{-at}x^{\mu-1}}{(1+x)^{\nu}} \exp\left(-p\left(x+\frac{1}{x}\right)\right) \left(1-\frac{zxe^{-t}}{1+x}\right)^{-\lambda} dt\,dx \tag{3.4}$$

$$\big(p \geqq 0;\ \Re(\nu) > \Re(\mu) > 0,\ \min\{\Re(s),\, \Re(a)\} > 0\big),$$

provided that the integrals in the right-hand sides of the assertions (3.3) *and* (3.4) *converge.*

Proof. By setting $\alpha = \mu + n$ and $\beta = \nu - \mu$ in the following integral representation of the extended Beta-function (see, *e.g.*,[1, p. 22, Equation (2.8)]):

$$B(\alpha,\, \beta; p) = e^{-2p} \int_0^{\infty} \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}} \exp\left(-p\left(x+\frac{1}{x}\right)\right) dx \qquad (\Re(p) > 0), \tag{3.5}$$

$$(p \geqq 0;\ \Re(\alpha) > 0,\ \Re(\beta) > 0)$$

we find that

$$B(\mu+n,\,\nu-\mu;p) = e^{-2p} \int_0^\infty \frac{x^{\mu+n-1}}{(1+x)^{\nu+n}} \exp\left(-p\left(x+\frac{1}{x}\right)\right) dx, \tag{3.6}$$

which by appealing to the (2.1) and using (1.3) immediately yields the first assertion (3.3) of Theorem 3.2.

Also, in view of (3.1) and (3.6), we also obtain

$$\begin{aligned}\Phi_{\lambda,\mu;\nu}(z,s,a;p) &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} \sum_{n=0}^\infty (\lambda)_n \frac{B(\mu+n,\,\nu-\mu\,;\,p)}{B(\mu,\,\nu-\mu)} \frac{(ze^{-t})^n}{n!} dt \\ &= \frac{e^{-2p}\,\Gamma(\nu)}{\Gamma(s)\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty \int_0^\infty \frac{t^{s-1}e^{-at}x^{\mu-1}}{(1+x)^\nu} \exp\left(-p\left(x+\frac{1}{x}\right)\right) \\ &\sum_{n=0}^\infty \frac{(\lambda)_n}{n!} \left(\frac{zxe^{-t}}{1+x}\right)^n dt\,dx,\end{aligned} \tag{3.7}$$

which in view of the binomial expansion

$$(1-zt)^{-a} = \sum_{n=0}^\infty (a)_n \frac{(zt)^n}{n!}$$

leads us to the second assertion (3.4) of Theorem 3.2. □

Theorem 3.3. *The following integral representation holds true:*

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) := \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t} \Phi^*_{\mu;\nu}(zt,s,a;p) dt \tag{3.8}$$

$\big(p \geqq 0;\ \Re(\lambda) > 0,\ \Re(a) > 0;\ \Re(s) > 0$ *when* $|z| \leqq 1\ (z \neq 1);\ \Re(s) > 1$ *when* $z = 1\big)$, *where* $\Phi^*_{\mu;\nu}(z,s,a;p)$ *is the limiting case defined by* (2.4).

Proof. Using the integral representation of the Pochhammer symbol $(\lambda)_n$:

$$(\lambda)_n = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+n-1} e^{-t}\, dt$$

in (2.1) and inverting the order of summation and integration (which is permissible under the conditions stated with Theorem 3.3), we get

$$\Phi_{\lambda,\mu;\nu}(z,s,a;p) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-t} \sum_{n=0}^\infty \frac{B(\mu+n,\,\nu-\mu\,;\,p)}{B(\mu,\,\nu-\mu)} \frac{(zt)^n}{n!(n+a)^s}\, dt. \tag{3.9}$$

Applying now (2.4), we get the desired integral representation. □

We next derive the following derivative formula for the function $\Phi_{\lambda,\mu;\nu}(z,s,a;p)$ defined by (2.1).

Theorem 3.4. *The following derivative formula for $\Phi_{\lambda,\mu;\nu}(z,s,a;p)$ holds true:*

$$\frac{d^n}{dz^n}\left\{\Phi_{\lambda,\mu;\nu}(z,s,a;p)\right\} = \frac{(\lambda)_n(\mu)_n}{(\nu)_n}\Phi_{\lambda+n,\mu+n;\nu+n}(z,s,a+n;p) \qquad (n \in \mathbb{N}_0)\,. \tag{3.10}$$

Proof. Differentiating (2.1) with respect to z, we obtain

$$\frac{d}{dz}\left\{\Phi_{\lambda,\mu;\nu}(z,s,a;p)\right\} = \sum_{n=1}^{\infty}\frac{(\lambda)_n}{(n-1)!}\frac{B(\mu+n,\,\nu-\mu\,;\,p)}{B(\mu,\,\nu-\mu)}\frac{z^{n-1}}{(n+a)^s}. \tag{3.11}$$

which upon replacing n by $n+1$ in the right-hand side of (3.11), and using the identities that

$$B(\mu,\,\nu-\mu) = \frac{\nu}{\mu}\,B(\mu+1,\,\nu-\mu) \quad \text{and} \quad (a)_{n+1} = a\,(a+1)_n$$

gives the derivative formula that

$$\frac{d}{dz}\left\{\Phi_{\lambda,\mu;\nu}(z,s,a;p)\right\} = \frac{\lambda\,\mu}{\nu}\Phi_{\lambda+1,\mu+1;\nu+1}(z,s,a+1;p).$$

A repeated application of the this above procedure n-times gives us the general formula (3.10). □

4 Mellin Transform and Generating Relations

In this section, we obtain Mellin transform and certain generating relations for the extended Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\nu}(z,s,a;p)$ defined by (2.1).

The Mellin transform of a suitable integrable function $f(t)$ with index α is defined, as usual, by

$$\mathcal{M}\left\{f(\tau) : \tau \to \alpha\right\} := \int_0^{\infty}\tau^{\alpha-1}\,f(\tau)\,d\tau, \tag{4.1}$$

provided that the improper integral in (4.1) exists.

Theorem 4.1. *The Mellin transform of the function* $\Phi_{\lambda,\mu;\nu}(z,s,a;p)$ *defined by* (2.1) *is given by*

$$\mathcal{M}\{\Phi_{\lambda,\mu;\nu}(z,s,a;p)(z,s,a):p\to\alpha\} = \frac{\Gamma(\alpha)B(\mu+\alpha,\,\nu-\mu+\alpha)}{B(\mu,\,\nu-\mu)}\,\Phi_{\lambda,\mu+\alpha,\nu+2\alpha}(z,s,a) \tag{4.2}$$

$$(\Re(\alpha)>0 \quad \textit{and} \quad \Re(\lambda+\alpha)>0).$$

Proof. Using the definition (4.1) of the Mellin transform, we find from (2.1) that

$$\begin{aligned}\mathcal{M}\{\Phi_{\lambda,\mu;\nu}(z,s,a;p):p\to\alpha\} &:= \int_0^\infty p^{\alpha-1}\Phi_{\lambda,\mu;\nu}(z,s,a;p)dp\\ &= \int\limits_0^\infty p^{\alpha-1}\left(\sum_{n=0}^\infty \frac{(\lambda)_n}{n!}\frac{B(\mu+n,\,\nu-\mu\,;\,p)}{B(\mu,\,\nu-\mu)}\frac{z^n}{(n+a)^s}\right)dp\\ &= \sum_{n=0}^\infty \frac{(\lambda)_n}{n!}\frac{z^n}{(n+a)^s}\frac{1}{B(\mu,\,\nu-\mu)}\int_0^\infty p^{\alpha-1}B(\mu+n,\,\nu-\mu\,;\,p)\,dp.\end{aligned}$$

Applying the following result given by ([1, p. 21, Eq. (2.1)])

$$\int\limits_0^\infty p^{\alpha-1}B(x,\,y\,;\,p)\,dp = \Gamma(\alpha)B(x+\alpha,\,y+\alpha) \tag{4.3}$$

$$(\Re(\alpha)>0,\ \ \Re(x+\alpha)>0,\ \ \Re(y+\alpha)>0,$$

we obtain

$$\begin{aligned}\mathcal{M}\{\Phi_{\lambda,\mu;\nu}(z,s,a;p):p\to\alpha\} &=\\ &= \Gamma(\alpha)\sum_{n=0}^\infty \frac{(\lambda)_n}{n!}\frac{z^n}{(n+a)^s}\frac{B(\mu+\alpha+n,\,\nu-\mu+\alpha)}{B(\mu,\,\nu-\mu)}\\ &= \frac{\Gamma(\alpha)\,B(\mu+\alpha,\,\nu-\mu+\alpha)}{B(\mu,\,\nu-\mu)}\sum_{n=0}^\infty \frac{(\lambda)_n(\mu+\alpha)_n}{(\nu+2\alpha)_n}\frac{z^n}{n!(n+a)^s},\end{aligned} \tag{4.4}$$

which upon using (2.1) gives the desired Mellin transform representation. □

Remark 4.1. The special case of (4.2) when $\alpha = 1$ yields the following integral of the function $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$ with respect to the variable p in terms of the generalized Hurwitz-Lerch Zeta function defined by (1.4):

$$\int_0^\infty \Phi_{\lambda,\mu;\nu}(z, s, a; p)\, dp = \frac{\Gamma(\alpha)B(\mu + 1,\, \nu - \mu + 1)}{B(\mu,\, \nu - \mu)}\, \Phi_{\lambda,\mu+1,\nu+2}(z, s, a). \quad (4.5)$$

We next derive the following generating relations for the function $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$ defined by (2.1).

Theorem 4.2. *The following generating function for* $\Phi_{\lambda,\mu;\nu}(z, s, a; p)$ *in* (2.1) *holds true:*

$$\sum_{n=0}^\infty (\lambda)_n\ \Phi_{\lambda+n,\mu;\nu}(z, s, a; p)\, \frac{t^n}{n!} = (1 - t)^{-\lambda}\ \Phi_{\lambda+n,\mu;\nu}\left(\frac{z}{1 - t}, s, a; p\right) \quad (4.6)$$

$$(p \geqq 0,\ \lambda \in \mathbb{C}\ \textit{and}\ |t| < 1).$$

Proof. Let the left-hand side of the assertion (4.6) be denoted by S. Then, from the definition (2.1), we have

$$\begin{aligned} S &= \sum_{n=0}^\infty (\lambda)_n \left\{ \sum_{k=0}^\infty (\lambda + n)_k\, \frac{B(\mu + k,\, \nu - \mu\,;\, p)}{B(\mu,\, \nu - \mu)}\, \frac{z^k}{k!(k + a)^s} \right\} \frac{t^n}{n!} \\ &= \sum_{k=0}^\infty (\lambda)_k\, \frac{B(\mu + k,\, \nu - \mu\,;\, p)}{B(\mu,\, \nu - \mu)} \left\{ \sum_{n=0}^\infty (\lambda + k)_n\, \frac{t^n}{n!} \right\} \frac{z^k}{k!(k + a)^s} \end{aligned}$$

upon reversal of the order of summation and use of the identity $(\lambda)_n(\lambda+n)_k = (\lambda)_k(\lambda + k)_n$.

Using the binomial expansion

$$(1 - t)^{-\lambda-k} = \sum_{n=0}^\infty (\lambda + k)_n\, \frac{t^n}{n!} \qquad (|t| < 1),$$

and interpreting in terms of (2.1) as a function of the form $\Phi_{\lambda+n,\mu;\nu}\left(\frac{z}{1-t}, s, a; p\right)$, we are thus lead to the assertion (4.6) of Theorem 4.2. □

Our next result on the generating function is contained in the following:

Theorem 4.3. *The following generating function for* $\Phi_{\lambda,\mu;\nu}(z,s,a;p)$ *in* (2.1) *holds true:*

$$\sum_{n=0}^{\infty} \frac{(s)_n}{n!}\, \Phi_{\lambda,\mu;\nu}(z,s+n,a;p)\, t^n = \Phi_{\lambda,\mu;\nu}(z,s,a-t;p) \tag{4.7}$$

$$(p \geqq 0,\ \lambda \in \mathbb{C} \text{ and } |t| < |a|; s \neq 1).$$

Proof. Using (2.1) in the right-hand side of the assertion (4.7), we have

$$\begin{aligned}
\Phi_{\lambda,\mu;\nu}(z,s,a-t;p) &= \sum_{k=0}^{\infty}(\lambda)_k \frac{B(\mu+k,\,\nu-\mu\,;\,p)}{B(\mu,\,\nu-\mu)} \frac{z^k}{k!(k+a-t)^s} \\
&= \sum_{k=0}^{\infty}(\lambda)_k \frac{B(\mu+k,\,\nu-\mu\,;\,p)}{B(\mu,\,\nu-\mu)} \frac{z^k}{k!(k+a)^s} \left(1-\frac{t}{k+a}\right)^{-s} \\
&= \sum_{k=0}^{\infty}(\lambda)_k \frac{B(\mu+k,\,\nu-\mu\,;\,p)}{B(\mu,\,\nu-\mu)} \frac{z^k}{k!(k+a)^s} \left\{\sum_{n=0}^{\infty} \frac{(s)_n}{n!} \frac{t^n}{(k+a)^n}\right\} \\
&= \sum_{n=0}^{\infty} \frac{(s)_n}{n!} \left\{\sum_{k=0}^{\infty}(\lambda)_k \frac{B(\mu+k,\,\nu-\mu\,;\,p)}{B(\mu,\,\nu-\mu)} \frac{z^k}{k!(k+a)^{s+n}}\right\} t^n.
\end{aligned}$$

The desired assertion (4.7) of Theorem 4.3 follows now on making use of (2.1).

□

5 Application to the Probability Distributions

In this concluding section, we consider a general probability distribution involving the extended generalized Hurwitz-Lerch Zeta function (2.1) defined as follow:

Definition 5.1. *A random variable* ξ *is said to be the generalized Hurwitz distributed if its probability density function is given by*

$$f_\xi(a) =: \begin{cases} \dfrac{s\Phi_{\lambda,\mu;\nu}(z,s+1,a;p)}{\Phi_{\lambda,\mu;\nu}(z,s,1;p)}, & a \geqq 1 \\[2ex] 0 & \textit{otherwise}, \end{cases} \tag{5.1}$$

since one can easily verify that

$$\int_1^{\infty} f_\xi(a)\, da = 1,$$

where it is tacitly assumed that the arguments z, s, p and the parameters λ, μ and ν are fixed and suitably constrained so that the probability density function $f_\xi(a)$ remains nonnegative.

Theorem 5.1. *Let ξ be a continuous random variable with its probability density function defined by* (5.1). *Then, the moment generating function $M(t)$ of the random variable ξ is given by*

$$M(t) = E_s[e^{\xi t}] = \sum_{n=0}^{\infty} E_s[\xi^n] \frac{t^n}{n!}, \tag{5.2}$$

where the moments $E_s[\xi^n]$ of order n are given by

$$E_s[\xi^n] = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{\Gamma(s-k)}{\Gamma(s)} \frac{\Phi_{\lambda,\mu;\nu}(z, s-k, 1; p)}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)}. \tag{5.3}$$

Proof. The assertion in (5.2) can be derived easily by using the series expansion of $e^{\xi t}$. To establish (5.3), we observe that

$$\frac{d}{da} \{\Phi_{\lambda,\mu;\nu}(z, s, a; p)\} = -s\Phi_{\lambda,\mu;\nu}(z, s+1, a; p), \tag{5.4}$$

which follows readily from (2.1), and thus from the definition of $E_s[\xi^n]$, we have

$$\begin{aligned} E_s[\xi^n] &= \int_1^\infty a^n f_\xi(a) da \\ &= \frac{s}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^n \Phi_{\lambda,\mu;\nu}(z, s+1, a; p) da \\ &= -\frac{1}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^n \frac{d}{da} \{\Phi_{\lambda,\mu;\nu}(z, s, a; p)\} da \\ &= \left[-\frac{a^n \Phi_{\lambda,\mu;\nu}(z, s, a; p)}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \right]_{a=1}^{\infty} + \\ &\qquad + \frac{n}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^{n-1} \Phi_{\lambda,\mu;\nu}(z, s, a; p) da \\ &= 1 - \lim_{a\to\infty} \left\{ \frac{a^n \Phi_{\lambda,\mu;\nu}(z, s, a; p)}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \right\} + \\ &\qquad + \frac{n}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^{n-1} \Phi_{\lambda,\mu;\nu}(z, s, a; p) da \\ &= 1 + \frac{n}{\Phi_{\lambda,\mu;\nu}(z, s, 1; p)} \int_1^\infty a^{n-1} \Phi_{\lambda,\mu;\nu}(z, s, a; p)\, da, \end{aligned} \tag{5.5}$$

where, in addition to the derivative property (5.4), we have used the following limit formula:

$$\begin{aligned}\lim_{a\to\infty}\{a^n\Phi_{\lambda,\mu;\nu}(z,s,a;p)\} &= \lim_{a\to\infty}\left\{\frac{a^n}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\ F_p(\lambda,\mu;\nu;ze^{-t})dt\right\}\\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\lim_{a\to\infty}\left\{a^n e^{-at}\right\}\ F_p(\lambda,\mu;\nu;ze^{-t})dt\\ &= 0. \end{aligned}\tag{5.6}$$

Consequently, we have the following reduction formula for $E_s[\xi^n]$:

$$E_s[\xi^n] = 1 + \frac{\Phi_{\lambda,\mu;\nu}(z,s-1,1;p)}{\Phi_{\lambda,\mu;\nu}(z,s,1;p)}\frac{n}{s-1}E_{s-1}[\xi^{n-1}], \tag{5.7}$$

and by iterating the recurrence (5.5), we arrive at the desired result (5.3). □

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