Correction of Dynamical Network's Viability by Decentralization by Price

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A connectionist system of a finite set of autonomous agents evolving independently over a common centralized environment of scarce resources is discussed and connected with the results of the agents' interactions by the connection operator, also evolving independently. The system forms a dynamical network.

The network is viable if a joint evolution satisfies the centralized scarcity constraints set by the environment. The focus of this paper is on the problem of restoring the network's viability, which is intrinsic as the decentralized behaviors (dynamics) of the agents and of the connection operator are not necessarily consistent with the centralized constraints. For restoring the viability, the decentralized dynamics are corrected using viability multipliers, which are regarded as correction prices. The correction prices provide the information about changes in the dynamics, necessary to govern evolutions satisfying the constraints. In this aspect, the viability of the network is restored by the mechanism of decentralization by price.

1. Introduction

In this paper, we address systems consisting of a fixed number of individual agents, each of which is characterized by his behavioral position in a space at a time. Agents' behavior is modeled by the agents' states, which evolve within a common environment according to the specified rule of state dynamics. The agents are connected by a linear connection operator that also evolves. This connection operator maps the agents' states into their collective results. Since the agents' states evolve in a common environment, the agents' collective results necessarily face constraints which, if satisfied, ensure viability of the environment. For simplicity, we assume the constraints to be unchanging and we refer to them as the viability constraints of the system.

We assume the agents behave according to their individual purposes, independently of others, since their knowledge about the whole system is limited or due to their own objectives and attitudes. The connection operator, in turn, behaves independently of the agents, therefore representing an autonomous connecting unit. Thus, the network's dynamics are *decentralized*, while the collective viability constraints are *centralized*.

The network is described as follows. Consider finite-dimensional vector spaces $X_1, X_2, \ldots, X_n$, Y, and Z, and the product $X = X_1 \times \cdots \times X_n$. Denote by $\mathcal{L}(X, Y)$ the space of all linear operators from X to Y. Introduce a linear operator $V \in \mathcal{L}(X, Y)$, n maps $f_i : X_i \mapsto X_i$, a map $\beta : \mathcal{L}(X, Y) \mapsto \mathcal{L}(X, Y)$, and a map $g : Y \mapsto Z$.

The network's agents $1, \ldots, n$ are connected by the connection operator V through the agents' states $x = (x_1, \ldots, x_n) \in X$ with the agents' collective result $V x$. The agents' state evolutions are governed by the dynamics generated by the equations $x_i'(t) = f_i(x_i(t))$ and the evolution of the network's connection operator is governed by $V'(t) = \beta(V(t))$. The evolutions set by the network's data must be subject to viability constraints set by the environment and require that at any time $t \geq 0$, the consequence of agents' actions and their connections, that is, the map g applied to the agents' collective result $V(t)\, x(t)$, be restricted to remain in a subset $\mathcal{M}$ of Z.

The network is described by the dynamics

$$\begin{cases} \forall\ i = 1 \ldots n, & x_i'(t) = f_i(x_i(t)) \\ V'(t) = \beta(V(t)) \end{cases} \tag{1}$$

and its viability constraints are given by

$$\forall\ t \geq 0,\ \ g(V(t)\, x(t)) \in \mathcal{M}. \tag{2}$$

To illustrate, consider the network of four agents 1, 2, 3, 4 with the corresponding states $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ connected by the connection operator

$$V(t) = \begin{pmatrix} v_{11}(t) & v_{12}(t) \\ v_{21}(t) & 0 \\ 0 & v_{32}(t) \\ v_{41}(t) & v_{42}(t) \end{pmatrix}$$

to their collective results $y_1(t)$, $y_2(t)$ (see Figure 1). The results are connected with the constrained results $z_1(t)$, $z_2(t)$, $z_3(t)$ (marked by the gray nodes in Figure 1) by map $g(y_1(t), y_2(t)) = (y_1(t), y_1(t) + y_2(t), y_2(t))$.

The scheme of the network is similar to the simplified view of a neural network with the input layer consisting of the agents' states, the hidden layer comprising the agents' collective results, and the output layer consisting of the constrained results [1].

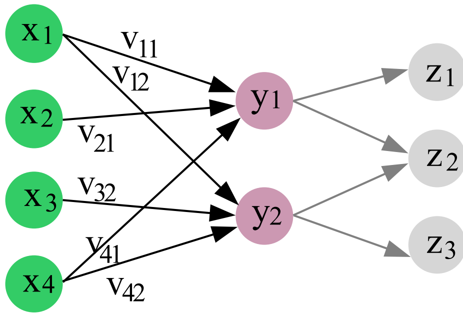

Figure 1. Scheme of the network of agents' states and their collective results.

By the *environment of the network*, we understand the set of all initial states $(x, V) \in X \times \mathcal{L}(X, Y)$ satisfying the viability constraints. We say that the environment of the network is viable under the network's dynamics if from any of the initial states at least one evolution governed by the dynamics that satisfies the viability constraints is started. The network is said to be viable if the environment of the network is viable under the network's dynamics.

The network most likely does not remain viable forever when the dynamics of the agents and that of the connection operator are left to evolve by themselves. This is due to the absence of mutual dependencies between the dynamics that are required to satisfy the centralized viability constraints.

The main question we deal with in this paper is that of restoring the viability of the network representing a decentralized model. We control the network's viability by modifying the decentralized settings in order to transfer to the centralized ones, satisfying the viability constraints.

The recent literature on the control of network viability [2, 3] concerns the systems of individual agents whose states' evolutions are restricted by viability constraints, and handles the question of restoring the viability assuming that the agents' connection operator is not changing [4] or providing only a general control frame for multilinear connection operators [3]. In practical applications, some of which are listed below, there is an interaction between the agents and their connection operator, and it is reasonable to correct the dynamics of both collectively. Therefore, in this paper, we bring into the model the connection operator that changes dynamically and treat the problem of restoring the viability as a problem of regulation of dynamics of both the agents' states and the connection operator. We use the approach to restore the viability provided in viability theory.

Fields where the problem of our interests is at their heart include economics, where evolving economic systems faced with scarcity constraints are studied. The fundamental model of resource allocations is replaced by a decentralized dynamic framework where the prices follow the regulation law represented as a function of the allocations [2, 5]. Other recent researches that study evolving economic systems are [6] and [7], where prices are used in the adjustment processes.

Another field which is of great interest is that of neural networks and cognitive sciences. In this case, the neural networks and cognitive systems are regarded as dynamical systems controlled by synaptic matrices [8].

Dynamical connectionist networks and dynamical cooperative games are also the fields where the problem has a central role. Here, the authors of [3] provide a class of control systems able to govern the evolution of actions, coalitions, and multilinear connection operators under which the architecture of a network remains viable. The controls are tensor products of the coalitions' actions and of multipliers of the viability constraints space, which allows the concept of Hebbian learning rules in neural networks to be encapsulated in the dynamical framework. They also use the viability and capturability approach to study the problem of characterizing the dynamic core of a dynamic cooperative game defined in a characteristic function form. Another recent work is [9], where the control of dynamics of a communication network was realized using a stochastic approach.

Recently, a lot of research attention has been given to sociological sciences. There, a society can be interpreted as a set of individuals that are subjected to survival or social constraints. Laws and cultural codes can be devised to provide each individual with psychological or economical means and guidelines that play the role of regulation controls [10, 11].

In this paper, we tackle the problem of restoring the viability by correcting the network's dynamics using regulatory parameters, which introduce the missing mutual dependencies. The parameters represent control units regulating the dynamics of the network's components—the agents and their connection operator. We denote the regulatory parameters by $p(t)$ and $P(t)$, where $p(t)$ is a vector and $P(t)$ is a linear mapping. The viable corrected network has the form of

$$\begin{cases} x'(t) = f(x(t)) - p(t) \\ V'(t) = \beta(V(t)) - P(t) \end{cases} \tag{3}$$

where $p(t) = \mathbf{0}$ and $P(t) = \mathbf{O}$, the zero vector and the zero linear operator, which cover the initial decentralized dynamics.

Parameter $p(t)$ is called the *viability multiplier* and parameter $P(t)$ is called the *viability connection operator*.

We prove that there is a common regulatory parameter $q(t)$, called a *correction price*, such that the viability multiplier $p(t)$ and the viability connection operator $P(t)$ are derived through another linear operator applied to $q(t)$.

We show that, under adequate assumptions, a parameter $q_o(t)$ that minimizes the norm $\|q(t)\|$ of the correction price can be selected from the correction prices regulating the network's viability. In this sense $q_o(t)$ defines optimal $p_o(t)$ and $P_o(t)$.

The correction price provides the information about the changes in the network's decentralized dynamics necessary to govern evolutions satisfying the centralized constraints. This is the meaning of restoring the viability of decentralized dynamics of the network by the *decentralization by price*. The corrected network shown in equation (3) is said to be decentralized by price.

2. Prerequisites from Viability Theory

Consider a model that consists of n agents, each of which is characterized by its state. An i^{th} agent's state $x_i(t)$ ranges over a vector space X_i with time t. The vector space X_i is referred to as the agent i's state space, and the finite dimensional vector space $X = \Pi_{i=1}^{n} X_i$ of elements $\{x = (x_1, \ldots, x_n)\}$ is referred to as a collective state space.

Each agent's state evolves independently from other agents. The evolution of the state of an agent i is governed by the dynamics of the state: $x_i'(t) = f_i(x_i(t))$. The map $f_i : X_i \mapsto X_i$ depends on the state x_i and not on the other agents' states, which reflects the independence of the agents' dynamics.

A subset K of the state space X is regarded as an environment of viability of the system, in which the agents' state $x(t) = (x_1(t), \ldots, x_n(t))$ must remain at any time $t \geq 0$. In our framework, the environment of viability is described through the viability constraints as follows. Given a finite dimensional vector space Z and a subset $\mathcal{M}$ of Z, the viability constraints defined by a map $h : X \mapsto Z$ are

$$h(x) \in \mathcal{M}.$$

Then, the environment of viability K can be written explicitly as $K = \{x \mid h(x) \in \mathcal{M}\}$.

Thus, the system of the dynamics of the agents' states and the viability constraints is written as

$$\forall\, i = 1 \ldots n, \qquad x_i'(t) = f_i(x_i(t)) \tag{4}$$

$$\forall\, t \geq 0, \qquad h(x(t)) \in \mathcal{M}. \tag{5}$$

Definition 1. (Viable Environment) The environment K defined by the viability constraints in equation (5) is viable under the dynamics in equation (4) if from any initial state $(x_1, \ldots, x_n) \in K$ starts at least one evolution governed by the dynamics that is viable in K.

Since there is no reason why the system, left to evolve by itself, shall always remain viable, the question of restoring the viability of the system arises. The question is resolved using the method of the viability multipliers as we describe in Section 2.1.

2.1 The Viability Theorem and Viability Multipliers

Let X be a finite dimensional vector space. We denote by P a cone, by $\overline{P}$ its closure, and by $\overline{co}\,P$ its closed convex hull. The polar cone of P is denoted by $P^- = \{p \in X^* \mid \forall\, x \in P, \langle p, x\rangle \le 0\}$.

Definition 2. (Tangent Cone) Consider a subset K of a finite dimensional vector space X and a vector x in K. The *tangent cone* (or the contingent cone of Bouligand) $T_K(x)$ to set K at x is the closed cone

$$T_K(x) = \left\{u \in X \,\middle|\, \liminf_{h\to 0+} \frac{d(x+hu, K)}{h} = 0\right\}, \tag{6}$$

which coincides with the whole space X if x belongs to the interior of K.

For a convex set K, the tangent cone coincides with the tangent cone of convex analysis, which is the closed cone spanned by $K - x$: $T_K(x) = \overline{\bigcup_{h>0} \frac{K-x}{h}}$.

Thus, set $\overline{co}\,(T_K(x))$ is the closed convex hull of the tangent cone $T_K(x)$.

Definition 3. (Normal Cone) The *normal cone* to a subset K of the vector space X at a point $x \in X$, denoted by $N_K(x)$, is defined to be

$$N_K(x) := T_K(x)^- = (\overline{co}\,(T_K(x)))^-. \tag{7}$$

Definition 4. (Sleekness) A subset K of the vector space X is said to be *sleek* if the graph of the mapping $x \to N_K(x)$ is closed.

Let the vector space X be supplied with a scalar product l with the norm λ, $\lambda(x) = \|x\|$, and let L be the duality map on X associated with the scalar product.

Definition 5. (Marchaud Set-Valued Map) A set-valued map $F : X \to Y$ is called *Marchaud* if it has a closed graph, convex values, and a linear growth defined by

$$\lambda(F(x)) := \sup_{v \in F(x)} \lambda(v) \le c(\lambda(x) + 1) \text{ for some constant } c.$$

We denote the agents' dynamics as a whole by $x'(t) = f(x(t))$, where $x(t) = (x_1(t), x_2(t) \ldots, x_n(t))$ is the collective state of agents and $f(x) = (f_1(x_1), \ldots, f_n(x_n))$.

Theorem 1. (Nagumo Viability Theorem) Let K be a closed subset of vector space X and $f : X \mapsto X$ be a continuous map with a linear growth. Then, K is viable under the differential equation $x' = f(x)$ if and only if for any $x \in K$, the dynamics and the constraints are linked by the following relation:

$$\forall\, x \in K, \qquad f(x) \in \overline{co}\,(T_K(x)). \tag{8}$$

The general approach to restoring viability is to replace function f in the differential equation $x' = f(x)$ by a correction map $\tilde{f}$ that satisfies the requirements of the Nagumo viability theorem.

We define a *viability discrepancy* to be the distance between the initial and the corrected dynamics and we denote it by $c(x) = \lambda\big(f(x) - \tilde{f}(x)\big)$. Obviously, the minimal viability discrepancy $c_o(x)$ is achieved for the best approximation projection of $f(x)$ on the closed convex hull of the tangent cone, $\tilde{f}(x) = \Pi_{\overline{co}\,(T_K(x))} f(x)$, since

$$c_o(x) = \lambda\big(f(x) - \Pi_{\overline{co}\,(T_K(x))} f(x)\big) = \inf_{u \in \overline{co}\,(T_K(x))} \lambda(f(x) - u).$$

Because $\overline{co}\,(T_K(x))$ is a closed convex cone and $N_K(x)$ is its polar cone, then the Moreau projection theorem (see [12]) implies that $f(x)$ can be written as $f(x) = \Pi_{\overline{co}\,(T_K(x))} f(x) + L^{-1}\, \Pi_{N_K(x)}\, L f(x)$. If we set

$$p_o = \Pi_{N_K(x)}\, L f(x), \tag{9}$$

the correction function can be represented as $\tilde{f}(x) = f(x) - L^{-1}\, p_o(x)$, where $p_o(x)$ is considered as a regulatory parameter.

Motivated by this representation, we consider a general correction in the form of

$$x'(t) = f(x(t)) - L^{-1}\, p(t), \tag{10}$$

where $p(t) \in X^*$ is a regulatory parameter belonging to the set of all the regulatory parameters providing the viability corrections—the *regulation map*:

$$R_K(x(t)) = \left\{p \in X^* \,\middle|\, f(x(t)) - L^{-1}\, p \in \overline{co}\,(T_K(x(t)))\right\}. \tag{11}$$

Such regulatory parameters are referred to as *viability multipliers*.

Clearly, the viability multiplier $p_o(x) = \Pi_{N_K(x)}\, L f(x)$ corresponds to the minimal viability discrepancy and belongs to the regulation map, $p_o(x) \in R_K(x)$. However, using only the Nagumo theorem, we cannot prove the viability of the correction since, though the tangential conditions are satisfied, the continuity properties are lost by projecting the map $f(x)$ onto the tangent cone.

In order to prove the viability of the correction for equations (10) and (11), we are helped by the following fundamental theorem.

Theorem 2. (Fundamental Viability Theorem) Consider the differential inclusion $x' \in F(x)$, where F is a set-valued map. If $F(x)$ is Marchaud, then K is viable under F if and only if $\forall\, x \in K,\, 0 \in F(x) - \overline{co}\,(T_K(x))$.

In the following theorem we present the requirements under which the correction for equations (10) and (11) restores the viability.

Theorem 3. (Restoring Viability) Denote by B a unit ball in X^*. Assume f to be continuous with linear growth and K to be sleek. Then, the environment K of the system from equations (4) and (5) is viable under the new dynamics in equations (10) and (11). Furthermore, it is viable under the correction with minimal viability discrepancy:

$$x'(t) = f(x(t)) - L^{-1}\, p_o(x(t)).$$

Proof. Since the map $x \mapsto f(x)$ is continuous and $x \to \overline{co}(T_K(x))$ is lower semi-continuous, we infer that the set-valued map $x \to c_o(x)\, B$, where $c_o(x) = d(f(x), \overline{co}(T_K(x)))\, B$, is upper semi-continuous thanks to the maximum theorem. The set-valued map $G : X \to X$ defined by $G(x) := f(x) - L^{-1}(c_o(x)\, B \bigcap N_K(x))$ is Marchaud because its graph is closed, its images are convex, and it has linear growth since

$$\forall\, x \in K,\;\; d(f(x), \overline{co}(T_K(x))) \leq \lambda(f(x)) \leq c(\lambda(x) + 1).$$

It remains to be proved that $G(x) \bigcap \overline{co}(T_K(x)) \;\neq \emptyset$.

Indeed, the Moreau theorem implies that the viability multiplier $p_o(x)$ minimizing the viability discrepancy is the projection $p_o(x) = \Pi_{N_K(x)}(L f(x))$ onto the normal cone $N_K(x)$ of $f(x)$, and $u_o(x) = f(x) - L^{-1}\, p_o(x)$ is equal to $\Pi_{\overline{co}\,(T_K(x))}\, f(x)$. So the viability multiplier $p_o(x)$ satisfies $p_o(x) \in c_o(x)\, B \bigcap N_K(x)$, and hence $u_o(x) = f(x) - L^{-1}\, p_o(x)$ belongs to $G(x) \bigcap \overline{co}\,(T_K(x))$. Thus, the assumptions of the fundamental viability theorem (Theorem 2) are satisfied, and we have proved that K is viable under the corrected differential inclusion $x'(t) \in G(x(t))$. □

Motivated by the economic interpretation [5], the correction in equations (10) and (11) is called viability correction by the decentralization by price.

2.2 Restoring Viability

The correction results described above deal with viability constraints written as $x \in K$. In the case of the explicit constraints $h(x) \in \mathcal{M}$, where $h : X \mapsto Z$ and $\mathcal{M} \subset Z$, the environment K can be defined in the form $K := \{x \in X \mid h(x) \in \mathcal{M}\}$.

Then, under the assumption that the function $h : X \mapsto Z$ is a continuously differentiable map such that its derivative $h'(x)$ is surjective and the set $\mathcal{M}$ is sleek, the tangent and the normal cones $T_K(x)$ and $N_K(x)$ can be described in terms of the tangent and the normal cones $T_{\mathcal{M}}(h(x))$ and $N_{\mathcal{M}}(h(x))$ by the formula

$$T_K(x) = h'(x)^{-1}\, T_{\mathcal{M}}(h(x)) \text{ and } N_K(x) = h'(x)^*\, N_{\mathcal{M}}(h(x)).$$

Hence, with additional assumptions on h and $\mathcal{M}$, the correction that restores the viability is defined in the following theorem.

Theorem 4. (Restoring Viability for Explicit Constraints) If $\mathcal{M} \subset Z$ is sleek, function $f : X \mapsto X$ is continuous with linear growth, and function $h : X \mapsto Z$ is a continuously differentiable map such that its differential $h'(x)$ is surjective, then $\mathcal{M}$ is a viability domain of

$$x' = f(x) - L^{-1}\, h'(x)^*\, q(x)$$

where $q(x)$ ranges over

$$R_{\mathcal{M}}(x) = \left\{ q(x) \in Y^* \,\middle|\, h'(x) f(x) - h'(x)\, L^{-1}\, h'(x)^*\, q(x) \in \overline{co}\, T_{\mathcal{M}}(h(x)) \right\}. \tag{12}$$

Particularly, taking $q_o(x) \in R_{\mathcal{M}}(x)$ can minimize the viability discrepancy

$$q_o(x) = \Pi_{N_{\mathcal{M}}(h(x))}\left(\left(h'(x)\, L^{-1}\, h'(x)^* \right)^{-1} h'(x) f(x) \right) \in R_{\mathcal{M}}(x). \tag{13}$$

Here, the viability multiplier $p \in X^*$ is equal to $h'(x)^*\, q(x)$, where $q(x) \in Z^*$ and the particular case when the minimal viability discrepancy is achieved corresponds to $p_o(x) = h'(x)^*\, q_o(x)$.

3. Network

We define the network as follows.

Definition 6. (Network) Given a linear operator $V \in \mathcal{L}(X, Y)$, maps $f : X \mapsto X$, $\beta : \mathcal{L}(X, Y) \mapsto \mathcal{L}(X, Y)$. Consider a system of n agents with the states $x = (x_1, \ldots, x_n) \in X$ that are governed by the decentralized dynamics $x_i'(t) = f_i(x_i(t))$ and connected by the connection operator V governed by the decentralized dynamics $V'(t) = \beta(V(t))$. The system defines a network of the agents' states connected to the agents' common results with the pattern of the connection $V x$.

We write the network as in equation (1):

$$\begin{cases} x'(t) = f(x(t)) \\ V'(t) = \beta(V(t)). \end{cases}$$

3.1 Network's Viability

When the agents' collective result $V(t)\,x\,(t)$ is restricted by the viability constraints in equation (2),

$$\forall\; t \geq 0, \qquad g(V(t)\,x(t)) \in \mathcal{M},$$

the question of the network's viability is raised.

By the environment of the network, we understand the set of pairs of agents' states and connection operators $(x,\, V)$ that satisfy the constraints. Evolutions governed by the network in equation (1)'s dynamics and satisfying viability constraints in equation (2) are called *viable evolutions*.

The environment of the network is said to be viable under the network's dynamics if for any initial state in the environment, there is at least one viable evolution starting from it. Then, we define a viable network as follows.

Definition 7. (Viable Network) The network is viable if the environment of the network is viable under the network's dynamics.

The network (equation (1)) with the viability constraints (equation (2)) that we study in this work represents a decentralized model that is characterized by the absence of mutual dependencies between the network's data—the agents and the connection operator. Therefore, there is nothing guaranteeing that the agents' states or the connection operator do not violate the centralized viability constraints. Hence, nothing guarantees the network's viability.

The main problem we relate in the present paper is the problem of viability of the network with decentralized dynamics evolving under the given (centralized) constraints. We solve the problem by correcting the network using the method of correction by decentralization by price.

3.2 Restoring the Network's Viability

We assume the spaces X and Y are supplied with the scalar products that define the duality maps $L : X \mapsto X^*$ and $M : Y \mapsto Y^*$. Note that the duality map H on $\mathcal{L}(X,\, Y)$ is equal to $H := L^{-1} \otimes M$.

Analogously to the correction by decentralization by price displayed in the formula in equations (10) and (11), we choose the parameters $p(t) \in X^*$ and $P(t) \in \mathcal{L}(Y^*,\, X^*)$ and write the correction of the network (equation (1)) with constraints (equation (2)) as the following:

$$\begin{cases} x'(t) = f(x(t)) - L^{-1}\,p(t) \\ V'(t) = \beta(V(t)) - H^{-1}\,P(t). \end{cases} \tag{14}$$

Theorem 5. (Restoring the Network's Viability) Consider the map $J_g(x, V) \in \mathcal{L}(Z, Z^*)$,

$$J_g(x, V) = \left[g'(Vx)\left[\lambda^2(x)\, M^{-1} + V\, L^{-1}\, V^*\right] g'(Vx)^*\right]^{-1},$$

the regulation map $R_{\mathcal{M}} : X \times \mathcal{L}(X, Y) \mapsto Z^*$, such that

$$R_{\mathcal{M}}(x, V) = \Big\{q \in Z^* \,\Big|\, g'(Vx)\,(\beta(V)\,x + Vf(x)) - J_g\left(x, V\right)^{-1} q \in \overline{co}\, T_{\mathcal{M}}(g(Vx))\Big\},$$

the element $q_o \in Z^*$,

$$q_o(x, V) = \Pi_{N_{\mathcal{M}(g(Vx))}}\left(J_g(x, V)\, g'(Vx)[\beta(V)\,x + Vf(x)]\right).$$

Assume maps f and β are continuous with linear growth, set $\mathcal{M} \subset Z$ is sleek, and map g is continuously differentiable such that derivative g' is surjective. Then, the network is viable under the corrected system

$$\begin{cases} x'(t) = f(x(t)) - L^{-1}\, V^*\, g'\,(Vx)^*\, q(t) \\ V'(t) = \beta(V(t)) - L\, x \otimes M^{-1}\, g'\,(Vx)^*\, q(t) \end{cases}$$

where the prices $q(x, V) \in R_{\mathcal{M}}(x, V)$. Particularly, the minimal correction price $q_o(x, V)$ belongs to $R_{\mathcal{M}}(x, V)$.

Note that operator $J_g(x, V)$ is a duality map on Z induced by the duality map on $X \times \mathcal{L}(X, Y)$.

As can be observed from these results, the centralized constraints in the network bring into the viable correction a factor of mutual dependencies between the agents' states and the connection operator. These mutual dependencies are encapsulated in the viability multiplier $p(t) \in X^*$ and the viability connection operator $P(t) \in \mathcal{L}(X, Y)$, which are defined by

$$\begin{aligned} p(t) &= V^*\, g'(Vx)^*\, q(t) \\ P(t) &= x \otimes g'(Vx)^*\, q(t). \end{aligned} \tag{15}$$

3.3 Example: Network of Agents Forming Coalitions

We now show the network's viability correction with the example of a network of agents (actors) forming coalitions. We refer to the model based on statistical physics allowing the reproduction of the interactions in formation of coalitions among agents. The detailed model and its social and political applications are described in [13].

In our framework, we consider the model as a multi-agent system with n agents, where each agent i belongs to one of two coalitions $\mathcal{A}$ or $\mathcal{B}$, in such a way that the state of the agent is $s_i = 1$ if the agent belongs to $\mathcal{A}$ and $s_i = -1$ if it belongs to $\mathcal{B}$. Interactions between any two agents depend on their bilateral mutual propensity, which is symmetric and can be positive or negative. The propensity of two different agents i and j is denoted by v_{ij}. The gain of agent i from its interactions with other agents is as follows:

$$H_i = \sum_{i \neq j} s_i v_{ij}.$$

The requirement of stability of the coalitions is satisfied when each agent i's gain is not less than its satisfactory minimum, $H_i \geq H_i^o$.

Stabilization of the coalitions is achieved due to the additional bilateral propensities p_{ij} between agents i and j, produced by supplementary exchanges between the agents. The additional propensities modify the overall propensity and the corrected gain becomes $H_i = \sum_{i \neq j} s_i(v_{ij} + p_{ij})$, which for chosen values of p_{ij} reaches the necessary satisfaction minimum.

This is the principle of the coalitions forming model. To be an example of our problem, the model must be a dynamic model and the coalitions must be fuzzy. In order to extend the model to the dynamic case, we assume the agents' states and the propensities are evolving with time according to given dynamics, and the coalitions formed by the agents are fuzzy. Let $x_i \in [0, 1]$ be the value of the agent i's fuzzy belonging to the coalitions, that is i is x_i in $\mathcal{A}$ and $1 - x_i$ in $\mathcal{B}$. Then, agent i's fuzzy state s_i can be expressed in the terms of the fuzzy belonging as $s_i = x_i - (1 - x_i) = -1 + 2x_i$.

Thus, we obtain a dynamic system of n agents with states $s(t) = (s_1(t), \; s_2(t), \ldots, s_n(t))$ and the connection operator $V(t) = \{v_{ij}(t)\}_{i,j}$ connecting between the agents' states and the agents' gains $H(t) = (H_1(t), H_2(t), \ldots, H_n(t))$. Then, the network is

$$\begin{cases} s'(t) = f(s(t)) \\ V'(t) = \beta(V(t)) \end{cases} \tag{16}$$

and its viability constraints standing for constraint of stability of the coalitions

$$\forall\, t \geq 0, \qquad V(t)\, s(t) \in \mathcal{M}. \tag{17}$$

For set $\mathcal{M}$ to be sleek, we assume each agent's satisfactory minimum to be $H_i^o - \theta_i$ for some $\theta_i \in \mathbb{R}$, $\theta_i > 0$. We then define $\mathcal{M} = \{(H_1, H_2, \ldots, H_n) \in \mathbb{R}^n \mid \forall\, i = 1 \ldots n, H_i \geq H_i^o - \theta_i\}$.

Assume maps f and β are continuous with linear growth. Since, in the present example, the spaces X, Y, $Z = \mathbb{R}^n$ are Euclidean spaces supplied with the canonical basis, the duality maps L and M are the identity maps, $V(t)^*$ is equal to the transpose matrix $V^T = V$, and map $g' = g'^{*} = I$. Then, according to the theorem on restoring the network's viability, since map $g = I$ is continuously differentiable and the derivative $g' = I$ is surjective, the network in equation (16) corrected by the viability multipliers $p(t)$ and viability connection operator $P(t)$ from the formula in equation (15) is viable:

$$\begin{cases} s'(t) = f(s(t)) - V(t)\, q(t) \\ V'(t) = \beta(V(t)) - s(t) \otimes q(t). \end{cases} \tag{18}$$

Here, the correction price $q(t)$ belongs to

$$R_{\mathcal{M}}(s(t),\, V(t)) = \\ \left\{ q(t) \in \mathbb{R}^n \,\middle|\, (\beta(V(t))\, s(t) + V(t)\, f(s(t))) - J_g(s\,(t),\, V\,(t))^{-1}\, q(t) \in \overline{co}\, T_{\mathcal{M}}(V(t)\, s(t)) \right\}$$

and the minimal correction price $q_o(s(t),\, V(t))$ is equal to

$$\Pi_{N_{\mathcal{M}}(V(t)\, s(t))}\left(J_g(s(t),\, V(t))[\beta(V(t))\, s(t) + V(t)\, f(s(t))] \right).$$

Note that in the corrected network (equation (18)), the elements of the viability connection operator $P(t) = s(t) \otimes q(t)$, which are equal to $s_j(t)\, q_i(t)$, play the role of the additional bilateral propensities $p_{i\,j}$ between agents i and j, which stabilize the coalitions.

In order to give a schematic picture of the correction, consider the particular case of three agents 1, 2, and 3 whose choices for the coalitions can be described by

$$s_1(t) = -1 + 2\cos^2(t), \\ s_2(t) = -1 + 2\sin^2(t), \\ s_3(t) = -1 + 2\cos^2(t),$$

and whose bilateral propensities are

$$v_{12}(t) = -\cos(t), \\ v_{13}(t) = \cos(t), \\ v_{23}(t) = -\cos(t).$$

The system is shown in Figure 2 at time $t = 0$.

The three agents with the states $s_1(t)$, $s_2(t)$, $s_3(t)$ and their connection operator $V(t)$ form the network as shown in Figure 3.

Assume $\theta_1 = \theta_2 = \theta_3 = \theta$ for some $\theta > 0$, then $\mathcal{M} = \left\{ (H_1,\, H_2,\, H_3) \in \mathbb{R}^3 \,\middle|\, H_1 \geq 2 - \theta,\, H_3 \geq 2 - \theta,\, H_2 \leq -2 + \theta \right\}$. Define $z(t) = V(t)\, s(t)$. At the initial time $t_0 = 0$, the agents' gains $z_0 = (H_1\,(0),\, H_2\,(0),\, H_3\,(0)) = (2,\, -2,\, 2)$ belong to the interior of $\mathcal{M}$. Over time, at some moment t_1 the agents' gains reach the boundary of $\mathcal{M}$ for the first time, $z_1 = (H_1\,(t_1),\, H_2\,(t_1),\, H_3\,(t_1)) = (2 - \theta,\, -2 + \theta,\, 2 - \theta)$, whereupon the network runs out of viability for some time.

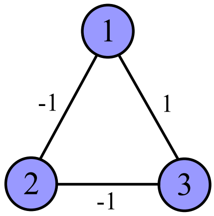

Figure 2. The model of three agents forming coalitions at $t = 0$.

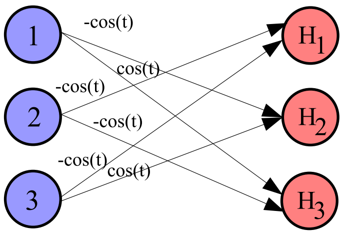

Figure 3. Scheme of the connectionist network corresponding to the model of three agents forming coalitions.

The correction of the network's viability in equation (18) at time t_1 is shown schematically in Figure 4.

In this example, we have seen the illustration of the practical application of the theorem, in which the regulatory parameters control the network's dynamics to keep them inside the viability domain.

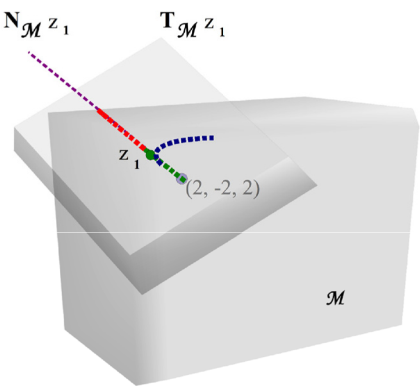

Figure 4. The collective agents' result $V(t)\, s(t)$ corrected at z_1 stays in the interior of set $\mathcal{M}$ (shown with the normal and tangent cones of the set at the point z_1).

3.4 Proof of the Theorem on Restoring the Network's Viability

In order to obtain the network's viability correction as in the form of equation (14), we shall use the viability multipliers approach. To this end, we consider a finite dimensional vector space $X \times \mathcal{L}(X, Y)$ of pairs (x, V) representing the states. The duality map T on the state space $X \times \mathcal{L}(X, Y)$ is $T := L \times H = L \times L^{-1} \otimes M$. Then, we write the network in equation (1) with the constraints in equation (2) in the form of the viability system in equations (4) and (5):

$$(x'(t), V'(t)) = (f(x(t)), \beta(V(t))) \tag{19}$$

$$\forall\, t \geq 0, \quad h(V(t), x(t)) \in \mathcal{M}, \tag{20}$$

where $h(x, V) := g(V\, x)$ is restricted to remain in a subset $\mathcal{M}$ of the constrained results' space Z.

According to Definition 7 of the viable network and Definition 1 of the viable environment, the network (equation (1)) with the constraints (equation (2)) is viable if and only if the environment $\mathcal{M}$ is a viability domain under the network's dynamics. Hence, in order to restore the network's viability, we can apply the results of Theorem 4 under the theorem's assumptions.

Since $\mathcal{M} \subset Z$ is sleek, the map $f \times \beta : X \times \mathcal{L}(X, Y) \mapsto X \times \mathcal{L}(X, Y)$ is continuous with linear growth, and $h : X \times \mathcal{L}(X, Y) \mapsto Z$ is a continuously differentiable map such that the differentiation operator $h'(x, V)$ is surjective, then the assumptions of Theorem 4 are satisfied. Substituting the term $\left[h'(x, V)\, T^{-1}\, h'(x, V)^*\right]^{-1}$ in the correction formulas (equations (12) and (13)) of the theorem by $J_h(x, V)$, we derive the viability correction of the network

$$\begin{cases} (x, V)' = (f(x), \beta(V)) - T^{-1}\, h'(x, V)^*\, q(x, V), \text{ where} \\ \qquad\qquad q(x, V) \in R_{\mathcal{M}}(x, V) \\ R_{\mathcal{M}}(x, V) = \left\{ q \in Z^* \,\middle|\, h'(x, V)\,(f(x), \right. \\ \qquad\qquad \beta(V)) - J_h(x, V)^{-1}\, q(x, V) \in \\ \qquad \left. \overline{co}\, T_{\mathcal{M}}(h(x, V)) \right\} \end{cases} \tag{21}$$

and particularly,

$$q_o(x, V) = \Pi_{N_{\mathcal{M}}(h(x,V))}(J_h(x, V)\, h'(x, V)\,(f(x), \beta(V))).$$

Note that since $h'(x, V)$ is a surjective linear operator mapping X to Z, the operator $J_h(x, V) = \left[h'(x, V)\, T^{-1}\, h'(x, V)^*\right]^{-1}$ is a duality map on Z induced by the duality map T on $X \times \mathcal{L}(X, Y)$.

In order to continue to the next step in the proof of Theorem 5, the following lemmas are required.

Lemma 1. Give a function of two variables $h : X \times \mathcal{L}(X, Y) \mapsto Z$ that maps a pair of a vector x and a linear operator V according to

$h(x, V) = g(V x)$, where $g: Y \mapsto Z$ is a differentiable function. Then, the differential $h'(x, V)$ in a general direction $(u, U) \in X \times \mathcal{L}(X, Y)$ complies with

$$h'(x, V)(u, U) = g'(V x)(U x + V u). \tag{22}$$

We then state and prove the following property.

Lemma 2. Define a linear operator $B: X \times X^* \otimes Y \mapsto Z$ by $B(u, U) = g'(V x)(U x + V u)$ for any $(u, U) \in X \times X^* \otimes Y$. Then, the transpose operator B^* maps Z^* to $X^* \times Y \otimes X^*$ such that for any arbitrary $q \in Z^*$, $B^* q = (V^* g'(V x)^* q, x \otimes g'(V x)^* q)$.

Proof. In order to prove the statement of the lemma, we apply properties of the transpose operation and the duality map.

Given an arbitrary $q \in Z^*$ and an instance (u, U) of $X \times \mathcal{L}(X, Y)$, consider the duality product $\langle B^* q, (u, U)\rangle$, where $B^* q \in (X \times \mathcal{L}(X, Y))^*$. Recall the property of the transpose operation stating for any $A \in \mathcal{L}(X, Y)$, $x \in X$ and $r \in Y^*$, that $\langle A^* r, x\rangle_X = \langle r, A x\rangle_Y$ and $\langle r, A a\rangle = \langle a \otimes r, A\rangle$, and obtain

$$\begin{aligned}
&\langle B^* q, (u, U)\rangle = \langle q, B(u, U)\rangle = \langle q, g'(Vx)(U x + V u)\rangle = \\
&\quad \langle q, g'(Vx)(V u)\rangle + \langle q, g'(Vx)(U x)\rangle = \\
&\quad \langle V^* g'(Vx)^* q, u\rangle + \langle g'(Vx)^* q, U x\rangle = \\
&\quad \langle V^* g'(Vx)^* q, u\rangle + \langle x \otimes g'(Vx)^* q, U\rangle.
\end{aligned}$$

As a consequence,

$$B^* q = (V^* g'(Vx)^* q, x \otimes g'(Vx)^* q).$$

This concludes the proof. □

Lemma 3. The operator $h'(x, V)$ applied to q is equal to $h'(x, V)^* q = (p, P)$, where

$$p = V^* g'(Vx)^* q \text{ and } P = x \otimes g'(Vx)^* q.$$

Proof. By Lemma 1, the derivation operator $h'(x, V)$ for an arbitrary derivation direction (u, U) is equal to $h'(x, V)(u, U) = g'(Vx)(U x + V u)$. Then, according to Lemma 2,

$$h'(x, V)^* q = (V^* g'(Vx)^* q, x \otimes g'(Vx)^* q).$$

This implies that $p = V^* g'(Vx)^* q$ and $P = x \otimes g'(Vx)^* q$. □

Lemma 4. Consider the linear operator $B: X \times X^* \otimes Y \mapsto Z$ defined by $B(u, U) = g'(Vx)(U x + V u)$ for any $(u, U) \in X \times X^* \otimes Y$. Then,

$$B\, T^{-1} B^* = g'(Vx)\left[\lambda^2(x)\, M^{-1} + V L^{-1} V^*\right] g'(Vx)^*. \tag{23}$$

Proof. As it is shown in Lemma 2, B^* maps any $q \in Z^*$ to $B^* q = (V^* g'(Vx)^* q, x \otimes g'(Vx)^* q)$.

Applying $T^{-1} = L^{-1} \times (L^{-1} \otimes M)^{-1}$ to $B^* q$, we obtain

$$T^{-1} B^* q = \left(L^{-1} V^* g'(Vx)^* q, (L^{-1} \otimes M)^{-1} x \otimes g'(Vx)^* q\right),$$

or equivalently,

$$T^{-1} B^* q = \left(L^{-1} V^* g'(Vx)^* q, L x \otimes M^{-1} g'(Vx)^* q\right). \tag{24}$$

Then, by applying B to $T^{-1} B^* q$, we infer that

$$B\left(L^{-1} V^* g'(Vx)^* q, L x \otimes M^{-1} g'(Vx)^* q\right) = \\ g'(Vx)\left[\left(L x \otimes M^{-1} g'(Vx)^* q\right) x + V\left(L^{-1} V^* g'(Vx)^* q\right)\right].$$

Since

$$\left(L x \otimes M^{-1} g'(Vx)^* q\right) x = \\ \langle L x, x \rangle M^{-1} g'(Vx)^* q = \lambda^2(x) M^{-1} g'(Vx)^* q,$$

we derive that

$$B T^{-1} B^* q = \\ g'(Vx)\left[\lambda^2(x) M^{-1} g'(Vx)^* q + V L^{-1} V^* g'(Vx)^* q\right] = \\ g'(Vx)\left[\lambda^2(x) M^{-1} + V L^{-1} V^*\right] g'(Vx)^* q,$$

from which we deduce that

$$B T^{-1} B^* = g'(Vx)\left[\lambda^2(x) M^{-1} + V L^{-1} V^*\right] g'(Vx)^*. \tag{25}$$

This concludes the proof. □

According to the correction formula in equation (21), the viability multiplier p and the viability connection operator P are subjected to $(p, P) = h'(x, V)^* q$, and therefore the value of (p, P) follows from Lemma 3. Hence, the correction formula follows from the fact that the duality map T on $X \times \mathcal{L}(X, Y)$ is equal to $T = X \times L^{-1} \otimes M$.

Using the notation of the correction formula, we obtain the network's regulation map

$$R_{\mathcal{M}}(x, V) = \Big\{q \in Z^* \,\Big|\, \\ h'(x, V)(f(x), \beta(V)) - J_h(x, V)^{-1} q \in \overline{co}\, T_{\mathcal{M}}(h(x, V))\Big\}$$

where

$$J_h(x, V) = \left[h'(x, V)\left(L \times L^{-1} \otimes M\right)^{-1} h'(x, V)^*\right]^{-1}$$

and $h(x, V) = g(V(x))$.

Lemmas 1, 2, and 4 imply that the duality map $J_g(x, V)$ on Z is equal to

$$J_g(x, V) = \left[g'(V x)\left[\lambda^2(x) M^{-1} + V L^{-1} V^*\right] g'(Vx)^*\right]^{-1}.$$

By Lemma 1, $h'(x, V)(u, U) = g'(V x)(U x + V u)$ for any (u, U), and then $h'(x, V)(f(x), \beta(V)) = g'(V x)(\beta(V) x + V f(x))$. Hence,

$$R_{\mathcal{M}}(x, V) = \{q \in Z^* \mid g'(Vx)(\beta(V) x + V f(x)) - J_g(x, V)^{-1} q \in \overline{co}\, T_{\mathcal{M}}(g(Vx))\}.$$

In the same way, we calculate the viability multiplier of minimal correction price $q_o(x, V)$. This concludes the proof of Theorem 5.

4. Conclusion

In this paper, we have discussed the correction of viability of the dynamical network defined over a finite set of autonomous agents connected with the results of the agents' interactions by a connection operator. The network represents a decentralized model where both agents and their connection operator evolve independently over a centralized environment of scarce resources that imposes viability constraints on the evolutions.

Due to the absence of mutual dependencies necessary to satisfy the centralized viability constraints, the network most likely runs out of viability for some time. We suggested restoring the network's viability in such a way that the decentralized nature of the system is kept. This is realized by regulation of both the agents' and the connection operator's dynamics using regulatory parameters that depend on a common value called correction price. The correction price provides all the information about the changes in the dynamics necessary to govern evolutions satisfying the collective constraints.

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