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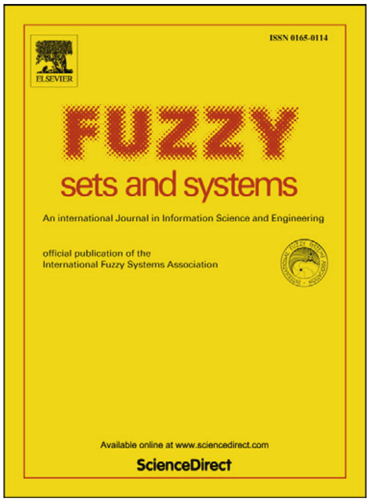

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A FIRST-ORDER EPISTEMIC QUANTUM COMPUTATIONAL SEMANTICS WITH RELATIVISTIC-LIKE EPISTEMIC EFFECTS

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Abstract. Quantum computation has suggested new forms of quantum logic, called *quantum computational logics*. In these logics well-formed formulas are supposed to denote pieces of quantum information: possible pure states of quantum systems that can store the information in question. At the same time, the logical connectives are interpreted as quantum logical gates: unitary operators that process quantum information in a reversible way, giving rise to quantum circuits. Quantum computational logics have been mainly studied as *sentential logics* (whose alphabet consists of atomic sentences and of logical connectives). In this article we propose a semantic characterization for a *first-order epistemic quantum computational logic*, whose language can express sentences like "Alice knows that everybody knows that she is pretty". One can prove that (unlike the case of logical connectives) both quantifiers and epistemic operators cannot be generally represented as (reversible) quantum logical gates. The "act of knowing" and the use of universal (or existential) assertions seem to involve some irreversible "theoretic jumps", which are similar to quantum measurements. Since all epistemic agents are characterized by specific *epistemic domains* (which contain all pieces of information accessible to them), the unrealistic phenomenon of *logical omniscience* is here avoided: knowing a given sentence does not imply knowing all its logical consequences.

Keywords: Quantum computation, quantum computational logics, epistemic operators.

1. Introduction

The theory of quantum computation has inspired the development of new forms of quantum logics that have been termed *quantum computational logics*. As is well known, the basic idea of the theory of quantum computers is using as a "positive resource" two characteristic concepts of quantum theory that had been for a long time described as "mysterious" and potentially paradoxical: *superposition* and *entanglement*. In quantum computation any *piece of information* is identified with a possible *state* of a quantum system (say, a photon-system) that can store and transmit the information in question. In the happiest situations a state corresponds to a *maximal* piece of

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information (about the system) that cannot be consistently extended to a richer knowledge. Such states are called *pure*. Due to the characteristic indeterminism of quantum theory, a pure state is at the same time a *maximal* and a *logically incomplete* piece of information that cannot *decide* some important properties of the corresponding physical system. Accordingly, from an intuitive point of view, one can say that any pure state describes a kind of *cloud of potential properties* that might become *actual* when a measurement is performed, giving rise to the so called *collapse of the wave-function*. The concept of *superposition* represents a mathematical realization of this intuitive idea. Any possible pure state of a quantum system S is identified with a unit-vector of an appropriate Hilbert space $\mathcal{H}_S$ and can be represented as a superposition of other unit-vectors that belong to a basis of the space. By adopting a notation introduced by Dirac, it is customary to write:

$$|\psi\rangle = \sum_i c_i |\varphi_i\rangle,$$

where c_i are complex numbers such that $\sum_i |c_i|^2 = 1$. The physical interpretation is the following: the system S that is in state $|\psi\rangle$ might satisfy the physical properties that are *certain* for the state $|\varphi_i\rangle$ with probability-value $|c_i|^2$. Apparently, any pure state $|\psi\rangle$ describes a *parallel* system of different pieces of quantum information ($|\varphi_i\rangle$). Just this parallelism is responsible for the extraordinary efficiency and speed of quantum computers.

Another powerful resource of quantum computation is due to the use of some “strange” pure states, called *entangled*, that turn out to violate the classical principle of *compositionality*. A paradigmatic case of entanglement may concern a composite physical system S consisting of two subsystems S_1 and S_2 (say, a two-electron system). The observer has a *maximal information* about S, represented by a pure state $|\psi\rangle$. What can be said about the states of the two subsystems? Due to the form of $|\psi\rangle$ and to the quantum-theoretic rules that concern the mathematical description of composite physical systems, such states cannot be pure: they are represented by two identical *mixed states*, which codify a “maximal degree” of uncertainty. Consequently, the information about the global systems (S) cannot be reconstructed as a function of the pieces of information about its parts (S_1, S_2). In such cases, information seems to flow from the *whole* to the *parts* (and not the other way around). Phenomena of this kind give rise to the so called *holistic* features of quantum theory. Interestingly enough, entangled states are currently used in teleportation-experiments and in quantum cryptography.

As expected, quantum computation cannot be identified with a “static” representation of pieces of information. What is important is the dynamic *process* of information that gives rise to quantum computations (performed by *quantum circuits*). Such process is mathematically realized by *quantum logical gates* (briefly, *gates*): special examples of unitary operators that transform pure states into pure states in a reversible way. Since in quantum theory the time-evolution of physical systems is mathematically described by unitary operators, one can say that quantum computations can be regarded as the time-evolution of some special quantum objects.

Quantum computational logics can be described as a logical abstraction from the theory of quantum circuits. The basic idea that underlies the semantic characterization of these logics can be sketched as follows:

- well formed formulas are supposed to denote pieces of quantum information: possible states of quantum systems that can store the information in question;
- the logical connectives correspond to some gates that can process quantum information.

In this way, connectives turn out to have way a dynamic character, representing possible computation-actions. At the same time, any formula can be regarded as a synthetic logical description of a quantum circuit, which may have a characteristic parallel structure.

Quantum computational logics have been mainly studied as *sentential logics* (whose alphabet consists of atomic sentences and of logical connectives). Different choices of the system of primitive connectives and of the basic semantic definitions give rise to different logics. We will refer here to a *holistic* version of the quantum computational semantics, where quantum entanglement is used as a "semantic resource": generally, the *meaning* of a compound expression determines the *contextual meanings* of its subexpressions (and not the other way around, as happens in the case of most *compositional* semantic approaches).

The logics characterized by this holistic semantics represent weak forms of quantum logic, where important classical properties of the "Boolean connectives" are generally violated. Like in fuzzy logics, conjunctions and disjunctions are not generally idempotent (according to the slogan "repetita iuvant!") and the non-contradiction principle is not valid. Furthermore, commutativity, associativity and distributivity for conjunctions and disjunctions do not generally hold.

In this article we propose a semantic characterization for a *first-order epistemic quantum computational logic*, whose language can express sentences like "Alice knows that everybody knows that she is pretty". As is well known, most semantic approaches to epistemic logics that can be found in the literature have been developed in the framework of a Kripke-style semantics. We will follow here a different approach, whose aim is representing both quantifiers and epistemic operators as "genuine" quantum concepts (living in a Hilbert-space environment). In this perspective, the following question arises: to what extent is it possible to interpret the quantifiers and the epistemic operators as special examples of quantum operations? Interestingly enough, these logical operators turn out to have a similar semantic behavior, giving rise to a kind of "reversibility-breaking": one can prove that (unlike the case of logical connectives) both quantifiers and epistemic operators cannot be generally represented as quantum logical gates (which are reversible unitary operations). The "act of knowing" and the use of universal (or existential) assertions seem to involve some irreversible "theoretic jumps", which are similar to quantum measurements (where the collapse of the wave-function comes into play).

A characteristic feature of the epistemic quantum computational semantics is the use of the notion of *truth-perspective*: each *epistemic agent* (say, Alice, Bob, ...) is supposed to be associated to a truth-perspective that is mathematically determined by the choice of a particular orthonormal basis of the two-dimensional Hilbert space $\mathbb{C}^2$. Truth-perspective changes give rise to some interesting relativistic-like epistemic effects: if Alice and Bob have different truth-perspectives, Alice might *see* a kind of *deformation* in Bob's logical behavior. Epistemic agents are also characterized by specific *epistemic domains* that contain all pieces of information accessible to them. Due to the limits of such domains the unrealistic phenomenon of *logical omniscience* is here avoided: Alice might know a given sentence without knowing all its logical consequences.[1]

As happens in the case of knowledge operators, quantifiers also can be interpreted as special examples of generally irreversible quantum operations. Unlike most semantic approaches, the *models* of the first-order quantum computational semantics do not refer to any *domain of individuals* dealt with as a *closed set* (in a classical sense). The interpretation of a universal formula does not require here any "ideal tests" that should be performed on *all* elements of a collection of objects (which might be infinite or indeterminate).

2. The mathematical environment

It is expedient to recall some basic concepts of quantum computation that play an important role in the quantum computational semantics (see, for instance, [10, 14, 1]). The general mathematical environment is the n-fold tensor product of the Hilbert space $\mathbb{C}^2$:

$$\mathcal{H}^{(n)} := \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n-times},$$

where all pieces of quantum information live. The elements $|1\rangle = (0,1)$ and $|0\rangle = (1,0)$ of the canonical orthonormal basis $B^{(1)}$ of $\mathbb{C}^2$ represent, in this framework, the two classical bits, which can be also regarded as the canonical truth-values *Truth* and *Falsity*, respectively. The canonical basis of $\mathcal{H}^{(n)}$ is the set

$$B^{(n)} = \left\{ |x_1\rangle \otimes \ldots \otimes |x_n\rangle : |x_1\rangle, \ldots, |x_n\rangle \in B^{(1)} \right\}.$$

[1] A different approach to epistemic quantum logics has been developed in some important contributions by A. Baltag and S. Smets (see, for instance, [2, 3, 4]). In this approach information is supposed to be stored by quantum objects; at the same time, epistemic agents are supposed to communicate in a classical way. On this basis, epistemic operators are dealt with as classical modalities in a Kripkean framework.

As usual, we will briefly write $|x_1, \ldots, x_n\rangle$ instead of $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$. By definition, a *quregister* is a unit vector of $\mathcal{H}^{(n)}$; while a *qubit* (or *qubit-state*) is a quregister of $\mathcal{H}^{(1)}$. Quregisters thus correspond to pure states (maximal pieces of information about the quantum systems that are supposed to store a given amount of quantum information). We shall also make reference to *mixed states* (or *mixtures of quregisters*), represented by density operators ρ of $\mathcal{H}^{(n)}$. Of course, any quregister $|\psi\rangle$ corresponds to a special example of a density operator: the projection operator $P_{|\psi\rangle}$ that projects over the closed subspace determined by $|\psi\rangle$. We will denote by $\mathsf{D}(\mathcal{H}^{(n)})$ the set of all density operators of $\mathcal{H}^{(n)}$, while $\mathsf{D} = \bigcup_n \left\{ \mathsf{D}(\mathcal{H}^{(n)}) \right\}$ will represent the set of all possible pieces of quantum information, briefly called *qumixes*.

The choice of an orthonormal basis for the space $\mathbb{C}^2$ is, obviously, a matter of convention. One can consider infinitely many bases that are determined by the application of a unitary operator T to the elements of the canonical basis. ¿From an intuitive point of view, we can think that the operator T gives rise to a change of *truth-perspective*. While in the classical case, the truth-values *Truth* and *Falsity* are identified with the two classical bits $|1\rangle$ and $|0\rangle$, assuming a different basis corresponds to a different idea of *Truth* and *Falsity*. Since any basis-change in $\mathbb{C}^2$ is determined by a unitary operator, we can identify a *truth-perspective* with a unitary operator T of $\mathbb{C}^2$. We will write:

$$|1_{\mathsf{T}}\rangle = \mathsf{T}|1\rangle;\ |0_{\mathsf{T}}\rangle = \mathsf{T}|0\rangle,$$

and we will assume that $|1_{\mathsf{T}}\rangle$ and $|0_{\mathsf{T}}\rangle$ represent, respectively, the truth-values *Truth* and *Falsity* of the truth-perspective T. The *canonical truth-perspective* is, of course, determined by the identity operator $\mathtt{I}$ of $\mathbb{C}^2$. We will indicate by $B^{(1)}_{\mathsf{T}}$ the orthonormal basis determined by T; while $B^{(1)}_{\mathtt{I}}$ will represent the canonical basis. From a physical point of view, we can suppose that each truth-perspective is associated to an apparatus that allows one to measure a given observable.

Any unitary operator T of $\mathcal{H}^{(1)}$ can be naturally extended to a unitary operator $\mathsf{T}^{(n)}$ of $\mathcal{H}^{(n)}$ (for any $n \geq 1$):

$$\mathsf{T}^{(n)}|x_1, \ldots, x_n\rangle = \mathsf{T}|x_1\rangle \otimes \ldots \otimes \mathsf{T}|x_n\rangle.$$

Accordingly, any choice of a unitary operator T of $\mathcal{H}^{(1)}$ determines an orthonormal basis $B^{(n)}_{\mathsf{T}}$ for $\mathcal{H}^{(n)}$ such that:

$$B^{(n)}_{\mathsf{T}} = \left\{ \mathsf{T}^{(n)}|x_1, \ldots, x_n\rangle : |x_1, \ldots, x_n\rangle \in B^{(n)}_{\mathtt{I}} \right\}.$$

Instead of $\mathsf{T}^{(n)}|x_1, \ldots, x_n\rangle$ we will also write $|x_{1_{\mathsf{T}}}, \ldots, x_{n_{\mathsf{T}}}\rangle$.

The elements of $B^{(1)}_{\mathsf{T}}$ will be called the T-*bits* of $\mathcal{H}^{(1)}$; while the elements of $B^{(n)}_{\mathsf{T}}$ will represent the T-*registers* of $\mathcal{H}^{(n)}$. On this ground the notions of *truth*, *falsity* and *probability* with respect to any truth-perspective T can be defined in a natural way.

Definition 2.1. (T-true and T-false registers)

- $|x_{1_\mathsf{T}}, \ldots, x_{n_\mathsf{T}}\rangle$ *is a* T-true register *iff* $|x_{n_\mathsf{T}}\rangle = |1_\mathsf{T}\rangle$;
- $|x_{1_\mathsf{T}}, \ldots, x_{n_\mathsf{T}}\rangle$ *is a* T-false register *iff* $|x_{n_\mathsf{T}}\rangle = |0_\mathsf{T}\rangle$.

In other words, the T-*truth-value* of a T-register (which corresponds to a sequence of T-bits) is determined by its last element.[2]

Definition 2.2. (T-truth and T-falsity)

- *The* T-truth *of* $\mathcal{H}^{(n)}$ *is the projection operator* ${}^{\mathsf{T}}P_1^{(n)}$ *that projects over the closed subspace spanned by the set of all* T- *true registers;*
- *the* T-falsity *of* $\mathcal{H}^{(n)}$ *is the projection operator* ${}^{\mathsf{T}}P_0^{(n)}$ *that projects over the closed subspace spanned by the set of all* T- *false registers.*

In this way, truth and falsity are dealt with as mathematical representatives of possible physical properties. Accordingly, by applying the Born-rule, one can naturally define the probability-value of any qumix with respect to the truth-perspective T.

Definition 2.3. (T-Probability)
For any $\rho \in \mathsf{D}(\mathcal{H}^{(n)})$,

$$\mathrm{p}_\mathsf{T}(\rho) := \mathtt{Tr}({}^{\mathsf{T}}P_1^{(n)}\rho),$$

where $\mathtt{Tr}$ *is the trace-functional.*

We interpret $\mathrm{p}_\mathsf{T}(\rho)$ as the probability that the information ρ satisfies the T-Truth. In the particular case of qubits, we will obviously obtain:

$$\mathrm{p}_\mathsf{T}(a_0|0_\mathsf{T}\rangle + a_1|1_\mathsf{T}\rangle) = |a_1|^2.$$

As is well known, quantum information is processed by *quantum logical gates* (briefly, *gates*): unitary operators that transform quregisters into quregisters in a reversible way. Let us recall the definition of some gates that play a special role both from the computational and from the logical point of view.

Definition 2.4. (The Negation)
For any $n \geq 1$, *the* negation *on* $\mathcal{H}^{(n)}$ *is the linear operator* $\mathtt{NOT}^{(n)}$ *such that, for every element* $|x_1, \ldots, x_n\rangle$ *of the canonical basis,*

$$\mathtt{NOT}^{(n)}|x_1, \ldots, x_n\rangle = |x_1, \ldots, x_{n-1}\rangle \otimes |1 - x_n\rangle.$$

In particular, we obtain:

$$\mathtt{NOT}^{(1)}|0\rangle = |1\rangle;\ \mathtt{NOT}^{(1)}|1\rangle = |0\rangle.$$

Hence, the gate $\mathtt{NOT}^{(n)}$ represents a natural generalization of the classical negation.

[2]As we will shortly see, the application of a classical gate to a register $|x_1, \ldots, x_n\rangle$ transforms the bit $|x_n\rangle$ into the target-bit $|x_n\rangle$, which determines the final truth-value. This justifies the choice of Def. 2.1.

Definition 2.5. (The Toffoli-gate)
For any $m, n, p \geq 1$, *the* Toffoli-gate *is the linear operator* $\mathtt{T}^{(m,n,p)}$ *defined on* $\mathcal{H}^{(m+n+p)}$ *such that, for every element* $|x_1, \ldots, x_m\rangle \otimes |y_1, \ldots, y_n\rangle \otimes |z_1, \ldots, z_p\rangle$ *of the canonical basis,*

$$\mathtt{T}^{(m,n,p)}|x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p\rangle = |x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_{p-1}\rangle \otimes |x_m \cdot y_n \widehat{+} z_p\rangle,$$

where $\cdot$ *is the product, while* $\widehat{+}$ *represents the addition modulo* 2.

For $m = n = p = 1$, we obtain:

$$\mathtt{T}^{(1,1,1)}|x, y, z\rangle = |x, y, x \cdot y \widehat{+} z\rangle.$$

Consequently, when $z = 0$, the gate $\mathtt{T}^{(1,1,1)}$ gives rise to a reversible representation of the classical truth-table for the conjunction:

$$|1,1,0\rangle \mapsto |1,1,1\rangle;\ |1,0,0\rangle \mapsto |1,0,0\rangle;\ |0,1,0\rangle \mapsto |0,1,0\rangle;\ |0,0,0\rangle \mapsto |0,0,0\rangle.$$

Definition 2.6. (The Hadamard-gate)
For any $n \geq 1$, *the* Hadamard-gate *on* $\mathcal{H}^{(n)}$ *is the linear operator* $\sqrt{\mathtt{I}}^{(n)}$ *such that for every element* $|x_1, \ldots, x_n\rangle$ *of the canonical basis:*

$$\sqrt{\mathtt{I}}^{(n)}|x_1, \ldots, x_n\rangle = |x_1, \ldots, x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}\left((-1)^{x_n}|x_n\rangle + |1 - x_n\rangle\right).$$

In particular we obtain:

$$\sqrt{\mathtt{I}}^{(1)}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle); \sqrt{\mathtt{I}}^{(1)}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Both the negation-gate and the Toffoli-gate are examples of *classical gates*, that transform registers into registers. The Hadamard-gate is instead a *genuine quantum gate* that can create superpositions, giving rise to characteristic parallel computational structures.

All gates can be naturally transposed from the canonical truth-perspective to any truth-perspective T. Let $G^{(n)}$ be any gate defined with respect to the canonical truth-perspective. The *twin-gate* $G^{(n)}_{\mathsf{T}}$, defined with respect to the truth-perspective T, is determined as follows:

$$G^{(n)}_{\mathsf{T}} := \mathsf{T}^{(n)} G^{(n)} \mathsf{T}^{(n)^\dagger},$$

where $\mathsf{T}^{(n)^\dagger}$ is the adjoint of $\mathsf{T}^{(n)}$.

All T-gates can be canonically extended to the set D of all qumixes [12]. Let G_{T} be any gate defined on $\mathcal{H}^{(n)}$. The corresponding *qumix gate* (also called *unitary quantum operation*) ${}^{\mathsf{D}}G_{\mathsf{T}}$ is defined as follows for any $\rho \in \mathsf{D}(\mathcal{H}^{(n)})$:

$${}^{\mathsf{D}}G_{\mathsf{T}}\rho = G_{\mathsf{T}}\, \rho\, G^{\dagger}_{\mathsf{T}}.$$

For the sake simplicity, also the qumix gates ${}^{\mathsf{D}}G_{\mathsf{T}}$ will be briefly called *gates*.

The Toffoli-gate ${}^{\mathsf{D}}\mathtt{T}^{(m,n,p)}_{\mathsf{T}}$ allows us to define a reversible operation $\mathtt{AND}^{(m,n)}_{\mathsf{T}}$ that represents a *holistic conjunction*.

Definition 2.7. (The holistic conjunction)
For any $m, n \geq 1$ the holistic conjunction $\mathtt{AND}^{(m,n)}_{\mathsf{T}}$ *with respect to the truth-perspective* T *is defined as follows for any qumix* $\rho \in \mathsf{D}(\mathcal{H}^{(m+n)})$*:*

$$\mathtt{AND}^{(m,n)}_{\mathsf{T}}(\rho) := {}^{\mathsf{D}}\mathtt{T}^{(m,n,1)}_{\mathsf{T}}(\rho \otimes {}^{\mathsf{T}}P^{(1)}_0),$$

where the T*-falsity* ${}^{\mathsf{T}}P^{(1)}_0$ *plays the role of an* ancilla.

When $\mathsf{T} = \mathtt{I}$, we will also write: $\mathtt{AND}^{(m,n)}$ (instead of $\mathtt{AND}^{(m,n)}_{\mathtt{I}}$) and $\mathtt{p}$ (instead of $\mathtt{p}_{\mathtt{I}}$).

If $m = n = 1$ and ρ corresponds to the register $P_{|x,y\rangle}$ (of the space $\mathcal{H}^{(2)}$), we obtain:

$$\mathtt{AND}^{(1,1)}(P_{|x,y\rangle}) = P_{\mathtt{T}^{(1,1,1)}|x,y,0\rangle}.$$

Hence, $\mathtt{AND}^{(1,1)}(P_{|x,y\rangle})$ represents the classical conjunction of the two bits $|x\rangle$ and $|y\rangle$.

It is worth-while noticing that generally

$$\mathtt{AND}^{(m,n)}_{\mathsf{T}}(\rho) \neq \mathtt{AND}^{(m,n)}_{\mathsf{T}}(Red^{(1)}_{[m,n]}(\rho) \otimes Red^{(2)}_{[m,n]}(\rho)),$$

where $Red^{(1)}_{[m,n]}(\rho)$ (which belongs to the space $\mathcal{H}^{(m)}$) and $Red^{(2)}_{[m,n]}(\rho)$ (which belongs to the space $\mathcal{H}^{(n)}$) represent the two *reduced states* that describe, respectively, the first and the second subsystem of the composite system described by the global state ρ (which belongs to the space $\mathcal{H}^{(m+n)}$).[3] Roughly speaking, we might say that the holistic conjunction defined on a global information consisting of two parts does not generally coincide with the conjunction of the two separate parts. As an example, we can consider the following qumix (which represents an *entangled pure state*):

$$\rho = P_{\frac{1}{\sqrt{2}}(|0,0\rangle+|1,1\rangle)}.$$

We have:

$$\mathtt{AND}^{(1,1)}(\rho) = {}^{\mathsf{D}}\mathtt{T}^{(1,1,1)}(P_{\frac{1}{\sqrt{2}}(|0,0\rangle+|1,1\rangle)} \otimes P^{(1)}_0) = P_{\frac{1}{\sqrt{2}}(|0,0,0\rangle+|1,1,1\rangle)},$$

which also represents an entangled pure state.

At the same time we have:

$$\mathtt{AND}^{(1,1)}(Red^{(1)}_{[1,1]}(\rho) \otimes Red^{(2)}_{[1,1]}(\rho)) = \mathtt{AND}^{(1,1)}(\frac{1}{2}\mathtt{I}^{(1)} \otimes \frac{1}{2}\mathtt{I}^{(1)}),$$

which is a proper mixture.

[3] We recall that according to the quantum theoretic formalism any possible state of a composite physical system S consisting of n subsystems $(S_1, \ldots, S_n)$ is a density operator ρ of the tensor-product space $\mathcal{H}_S = \mathcal{H}_{S_1} \otimes \ldots \otimes \mathcal{H}_{S_n}$ (where each $\mathcal{H}_{S_i}$ is the Hilbert space associated to the system S_i). The state ρ determines n *reduced states*: $Red^{(1)}(\rho), \ldots, Red^{(n)}(\rho)$, where each $Red^{(i)}(\rho)$ is a density operator of $\mathcal{H}_{S_i}$ that represents the state of S_i. Generally, we have: $\rho \neq Red^{(1)}(\rho) \otimes \ldots \otimes Red^{(n)}(\rho)$. In other words, the state of the global system cannot be generally represented as a *factorized* state determined by the tensor product of the states of its parts.

Furthermore, we have:

$$\mathtt{p}(\mathtt{AND}^{(1,1)}(\rho)) = \frac{1}{2};\ \ \mathtt{p}(\mathtt{AND}^{(1,1)}(Red^{(1)}_{[1,1]}(\rho) \otimes Red^{(2)}_{[1,1]}(\rho))) = \frac{1}{4}.$$

3. A first-order epistemic quantum computational language

Let us first introduce the language that will be used. This language, indicated by $\mathcal{L}$, contains:

- sentential constants $(\mathbf{q}, \mathbf{q}_1, \mathbf{q}_2, \ldots)$ including two privileged sentences $\mathbf{t}$ and $\mathbf{f}$ that represent the truth-values *Truth* and *Falsity*, respectively;
- individual names $(\mathbf{a}, \mathbf{b}, \ldots)$ and individual variables $(x, y, \ldots)$;
- m-ary predicates $\mathbf{P}^m_i$ (with $1 \leq m$);
- the following logical connectives: the negation $\neg$ (which corresponds to the gate *Negation*), the square root of the identity $\sqrt{id}$ (which corresponds to the *Hadamard*-gate), a ternary connective T (which corresponds to the *Toffoli*-gate);
- the universal quantifier $\forall$;
- the epistemic operator K (*to know*).

We will use $t, t_1, \ldots$ as metavariables for individual terms (either names or variables). The notions of *formula* and of *sentence* are defined in the expected way.

- Sentential constants and expressions having the form $\mathbf{P}^m_i t_1 \ldots t_m$ are (atomic) formulas;
- if α, β, γ are formulas, then the expressions $\neg\alpha$, $\sqrt{id}\,\alpha$, $\mathsf{T}(\alpha, \beta, \gamma)$ are formulas;
- for any formula $\alpha(x)$, the expression $\forall x\alpha(x)$ is a formula;
- for any term t and any formula α, the expression $Kt\alpha$ (t *knows* α) is a formula.
  Any expression Kt represents an *epistemic connective*.

Sentences are formulas that do not contain any free variable.

The binary logical conjunction $\wedge$ can be defined by means of the following metalinguistic definition:

$$\alpha \wedge \beta := \mathsf{T}(\alpha, \beta, \mathbf{f})$$

(where the false sentence $\mathbf{f}$ plays the role of a *syntactical ancilla*). This definition clearly reflects, at a syntactical level, the definition of the holistic conjunction in terms of the Toffoli-gate ($\mathtt{AND}^{(m,n)}_{\mathsf{T}}(\rho) := {}^{\mathsf{D}}\mathtt{T}^{(m,n,1)}_{\mathsf{T}}(\rho \otimes {}^{\mathsf{T}}P^{(1)}_0)$).

The binary inclusive disjunction $\vee$ and the existential quantifier $\exists$ are metalinguistically defined as follows:

$$\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta);\ \ \exists x\alpha := \neg\forall x\neg\alpha.$$

Any formula α can be naturally decomposed into its parts, giving rise to a special configuration called the *syntactical tree* of α. Such configuration (indicated by $STree^{\alpha}$) can be represented as a finite sequence of *levels*:

$$\begin{array}{c} Level_h^{\alpha} \\ \vdots \\ Level_1^{\alpha} \end{array}$$

where:

- each $Level_i^{\alpha}$ (with $1 \leq i \leq h$) is a sequence $(\beta_{i_1}, \ldots, \beta_{i_r})$ of subformulas of α;
- the *bottom level* $Level_1^{\alpha}$ is (α);
- the *top level* $Level_h^{\alpha}$ is the sequence $(at_1^{\alpha}, \ldots, at_k^{\alpha})$ of the atomic subformulas occurring in α;
- for any i (with $1 \leq i < h$), $Level_{i+1}^{\alpha}$ is the sequence obtained by dropping the *principal logical connective*, the *principal epistemic connective* and the *principal quantifier* in all molecular formulas occurring at $Level_i^{\alpha}$, and by repeating all the atomic formulas that occur at $Level_i^{\alpha}$.

By *Height* of α (indicated by $Height(\alpha)$) we mean the number h of levels of the syntactical tree of α.

Example 3.1.

$$\alpha = \mathbf{P}^1\mathbf{a} \wedge \neg\mathbf{P}^1\mathbf{a} = \mathsf{T}(\mathbf{P}^1\mathbf{a}, \neg\mathbf{P}^1\mathbf{a}, \mathbf{f}).$$

The syntactical tree of α is the following sequence of sequences of subformulas of α:

$$\begin{aligned} Level_3^{\alpha} &= (\mathbf{P}^1\mathbf{a}, \mathbf{P}^1\mathbf{a}, \mathbf{f}) \\ Level_2^{\alpha} &= (\mathbf{P}^1\mathbf{a}, \neg\mathbf{P}^1\mathbf{a}, \mathbf{f}) \\ Level_1^{\alpha} &= (\mathsf{T}(\mathbf{P}^1\mathbf{a}, \neg\mathbf{P}^1\mathbf{a}, \mathbf{f})) \end{aligned}$$

We have: $Height(\alpha) = 3$.

We will now define the notion of *atomic structure* of a formula α (which will play an important semantic role). Consider first a simple example: the case of an atomic formula $\mathbf{P}^1 t$. The underlying semantic idea is that the information corresponding to $\mathbf{P}^1 t$ can be stored by three qumixes: the first qumix is supposed to store the information described by the predicate $\mathbf{P}^1$; the second qumix stores the information described by the term t; the third qumix stores the "truth-degree" according to which the object denoted by t satisfies the property denoted by $\mathbf{P}^1$.

Notice that, according to this idea, the same type of information is supposed to store both predicates and individual terms. Unlike classical set-theoretic semantics, we do not refer to any *ontological hierarchy*.

In the case of an atomic formula having the form $\mathbf{P}^m t_1 \ldots t_m$, we will need $m + 2$ qumixes; while for a sentential constant, one qumix will be sufficient.

Accordingly, we can assume that the atomic structure of $\mathbf{P}^m t_1 \ldots t_m$ is $(m+2)$; while (1) is the atomic structure of a sentential constant.

In the general case, the notion of *atomic structure* of a formula α is defined as follows.

Definition 3.1. (Atomic structure)
Consider a formula α such that:

$$Level_h^\alpha = (at_1^\alpha, \ldots, at_k^\alpha),$$

where h is the Height of α. The atomic structure *of α is a sequence of natural numbers*

$$AtStr(\alpha) = (n_1, \ldots, n_k),$$

such that:

$$n_i = \begin{cases} 1, & \text{if } at_i^\alpha \text{ is a sentential constant;} \\ 2+m, & \text{if } at_i^\alpha = \mathbf{P}^m t_1 \ldots t_m. \end{cases}$$

If $AtStr(\alpha) = (n_1, \ldots, n_k)$, the number $n_1 + \ldots + n_k$ is called the *atomic complexity* of α (indicated by $At(\alpha)$).

Semantically, the atomic structure of α is important because it determines the Hilbert space $\mathcal{H}^\alpha$ that represents the *semantic space* of α, where any possible meaning for α shall live. Let $AtStr(\alpha) = (n_1, \ldots, n_k)$. We write: $\mathcal{H}^\alpha = \mathcal{H}^{(n_1)} \otimes \ldots \otimes \mathcal{H}^{(n_k)} = \mathcal{H}^{(n_1+\ldots+n_k)} = \mathcal{H}^{At(\alpha)}$.

Example 3.2. *Consider again the formula $\alpha = \mathsf{T}(\mathbf{P}^1\mathbf{a}, \neg\mathbf{P}^1\mathbf{a}, \mathbf{f})$. We have: $AtStr(\alpha) = (3,3,1)$; $At(\alpha) = 7$; $\mathcal{H}^\alpha = \mathcal{H}^{(7)}$.*

4. A Holistic quantum computational semantics

The basic intuitive idea of the holistic quantum computational semantics can be sketched as follows [9, 8]. For any choice of a truth-perspective, any model of the language assigns to any formula α a *global informational meaning* that lives in $\mathcal{H}^\alpha$ (the semantic space of α). This meaning determines the *contextual meanings* of all subformulas of α (from the whole to the parts!). It may happen that one and the same model assigns to a given formula α different contextual meanings in different contexts. One obtains, in this way, a semantic situation that is quite similar to what happens in the case of *entanglement-phenomena*.

It is expedient to consider first the semantics for a fragment $\mathcal{L}^-$ of $\mathcal{L}$ consisting of all formulas that do not contain any occurrence either of $\forall$ or of K. In such a case, for any choice of a truth-perspective T, the syntactical tree of any formula α uniquely determines a sequence of gates, all defined on the semantic space of α.

As an example, consider again the formula

$$\alpha = \mathbf{P}^1\mathbf{a} \wedge \neg\mathbf{P}^1 = \mathsf{T}(\mathbf{P}^1\mathbf{a}, \neg\mathbf{P}^1\mathbf{a}, \mathbf{f}).$$

In the syntactical tree of α the second level has been obtained from the third level by repeating the first occurrence of $\mathbf{P}^1\mathbf{a}$, by negating the second occurrence of $\mathbf{P}^1\mathbf{a}$ and by repeating $\mathbf{f}$, while the first level has been obtained by applying the connective T to the sequence of formulas occurring at the second level. Accordingly, one can say that, for any choice of a truth-perspective T, the syntactical tree of α uniquely determines the following sequence consisting of two gates, both defined on the semantic space of α:

$$\left({}^{\mathsf{D}}\mathtt{I}_{\mathsf{T}}^{(3)} \otimes {}^{\mathsf{D}}\mathtt{NOT}_{\mathsf{T}}^{(3)} \otimes {}^{\mathsf{D}}\mathtt{I}_{\mathsf{T}}^{(1)},\ {}^{\mathsf{D}}\mathtt{T}_{\mathsf{T}}^{(3,3,1)}\right).$$

Such a sequence is called the T-*gate tree* of α. This procedure can be naturally generalized to any formula α. The general form of the T- gate tree of α will be:

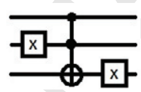
$$({}^{\mathsf{D}}G^{\alpha}_{\mathsf{T}_{(h-1)}}, \ldots, {}^{\mathsf{D}}G^{\alpha}_{\mathsf{T}_{(1)}}),$$

where h is the Height of α.

¿From an intuitive point of view, any formula α of $\mathcal{L}$ can be regarded as a synthetic logical description of a quantum circuit that may assume as inputs qumixes living in the semantic space of α. For instance, the circuit described by the formula

$$\alpha = \neg(\mathbf{q} \wedge \neg\mathbf{q}) = \neg\ \mathsf{T}\ (\mathbf{q}, \neg\mathbf{q}, \mathbf{f})$$

(which asserts the non-contradiction principle) can be represented as follows:



Thus, $\mathcal{L}$-formulas turn out to have a characteristic *dynamic* character, representing systems of *computation-actions*.

Before defining the concept of *holistic model*, it is expedient to introduce the weaker notion of *holistic map* for the language $\mathcal{L}^-$.

Definition 4.1. (Holistic map)
A holistic map *for $\mathcal{L}^-$ (associated to a truth-perspective T) is a map $\mathtt{Hol}_{\mathsf{T}}$ that assigns a meaning $\mathtt{Hol}_{\mathsf{T}}(Level_i^{\alpha})$ to each level of the syntactical tree of α, for any formula α. This meaning is a qumix living in the semantic space of α.*

On this basis, the meaning assigned by $\mathtt{Hol}_{\mathsf{T}}$ to the formula α is defined as follows: $\mathtt{Hol}_{\mathsf{T}}(\alpha) := \mathtt{Hol}_{\mathsf{T}}(Level_1^{\alpha})$.

Given a formula γ, any holistic map $\mathtt{Hol}_{\mathsf{T}}$ determines the *contextual meaning*, with respect to the context $\mathtt{Hol}_{\mathsf{T}}(\gamma)$, of any occurrence in γ of a subformula, of a predicate, of a term. The intuitive idea is the following: $\mathtt{Hol}_{\mathsf{T}}(\gamma)$ can be regarded as the state of a composite quantum system S that stores the information expressed by γ, while the subexpressions of γ correspond to the states of particular subsystems of S, which are determined by the global state $\mathtt{Hol}_{\mathsf{T}}(\gamma)$. Accordingly, the *contextual meaning* of a subexpression of γ can be naturally defined by using the notion of *reduced state*.

Definition 4.2. (Contextual meaning of an occurrence of a subformula)
Consider a formula γ such that $Level_i^\gamma = (\beta_{i_1}, \ldots, \beta_{i_r})$. We have: $\mathcal{H}^\gamma = \mathcal{H}^{\beta_{i_1}} \otimes \ldots \otimes \mathcal{H}^{\beta_{i_r}}$. Let $\mathtt{Hol}_\mathsf{T}$ be a holistic map. The contextual meaning *of the occurrence β_{i_j} with respect to the context $\mathtt{Hol}_\mathsf{T}(\gamma)$ is defined as follows:*

$$\mathtt{Hol}_\mathsf{T}^\gamma(\beta_{i_j}) := Red^{(j)}_{[At(\beta_{i_1}),\ldots,At(\beta_{i_r})]}(\mathtt{Hol}_\mathsf{T}(Level_i(\gamma))).$$

Of course, we obtain:

$$\mathtt{Hol}_\mathsf{T}^\gamma(\gamma) = \mathtt{Hol}_\mathsf{T}(\gamma).$$

Definition 4.3. (Contextual meaning of an occurrence of a predicate and of a term)
Consider a formula γ such that $Level_i^\gamma = (\beta_{i_1}, \ldots, \beta_{i_r})$ and let $\beta_{i_j} = \mathbf{P}^m t_1 \ldots t_m$. Consider a holistic map $\mathtt{Hol}_\mathsf{T}$. The contextual meanings *of the occurrences of $\mathbf{P}^m$ and of t_k (with $1 \leq k \leq m$) in β_{i_j} with respect to the context $\mathtt{Hol}_\mathsf{T}(\gamma)$ are defined as follows:*

$$\mathtt{Hol}_\mathsf{T}^{(\gamma,\beta_{i_j})}(\mathbf{P}^m) := Red^{(1)}_{[1,m+1]}(\mathtt{Hol}_\mathsf{T}^\gamma(\beta_{i_j}));$$

$$\mathtt{Hol}_\mathsf{T}^{(\gamma,\beta_{i_j})}(t_k) := Red^{(2)}_{[k,1,m+2-(k+1)]}(\mathtt{Hol}_\mathsf{T}^\gamma(\beta_{i_j})).$$

Definition 4.4. (Normal holistic map)
A holistic map $\mathtt{Hol}_\mathsf{T}$ is called normal *iff the following conditions are satisfied:*

(1) *for any formula γ, $\mathtt{Hol}_\mathsf{T}$ assigns the same contextual meaning to all occurrences of a subformula (of a predicate, of a term) in the syntactical tree of γ.*
(2) *The contextual meanings assigned by $\mathtt{Hol}_\mathsf{T}$ to the false sentence $\mathbf{f}$ and to the true sentence $\mathbf{t}$ are the T-Falsity ${}^\mathsf{T}P_0^{(1)}$ and the T-Truth ${}^\mathsf{T}P_1^{(1)}$, respectively.*

We can now define the concept of *holistic model* of the language $\mathcal{L}^-$. Unlike holistic maps, holistic models shall preserve the logical form of any formula α, by interpreting the logical connectives as the corresponding gates.

Definition 4.5. (Holistic model)
A holistic model *of $\mathcal{L}^-$ is a normal holistic map $\mathtt{Hol}_\mathsf{T}$ that satisfies the following condition for any formula α: if $({}^\mathsf{D}G^\alpha_{\mathsf{T}_{(h-1)}}, \ldots, {}^\mathsf{D}G^\alpha_{\mathsf{T}_{(1)}})$ is the T-gate tree of α and $1 \leq i < h$, then,*

$$\mathtt{Hol}_\mathsf{T}(Level_i^\alpha) = {}^\mathsf{D}G^\alpha_{\mathsf{T}_{(i)}}(\mathtt{Hol}_\mathsf{T}(Level_{i+1}^\alpha)).$$

In other words, the meaning of each level (different from the top level) is obtained by applying the corresponding gate to the meaning of the level that occurs immediately above.

On this basis, we put:

$$\mathtt{Hol}_\mathsf{T}(\alpha) := \mathtt{Hol}_\mathsf{T}(Level_1^\alpha),$$

for any formula α.

Notice that any $\mathtt{Hol}_{\mathsf{T}}(\alpha)$ represents a kind of autonomous semantic context that is not necessarily correlated with the meanings of other formulas. Generally we have:

$$\mathtt{Hol}^{\gamma}_{\mathsf{T}}(\beta) \neq \mathtt{Hol}^{\delta}_{\mathsf{T}}(\beta).$$

Thus, one and the same formula may receive different contextual meanings in different contexts (as, in fact, happens in the case of our normal use of natural languages).

Consider now a formula α whose atomic complexity is n. By definition of model we have: $\mathtt{Hol}_{\mathsf{T}}(\alpha) \in \mathsf{D}(\mathcal{H}^{(n)})$. From an intuitive point of view, the qumix $Red^{(2)}_{[n-1,1]}(\mathtt{Hol}_{\mathsf{T}}(\alpha))$ (which lives the space $\mathbb{C}^2$) can be regarded as a *generalized truth-value* of α (determined by the model $\mathtt{Hol}_{\mathsf{T}}$). At the same time, the number $\mathtt{p}_{\mathsf{T}}(\mathtt{Hol}(\alpha))$ represents the probability-value of α with respect to the truth-perspective T (determined by the model $\mathtt{Hol}_{\mathsf{T}}$). Accordingly, our semantics can be described as a *two-level many valued semantics*, where for any choice of a model $\mathtt{Hol}_{\mathsf{T}}$, any formula receives two correlated *semantic values*: a generalized truth-value (represented by a density operator of $\mathbb{C}^2$) and a probability-value (a real number in the interval $[0, 1]$).

Now the concepts of *truth*, *validity* and *logical consequence* can be defined in terms of the probability-function $\mathtt{p}_{\mathsf{T}}$.

Definition 4.6. (Truth)
A formula α is called true *with respect to a model $\mathtt{Hol}_{\mathsf{T}}$ (abbreviated as $\models_{\mathtt{Hol}_{\mathsf{T}}} \alpha$) iff $\mathtt{p}_{\mathsf{T}}(\mathtt{Hol}_{\mathsf{T}}(\alpha)) = 1$.*

Definition 4.7. (Validity)

1) *α is called* T-valid *($\models_{\mathsf{T}} \alpha$) iff for any model $\mathtt{Hol}_{\mathsf{T}}$, $\models_{\mathtt{Hol}_{\mathsf{T}}} \alpha$.*
2) *α is called* valid *($\models \alpha$) iff for any truth-perspective T, $\models_{\mathsf{T}} \alpha$.*

Definition 4.8. (Logical consequence)

1) *β is called a* T-logical consequence *of α ($\alpha \vDash_{\mathsf{T}} \beta$) iff for any formula γ such that α and β are subformulas of γ and for any model $\mathtt{Hol}_{\mathsf{T}}$,*
$$\mathtt{p}_{\mathsf{T}}(\mathtt{Hol}^{\gamma}_{\mathsf{T}}(\alpha)) \leq \mathtt{p}_{\mathsf{T}}(\mathtt{Hol}^{\gamma}_{\mathsf{T}}(\beta)).$$
2) *β is called a* logical consequence *of α ($\alpha \vDash \beta$) iff for any truth-perspective T, $\alpha \vDash_{\mathsf{T}} \beta$.*

When $\alpha \vDash_{\mathtt{I}} \beta$, we say that β is a *canonical logical consequence* of α.

The concept of logical consequence turns out to be invariant with respect to truth-perspective changes.

Lemma 4.1. [8]
$\alpha \vDash \beta$ iff $\alpha \vDash_{\mathtt{I}} \beta$ iff there is a truth-perspective T such that $\alpha \vDash_{\mathsf{T}} \beta$.

Although the holistic semantics is strongly context-dependent, one can prove that the logical consequence-relation is reflexive and transitive.

Theorem 4.1. [11]

(1) $\alpha \vDash \alpha$;
(2) $\alpha \vDash \beta$ *and* $\beta \vDash \delta \;\Rightarrow\; \alpha \vDash \delta$.

The concept of logical consequence, defined in this semantics, characterizes a special form of *holistic quantum computational logic*. One is dealing with a very weak form of quantum logic, where some standard logical arguments are generally violated [11]. We have, for instance:

(1) $\alpha \nvDash \alpha \wedge \alpha$
(2) $\alpha \wedge \beta \nvDash \beta \wedge \alpha$
(3) $\alpha \wedge (\beta \wedge \delta) \nvDash (\alpha \wedge \beta) \wedge \delta$
(4) $(\alpha \wedge \beta) \wedge \delta \nvDash \alpha \wedge (\beta \wedge \delta)$
(5) $\alpha \wedge (\beta \vee \delta) \nvDash (\alpha \wedge \beta) \vee (\alpha \wedge \delta)$
(6) $(\alpha \wedge \beta) \vee (\alpha \wedge \delta) \nvDash \alpha \wedge (\beta \vee \delta)$
(7) $\delta \vDash \alpha$ and $\delta \vDash \beta \;\not\Rightarrow\; \delta \vDash \alpha \wedge \beta$
(8) $\nvDash \neg(\alpha \wedge \neg\alpha)$
(9) $\alpha \wedge \neg\alpha \nvDash \beta$

Some important logical consequences that hold in this logic are the following:

(1) $\alpha \wedge \beta \vDash \alpha$; $\alpha \wedge \beta \vDash \beta$
(2) $\alpha \vDash \beta \;\Rightarrow\; \alpha \wedge \delta \vDash \beta$
(3) $\neg\neg\alpha \vDash \alpha$; $\alpha \vDash \neg\neg\alpha$.
(4) $\alpha \vDash \beta \;\Rightarrow\; \neg\beta \vDash \neg\alpha$
(5) $\mathbf{f} \vDash \beta$; $\beta \vDash \mathbf{t}$

Since the conjunction $\wedge$ is generally non-associative, brackets cannot be omitted in the case of multiple conjunctions. We will use the expression $\beta_1 \wedge \ldots \wedge \beta_n$ as a metalinguistic abbreviation for any possible bracket-configuration in a multiple conjunction whose members are the elements of the sequence $(\beta_1, \ldots, \beta_n)$.

5. An epistemic quantum computational semantics

We will now investigate the semantics for the language $\mathcal{L}^{-Ep}$, which represents the epistemic extension of $\mathcal{L}^-$ that includes all quantifier-free epistemic formulas of $\mathcal{L}$.[4] This semantics is based on the following intuitive idea: any occurrence of an epistemic operator K in a formula α is interpreted as a special example of a qumix-operation representing a *knowledge-operation* associated to a given epistemic agent, which is characterized by a particular truth-perspective. Of course "real" agents evolve in time, changing their knowledge; for the sake of simplicity, however, in this article we will abstract from time, assuming that all agents are referred to a particular "short" time-interval.

[4] A semantics for a sentential epistemic quantum computational language has been studied in [6, 7].

We will first introduce the notion of *knowledge-operation* of a Hilbert space $\mathcal{H}^{(n)}$ with respect to a truth-perspective T:

Definition 5.1. (Knowledge-operation)
A knowledge-operation *of the space $\mathcal{H}^{(n)}$ with respect to the truth-perspective T is a map*

$$\mathbf{K}_{\mathsf{T}}^{(n)} : \mathcal{B}(\mathcal{H}^{(n)}) \mapsto \mathcal{B}(\mathcal{H}^{(n)}),$$

where $\mathcal{B}(\mathcal{H}^{(n)})$ is the set of all bounded operators of $\mathcal{H}^{(n)}$. The following conditions are required:

(1) $\mathbf{K}_{\mathsf{T}}^{(n)}$ *is associated with an* epistemic domain $EpD(\mathbf{K}_{\mathsf{T}}^{(n)})$, *which is a subset of $\mathsf{D}(\mathcal{H}^{(n)})$;*
(2) *for any $\rho \in \mathsf{D}(\mathcal{H}^{(n)})$, $\mathbf{K}_{\mathsf{T}}^{(n)}\rho \in \mathsf{D}(\mathcal{H}^{(n)})$;*
(3) $\mathsf{p}_{\mathsf{T}}(\mathbf{K}_{\mathsf{T}}^{(n)}\rho) \leq \mathsf{p}_{\mathsf{T}}(\rho)$, *for any $\rho \in EpD(\mathbf{K}_{\mathsf{T}}^{(n)})$;*
(4) $\forall \rho \in \mathsf{D}(\mathcal{H}^{(n)}) : \rho \notin EpD(\mathbf{K}_{\mathsf{T}}^{(n)}) \;\Rightarrow\; \mathbf{K}_{\mathsf{T}}^{(n)}\rho = \overline{\rho_0}$ *(where $\overline{\rho_0}$ is a fixed density operator of $\mathsf{D}(\mathcal{H}^{(n)})$).*

As expected, the intuitive interpretation of $\mathbf{K}_{\mathsf{T}}^{(n)}\rho$ is the following: "the piece of information ρ is known". The knowledge described by $\mathbf{K}_{\mathsf{T}}^{(n)}$ is limited by a given epistemic domain (which is intended to represent the information accessible to a given agent, relatively to the space $\mathcal{H}^{(n)}$).[5] Whenever an information ρ does not belong to the epistemic domain of $\mathbf{K}_{\mathsf{T}}^{(n)}$, then $\mathbf{K}_{\mathsf{T}}^{(n)}\rho$ collapses into a fixed element $\overline{\rho_0}$ (which may be identified, for instance, with the maximally uncertain information $\frac{1}{2^n}\mathtt{I}^{(n)}$ or with the T-Falsity ${}^{\mathsf{T}}P_0^{(n)}$). At the same time, whenever ρ belongs to the epistemic domain of $\mathbf{K}_{\mathsf{T}}^{(n)}$, it seems reasonable to assume that the probability-values of ρ and $\mathbf{K}_{\mathsf{T}}^{(n)}\rho$ are correlated: the probability of the quantum information asserting that "ρ is known" should always be less than or equal to the probability of ρ. Hence, in particular, we have:

$$\mathrm{p}_{\mathsf{T}}(\mathbf{K}_{\mathsf{T}}^{(n)}\rho) = 1 \;\Rightarrow\; \mathrm{p}_{\mathsf{T}}(\rho) = 1.$$

But generally, not the other way around! In other words, pieces of quantum information that are certainly known are certainly true (with respect to the truth-perspective in question). This condition is clearly in agreement with a general principle of classical epistemic logics, according to which "knowledge implies truth, but no the other way around".

A knowledge-operation $\mathbf{K}_{\mathsf{T}}^{(n)}$ is called *non-trivial* iff for at least one density operator $\rho \in EpD(\mathbf{K}_{\mathsf{T}}^{(n)})$, $\mathrm{p}_{\mathsf{T}}(\mathbf{K}_{\mathsf{T}}^{(n)}\rho) < \mathrm{p}_{\mathsf{T}}(\rho)$. Notice that knowledge-operations do not generally preserve pure states [5].

[5]The *epistemic domain* of $\mathbf{K}_{\mathsf{T}}^{(n)}$ should not be confused with the *domain* of $\mathbf{K}_{\mathsf{T}}^{(n)}$, which coincides with the set of all bounded operators of the space. In particular, we have that $\mathbf{K}_{\mathsf{T}}^{(n)}\rho$ is defined, even if ρ does not belong to the epistemic domain of $\mathbf{K}_{\mathsf{T}}^{(n)}$.

Can knowledge-operations be described as special examples of gates? The following theorem gives a negative answer to this question.

Theorem 5.1. [5]
Non-trivial knowledge-operations cannot be represented as unitary quantum operations.

At the same time knowledge-operations can be represented as qumix operations that are generally irreversible. The *act of knowing* seems to be characterized by an intrinsic irreversibility, which is quite similar to quantum measurement-phenomena.

On this basis we can now define the notions of *epistemic situation* and of *epistemic realization* for the epistemic language $\mathcal{L}^{-Ep}$.

Definition 5.2. (Epistemic situation)
Let i *represent an epistemic agent (say,* Alice, Bob, *...). An* epistemic situation *for* i *is a pair*

$$EpSit_{\mathsf{i}} = (\mathsf{T}_{\mathsf{i}}, \mathbf{K}_{\mathsf{i}}),$$

where:

(1) T_{i} *represents the truth-perspective of* i*;*
(2) $\mathbf{K}_{\mathsf{i}}$ *is a map that assigns to any* $n \geq 1$ *a knowledge-operation* $\mathbf{K}^{(n)}_{\mathsf{T}_{\mathsf{i}}}$ *(defined on* $\mathcal{H}^{(n)}$*), which represents the knowledge of* i *with respect to the information-environment* $\mathsf{D}(\mathcal{H}^{(n)})$*.*

The concept of *normal holistic map* $\mathtt{Hol}_{\mathsf{T}}$ for the language $\mathcal{L}^{-Ep}$ and the *contextual meanings* $\mathtt{Hol}^{\gamma}_{\mathsf{T}}(t)$, $\mathtt{Hol}^{\gamma}_{\mathsf{T}}(\mathbf{P}^m)$, $\mathtt{Hol}^{\gamma}_{\mathsf{T}}(\beta)$ (for any term t, any predicate $\mathbf{P}^m$ and any formula β occurring in γ) are defined like in the case of the language $\mathcal{L}^{-}$.

Definition 5.3. (Epistemic realization)
An epistemic realization for the language $\mathcal{L}^{-Ep}$ *is a pair* $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$*, where* $\mathtt{Hol}_{\mathsf{T}}$ *is a normal holistic map for the language* $\mathcal{L}^{-Ep}$ *and* E_{T} *is an epistemic map that associates to any pair* (α, t) *consisting of a formula* α *and of a term* t *occurring in an epistemic connective* Kt *of* α *an epistemic situation*

$$\mathsf{E}_{\mathsf{T}}(\alpha, t) = (\mathsf{T}_{\mathtt{Hol}^{\alpha}_{\mathsf{T}}(t)}, \mathbf{K}_{\mathtt{Hol}^{\alpha}_{\mathsf{T}}(t)}).$$

As expected, $\mathsf{E}_{\mathsf{T}}(\alpha, t)$ represents the epistemic situation of the agent corresponding to the contextual meaning of the term t in the context $\mathtt{Hol}_{\mathsf{T}}(\alpha)$. Notice that generally

$$\mathsf{T} \neq \mathsf{T}_{\mathtt{Hol}^{\alpha}_{\mathsf{T}}(t)}.$$

In other words, the truth-perspective of the agent denoted by the term t (according to the map $\mathtt{Hol}_{\mathsf{T}}$) does not necessarily coincide with the truth-perspective of the holistic map $\mathtt{Hol}_{\mathsf{T}}$. In the next Section we will see how these truth-perspective differences may cause some interesting *relativistic-like epistemic effects*.

Any epistemic realization $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$ determines for any formula α a special gate tree, called the $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$ - *epistemic pseudo gate tree of* α. As an example, consider the following epistemic sentence:

$$\alpha = K\mathbf{a}\neg K\mathbf{b}\mathbf{P}^1\mathbf{a}$$

(say, *Alice knows that Bob does not know that she is pretty*).
We have: $\mathcal{H}^\alpha = \mathcal{H}^{(3)}$. The syntactical tree of α is:

$$Level_4^\alpha = (\mathbf{P}^1\mathbf{a})$$
$$Level_3^\alpha = (K\mathbf{b}\mathbf{P}^1\mathbf{a})$$
$$Level_2^\alpha = (\neg K\mathbf{b}\mathbf{P}^1\mathbf{a})$$
$$Level_1^\alpha = (K\mathbf{a}\neg K\mathbf{b}\mathbf{P}^1\mathbf{a})$$

Let $(\mathsf{T}_{\mathtt{Hol}_{\mathsf{T}}^\alpha(\mathbf{a})}, \mathbf{K}_{\mathtt{Hol}_{\mathsf{T}}^\alpha(\mathbf{a})})$ and $(\mathsf{T}_{\mathtt{Hol}_{\mathsf{T}}^\alpha(\mathbf{b})}, \mathbf{K}_{\mathtt{Hol}_{\mathsf{T}}^\alpha(\mathbf{b})})$ be the two epistemic situations associated by the epistemic realization $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$ to the two pairs $(\alpha, \mathbf{a})$ and $(\alpha, \mathbf{b})$. In such a case the $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$ - epistemic pseudo gate tree of α can be naturally identified with the following sequence of qumix operations:

$$(\mathbf{K}^{(3)}_{\mathsf{T}_{\mathtt{Hol}_{\mathsf{T}}^\alpha(\mathbf{b})}}, {}^{\mathsf{D}}\mathtt{NOT}^{(3)}_{\mathsf{T}}, \mathbf{K}^{(3)}_{\mathsf{T}_{\mathtt{Hol}_{\mathsf{T}}^\alpha(\mathbf{a})}}).$$

This procedure can be obviously generalized. For any formula α, the choice of an epistemic realization $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$ determines the $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$ - epistemic pseudo gate tree of α, indicated as follows:

$$({}^{\mathsf{D}}G^{(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})}_{\mathsf{T}_{(h-1)}}, \ldots, {}^{\mathsf{D}}G^{(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})}_{\mathsf{T}_{(1)}}).$$

Of course, epistemic pseudo gate trees are generally irreversible. It is worth-while noticing that, unlike the case of $\mathcal{L}^-$, epistemic pseudo gate trees are not uniquely determined by the formulas' syntactical trees. Any epistemic realization $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$ chooses for any α a particular interpretation of the epistemic connectives occurring in α.

Now the concept of *holistic model* for the language $\mathcal{L}^{-Ep}$ can be defined in the expected way. Like in the case of $\mathcal{L}^-$, any model $\mathtt{Hol}_{\mathsf{T}}$ shall preserve the logical form of any formula α , by interpreting the epistemic connectives occurring in α as convenient epistemic operations.

Definition 5.4. (Holistic model of $\mathcal{L}^{-Ep}$)
A holistic model *of* $\mathcal{L}^{-Ep}$ *is an epistemic realization* $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$ *that satisfies the following condition for any formula* α*:*
if $({}^{\mathsf{D}}G^{(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})}_{\mathsf{T}_{(h-1)}}, \ldots, {}^{\mathsf{D}}G^{(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})}_{\mathsf{T}_{(1)}})$ *is the* $(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})$ - *epistemic pseudo gate tree of* α *and* $1 \leq i < h$*, then,*

$$\mathtt{Hol}_{\mathsf{T}}(Level_i(\alpha)) = {}^{\mathsf{D}}G^{(\mathtt{Hol}_{\mathsf{T}}, \mathsf{E}_{\mathsf{T}})}_{\mathsf{T}_{(i)}}(\mathtt{Hol}_{\mathsf{T}}(Level_{i+1}(\alpha))).$$

In other words, the meaning of each level (different from the top level) is obtained by applying the corresponding gate (or pseudo gate) to the meaning of the level that occurs immediately above.

On this basis the concepts of *truth*, *validity* and *logical consequence* are defined like in the case of the language $\mathcal{L}^-$, *mutatis mutandis*.[6]

It is interesting to classify some special kinds of epistemic models that satisfy particular restrictions.

Definition 5.5. (Special models)
Let $\mathtt{EHol}_\mathsf{T} = (\mathtt{Hol}_\mathsf{T}, \mathsf{E}_\mathsf{T})$ *be a model of* $\mathcal{L}^{-Ep}$.

(1) $\mathtt{EHol}_\mathsf{T}$ *is called* harmonic *iff for all epistemic situations* $(\mathsf{T}_\mathsf{i}, \mathbf{K}_{\mathsf{T}_\mathsf{i}})$ *determined by* $\mathtt{EHol}_\mathsf{T}$, $\mathsf{T}_\mathsf{i} = \mathsf{T}$. *Hence, all agents considered by* $\mathtt{EHol}_\mathsf{T}$ *share the same truth-perspective* T.

(2) $\mathtt{EHol}_\mathsf{T}$ *is called* sound *iff for all epistemic situations* $(\mathsf{T}_\mathsf{i}, \mathbf{K}_{\mathsf{T}_\mathsf{i}})$ *determined by* $\mathtt{EHol}_\mathsf{T}$, *the qumixes* ${}^{\mathsf{T}_\mathsf{i}}P_1^{(1)}$ *and* ${}^{\mathsf{T}_\mathsf{i}}P_0^{(1)}$ *belong to the epistemic domain of* $\mathbf{K}_{\mathsf{T}_i}^{(1)}$. *Furthermore,* $\mathbf{K}_{\mathsf{T}_i}^{(1)}\, {}^{\mathsf{T}_\mathsf{i}}P_1^{(1)} = {}^{\mathsf{T}_\mathsf{i}}P_1^{(1)}$ *and* $\mathbf{K}_{\mathsf{T}_i}^{(1)}\, {}^{\mathsf{T}_\mathsf{i}}P_0^{(1)} = {}^{\mathsf{T}_\mathsf{i}}P_0^{(1)}$. *In other words, any agent* i *has access to the truth-values of his/her truth-perspective, assigning to them the "right" probability-values.*

(3) $\mathtt{EHol}_\mathsf{T}$ *is called* falsity-based *iff for any epistemic situation* $(\mathsf{T}_\mathsf{i}, \mathbf{K}_{\mathsf{T}_\mathsf{i}})$ *determined by* $\mathtt{EHol}_\mathsf{T}$, *the following condition is satisfied: for any* $\rho \notin EpD(\mathbf{K}_{\mathsf{T}_\mathsf{i}}^{(n)})$, $\mathbf{K}_{\mathsf{T}_\mathsf{i}}^{(n)}\rho = {}^{\mathsf{T}_\mathsf{i}}P_0^{(n)}$.

(4) $\mathtt{EHol}_\mathsf{T}$ *is called* perfect *iff any agent* i *of an epistemic situation* $(\mathsf{T}_\mathsf{i}, \mathbf{K}_{\mathsf{T}_\mathsf{i}})$ *determined by* $\mathtt{EHol}_\mathsf{T}$ *has a* perfect epistemic capacity, *satisfying the following conditions:*

(4.1) *the epistemic domain of* $\mathbf{K}_{\mathsf{T}_\mathsf{i}}^{(n)}$ *coincides with the set of all possible qumixes of* $\mathcal{H}^{(n)}$ *(for any* $n \geq 1$*);*

(4.2) *for any qumix* ρ $(\in \mathsf{D}(\mathcal{H}^{(n)}))$, $\mathbf{K}_{\mathsf{T}_\mathsf{i}}^{(n)}\rho = \rho$. *Hence,* i *assigns the "right" probability-values to all pieces of information.*

Notice that a perfect epistemic capacity does not imply *omniscience* (i.e. the capacity of *semantically deciding* any sentence). For, the *semantic excluded-middle principle*:

$$\text{either } \models_{\mathtt{EHol}_\mathsf{T}} \alpha \text{ or } \models_{\mathtt{EHol}_\mathsf{T}} \neg\alpha$$

does not generally hold (as happens in all forms of quantum logic).

Models that are at the same time harmonic, sound and falsity-based will be also called *simple*. By *simple epistemic (quantum computational) semantics* we will mean the special case of epistemic semantics based on the assumption that all models are simple.

When α is valid or β is a logical consequence of α in the simple semantics we will also write:

$$\models^{Simple} \alpha; \ \ \alpha \models^{Simple} \beta.$$

Let us finally recall some significant examples of epistemic arguments that are either valid or possibly violated in this semantics.

[6] In [7] we have considered a different concept of *logical consequence* for the case of a sentential epistemic quantum computational language.

1) $\mathcal{K}t\alpha \models^{Simple} \alpha$.
   In the simple semantics, knowing a formula implies the formula itself. Of course this relation does not hold in the general epistemic semantics, where non-harmonic models may refer to different truth-perspectives of different agents.
2) As a particular case of 1) we obtain:
   $\mathcal{K}t\alpha\mathcal{K}t\alpha \models^{Simple} \mathcal{K}t\alpha$.
   Knowing of knowing implies knowing. But not the other way around!
3) $\mathcal{K}t_1\mathcal{K}t_2\alpha \models^{Simple} \alpha$.
   In the simple semantics, knowing that another agent knows a given formula implies the formula in question. At the same time, we have:
   $\mathcal{K}t_1\mathcal{K}t_2\alpha \not\models^{Simple} \mathcal{K}t_1\alpha$.
   Alice might know that Bob knows a given formula, without knowing herself the formula in question!
4) $\models^{Simple} \mathcal{K}t\mathbf{t}$; $\models^{Simple} \mathcal{K}t\neg\mathbf{f}$.
   Hence, there are sentences that everybody knows.
5) $\mathcal{K}t(\alpha \wedge \beta) \not\models \mathcal{K}t\alpha$; $\mathcal{K}t(\alpha \wedge \beta) \not\models \mathcal{K}t\beta$.
   Knowing a conjunction does not generally imply knowing its members.
6) $\mathcal{K}t\alpha \wedge \mathcal{K}t\beta \not\models \mathcal{K}t(\alpha \wedge \beta)$.
   Knowledge is not generally closed under conjunction.
7) For any model $\mathtt{EHol}_\mathtt{T}$, $\not\models_{\mathtt{EHol}_\mathtt{T}} \mathcal{K}t(\alpha \wedge \neg\alpha)$.
   Contradictions are never known.
8) In the non-simple semantics (where models are not necessarily harmonic) the following situation is possible:
   $\models_{\mathtt{EHol}_\mathtt{T}} \mathcal{K}\mathbf{a}\mathcal{K}\mathbf{bf}$.
   In other words, according to the truth-perspective of Alice it is true that Alice knows that Bob knows the *Falsity* of Alice's truth-perspective. Roughly, we might say: Alice knows that Bob is wrong. However, Bob is not aware of being wrong!

The examples illustrated above seem to reflect pretty well some characteristic limitations of the real processes of acquiring information and knowledge. Owing to the limits of epistemic domains, knowledge is not generally closed under logical consequence. Hence, the unpleasant phenomenon of *logical omniscience* is here avoided: Alice might know a given sentence without knowing *all* its logical consequences. We have, in particular, that knowledge is not generally closed under logical conjunction, in accordance with what happens in the case of concrete memories both of human and of artificial intelligence. It is also admitted that an agent knows a conjunction, without knowing its members. Such situation, which might appear *prima facie* somewhat "irrational", seems to be instead deeply in agreement with our use of natural languages, where sometimes agents show to use correctly and to understand some *global* expressions without being able to understand their (meaningful) parts.

6. Physical examples and relativistic-like epistemic effects

We will now illustrate some examples of knowledge-operations that may be interesting from a physical point of view. One is dealing with special cases of *quantum channels*, which can be, generally, obtained from some unitary operators, tracing out the ancillary qubits that describe the environment.

Let us first recall the definition of *quantum channel* (which is based on the so called *Kraus first representation theorem* [13]).

Definition 6.1. (Quantum channel)
A quantum channel *on* $\mathcal{H}^{(n)}$ *is a linear map* $\mathcal{E}^{(n)} : \mathcal{B}(\mathcal{H}^{(n)}) \mapsto \mathcal{B}(\mathcal{H}^{(n)})$ *such that for some set* I *of indices there exists a set* $\{E_i\}_{i\in I}$ *of elements of* $\mathcal{B}(\mathcal{H}^{(n)})$ *satisfying the following conditions:*

(1) $\sum_i E_i^\dagger E_i = \mathtt{I}^{(n)}$;
(2) $\forall A \in \mathcal{B}(\mathcal{H}^{(n)}) : \mathcal{E}^{(n)}(A) = \sum_i E_i A E_i^\dagger$.

A set $\{E_i\}_{i\in I}$ such that $\sum_i E_i^\dagger E_i = \mathtt{I}^{(n)}$ is called a *system of Kraus operators*. One can prove that quantum channels are trace-preserving, and hence transform density operators into density operators.

Of course, unitary quantum operations ${}^{\mathsf{D}}G^{(n)}$ are special cases of quantum channels, for which $\{E_i\}_{i\in I} = \left\{G^{(n)}\right\}$. In the general case, however, quantum channels cannot be represented as unitary quantum operations: one is dealing with some characteristic irreversible transformations.

We will now define a class of quantum channels that have a special physical interest. Let a, b, c be complex numbers such that $|a|^2 + |b|^2 + |c|^2 \leq 1$. Consider the following system of Kraus operators of $\mathbb{C}^2$:

$$\begin{aligned} E_0 &= \sqrt{1-|a|^2-|b|^2-|c|^2}\,\mathtt{I} \\ E_1 &= |a|\sigma_x \\ E_2 &= |b|\sigma_y \\ E_3 &= |c|\sigma_z \end{aligned}$$

(where σ_x, σ_y, σ_z are the three Pauli matrices).
Define ${}^{a,b,c}\mathcal{E}^{(1)}$ as follows for any $\rho \in \mathsf{D}(\mathbb{C}^2)$:

$${}^{a,b,c}\mathcal{E}^{(1)}\rho = \sum_{i=0}^{3} E_i\,\rho\,E_i^\dagger.$$

We have:

$${}^{a,b,c}\mathcal{E}^{(1)}\rho = (1-|a|^2-|b|^2-|c|^2)\rho + |a|^2\sigma_x\rho\sigma_x + |b|^2\sigma_y\rho\sigma_y + |c|^2\sigma_z\rho\sigma_z.$$

One can prove that for any choice of a, b, c (such that $|a|^2+|b|^2+|c|^2 \leq 1$), the map ${}^{a,b,c}\mathcal{E}^{(1)}$ is a quantum channel of the space $\mathbb{C}^2$.

Let us now refer to the *Bloch-sphere* (whose radius is 1) that is in one-to one correspondence with $\mathsf{D}(\mathbb{C}^2)$.[7] Any map ${}^{a,b,c}\mathcal{E}^{(1)}$ induces the following

[7]We recall that the bijection f from the Bloch-sphere onto $\mathsf{D}(\mathbb{C}^2)$ is determined as follows: for any $x, y, z \in \mathbb{R}$ such that $|x|^2+|y|^2+|z|^2 \leq 1$, $f(x,y,z) = \frac{1}{2}(\mathrm{I}+\mathrm{x}\sigma_\mathrm{x}+\mathrm{y}\sigma_\mathrm{y}+\mathrm{z}\sigma_\mathrm{z})$.

vector-transformation (the sphere is deformed into an ellipsoid centered at the origin):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} (1-2|b|^2-2|c|^2)\,x \\ (1-2|a|^2-2|c|^2)\,y \\ (1-2|a|^2-2|b|^2)\,z \end{pmatrix}$$

For particular choices of a, b and c, one obtains some special cases of quantum channels.

- For $a=b=c=0$, one obtains the identity operator.
- For $b=c=0$, one obtains the *bit-flip channel* ${}^{a}\mathcal{BF}^{(1)}$ that flips the two canonical bits (represented as the projection operators ${}^{\mathtt{I}}P_0^{(1)}$ and ${}^{\mathtt{I}}P_1^{(1)}$) with probability $|a|^2$:

  $${}^{\mathtt{I}}P_0^{(1)} \mapsto (1-|a|^2)\,{}^{\mathtt{I}}P_0^{(1)} + (|a|^2)\,{}^{\mathtt{I}}P_1^{(1)};$$

  $${}^{\mathtt{I}}P_1^{(1)} \mapsto (1-|a|^2)\,{}^{\mathtt{I}}P_1^{(1)} + (|a|^2)\,{}^{\mathtt{I}}P_0^{(1)}.$$

  The sphere is mapped into an ellipsoid with x as symmetry-axis (see Fig. 1).

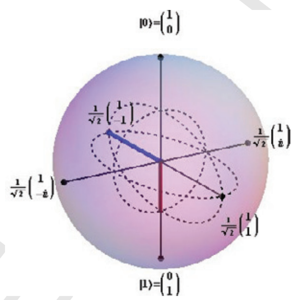


Figure 1. The bit-flip channel

- For $a=c=0$, one obtains the *bit-phase-flip channel* ${}^{b}\mathcal{BPF}^{(1)}$ that flips both bits and phase with probability $|b|^2$. The sphere is mapped into an ellipsoid with y as symmetry-axis.
- For $a=b=0$, one obtains the *phase-flip channel* ${}^{c}\mathcal{PF}^{(1)}$ that flips the phase with probability $|c|^2$. The sphere is mapped into an ellipsoid with z as symmetry-axis.
- For $|a|^2=|b|^2=|c|^2=\frac{p}{4}$ (with $p \le 1$), one obtains the *depolarizing channel* ${}^{p}\mathcal{D}^{(1)}$. If $p=1$, the polarization along any direction is equal to 0. The sphere is contracted by a factor $1-p$ and the center of the sphere is a fixed point.

The channels we have considered above have been defined with respect to the canonical truth-perspective $\mathtt{I}$. However, as expected, they can be naturally transposed to any truth-perspective $\mathtt{T}$. Given $\mathcal{E}^{(1)}$ such that

$\mathcal{E}^{(1)}\rho = \sum_{i=0}^{3} E_i\,\rho\,E_i^\dagger$, the twin-channel $\mathcal{E}_{\mathsf{T}}^{(1)}$ of $\mathcal{E}^{(1)}$ can be defined as follows:

$$\mathcal{E}_{\mathsf{T}}^{(1)}\rho := \sum_i \mathsf{T}E_i\mathsf{T}^\dagger\,\rho\,\mathsf{T}E_i^\dagger\mathsf{T}^\dagger.$$

So far we have only considered quantum channels of the space $\mathbb{C}^2$. At the same time, any operation $\mathcal{E}_{\mathsf{T}}^{(1)}$ (defined on $\mathbb{C}^2$) can be canonically extended to an operation $\mathcal{E}_{\mathsf{T}}^{(n)}$ defined on the space $\mathcal{H}^{(n)}$ (for any $n > 1$). Consider a density operator ρ of $\mathcal{H}^{(n)}$ and the reduced state $Red^{(2)}_{[n-1,1]}(\rho)$ (which describes the n-th subsystem of the composite system described by ρ). We have: $\mathrm{p}_{\mathsf{T}}(\rho) = \mathtt{Tr}({}^{\mathsf{T}}P_1^{(1)} Red^{(2)}_{[n-1,1]}(\rho))$. In other words, the T-probability of ρ only depends on the T-probability of the reduced state that describes the n-th subsystem. On this basis, it is reasonable to define $\mathcal{E}_{\mathsf{T}}^{(n)}$ as follows:

$$\mathcal{E}_{\mathsf{T}}^{(n)} = \mathtt{I}^{(n-1)} \otimes \mathcal{E}_{\mathsf{T}}^{(1)}.$$

Notice that, generally, a quantum channel does not represent a knowledge-operation. We have, for instance, for some density operators ρ:

$$\mathrm{p}_{\mathtt{I}}({}^a\mathcal{BF}^{(1)}\rho) \nleq \mathrm{p}_{\mathtt{I}}(\rho),$$

against the definition of knowledge-operation, if ρ is supposed to belong to the epistemic domain of ${}^a\mathcal{BF}^{(1)}$. At the same time, by convenient choices of the epistemic domains, our quantum channels can be transformed into knowledge-operations.

Definition 6.2. (A bit-flip knowledge-operation ${}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}$)
Let $a \neq 0$. Define ${}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}$ as follows:

(1) $EpD({}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}) \subseteq D = \{\rho \in \mathsf{D}(\mathcal{H}^{(n)}) \mid \mathrm{p}_{\mathsf{T}}(\rho) \geq \frac{1}{2}\}$.
In other words an agent (whose knowledge-operation is ${}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}$) has only access to pieces of information that are not "too far from the truth".

(2) $\rho \in EpD({}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}) \Rightarrow {}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}\rho = {}^a\mathcal{BF}_{\mathsf{T}}^{(n)}\rho$.

Theorem 6.1.

(i) *Any ${}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}$ is a knowledge-operation. In particular, ${}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}$ is a non-trivial knowledge operation if there exists at least one $\rho \in EpD({}^a\mathbf{KBF}_{\mathsf{T}}^{(n)})$ such that $\mathrm{p}_{\mathsf{T}}(\rho) > \frac{1}{2}$.*

(ii) *the set D is the maximal set such that the corresponding ${}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}$ is a knowledge-operation.*

(iii) *Let $|a|^2 \leq \frac{1}{2}$ and let $EpD({}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}) = D$. The following closure property holds: for any $\rho \in D$, ${}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}\rho \in D$.*

Proof. (i)-(ii) Suppose that $\rho \in EpD({}^a\mathbf{KBF}_{\mathsf{T}}^{(n)}) \subseteq D$ and let us represent the density operator $\mathsf{T}^\dagger Red^{(2)}_{[n-1,1]}(\rho)\mathsf{T}$ as $\frac{1}{2}(\mathrm{I} + \mathrm{x}\sigma_{\mathrm{x}} + \mathrm{y}\sigma_{\mathrm{y}} + \mathrm{z}\sigma_{\mathrm{z}})$. We have:

$\mathrm{p}_{\mathsf{T}}({}^{a}\mathbf{KBF}^{(n)}_{\mathsf{T}}\rho) = \mathtt{Tr}({}^{\mathsf{T}}P^{(n)}_{1}\,{}^{a}\mathbf{KBF}^{(n)}_{\mathsf{T}}\rho) = \mathtt{Tr}(\mathsf{T}\,{}^{\mathrm{I}}P^{(1)}_{1}\mathsf{T}^{\dagger}\sum_i \mathsf{T}E_i\mathsf{T}^{\dagger}\,Red^{(2)}_{[n-1,1]}(\rho)$
$\mathsf{T}E_i^{\dagger}\mathsf{T}^{\dagger}) = \mathtt{Tr}({}^{\mathrm{I}}P^{(1)}_{1}\sum_i E_i\mathsf{T}^{\dagger}\,Red^{(2)}_{[n-1,1]}(\rho)\,\mathsf{T}E_i^{\dagger}) = \frac{1-(1-2|a|^2)z}{2}$;
$\mathrm{p}_{\mathsf{T}}(\rho) = \mathtt{Tr}({}^{\mathsf{T}}P^{(n)}_{1}\rho) = \mathtt{Tr}(\mathsf{T}\,{}^{\mathrm{I}}P^{(1)}_{1}\mathsf{T}^{\dagger}Red^{(2)}_{[n-1,1]}(\rho)) = \frac{1-z}{2}$.
Hence, $\mathrm{p}_{\mathsf{T}}({}^{a}\mathbf{KBF}^{(n)}_{\mathsf{T}}\rho) \leq \mathrm{p}_{\mathsf{T}}(\rho) \Leftrightarrow (1-2|a|^2)z \geq z \Leftrightarrow z \in [-1,0] \Leftrightarrow \mathrm{p}_{\mathsf{T}}(\rho) \geq \frac{1}{2}$.
Thus, ${}^{a}\mathbf{KBF}^{(n)}_{\mathsf{T}}$ is a knowledge operation and the set D is the maximal set such that the corresponding ${}^{a}\mathbf{KBF}^{(n)}_{\mathsf{T}}$ is a knowledge operation.
(iii) $\mathrm{p}_{\mathsf{T}}({}^{a}\mathbf{KBF}^{(n)}_{\mathsf{T}}\rho) = \frac{1-(1-2|a|^2)z}{2} \geq \frac{1}{2}$, since $|a|^2 \leq \frac{1}{2}$ and $z \in [-1,0]$. □

In a similar way one can define knowledge-operations that correspond to the phase-flip channel, the bit-phase-flip channel and the depolarizing channel.

Truth-perspectives are, in a sense, similar to different reference-frames in relativity. Accordingly, one could try and apply a “relativistic” way of thinking in order to describe how a given agent can “see” the logical behavior of another agent.

As an example let us refer to two agents Alice and Bob, whose truth-perspectives are T_{Alice} and T_{Bob}, respectively. Let $\{|1_{Alice}\rangle, |0_{Alice}\rangle\}$ and $\{|1_{Bob}\rangle, |0_{Bob}\rangle\}$ represent the systems of truth-values of our two agents. Furthermore, for any canonical gate ${}^{\mathsf{D}}G^{(n)}$ (defined with respect to the canonical truth-perspective $\mathtt{I}$), let ${}^{\mathsf{D}}G^{(n)}_{Alice}$ and ${}^{\mathsf{D}}G^{(n)}_{Bob}$ represent the corresponding *twin-gates* for Alice and for Bob, respectively.

According to the rule assumed in Section 2, we have:

$${}^{\mathsf{D}}G^{(n)}_{Alice} = {}^{\mathsf{D}}(\mathsf{T}^{(n)}_{Alice}G^{(n)}\mathsf{T}^{(n)\dagger}_{Alice}).$$

In a similar way in the case of Bob.

We will adopt the following conventional terminology.

- When $|1_{Bob}\rangle = a_0|0_{Alice}\rangle + a_1|1_{Alice}\rangle$, we will say that Alice *sees* that Bob’s Truth is $a_0|0_{Alice}\rangle + a_1|1_{Alice}\rangle$. In a similar way, for Bob’s *Falsity*.
- When ${}^{\mathsf{D}}G^{(n)}_{Alice} = {}^{\mathsf{D}}(\mathsf{T}^{(n)}_{Alice}G^{(n)}\mathsf{T}^{(n)\dagger}_{Alice})$ and ${}^{\mathsf{D}}G^{(n)}_{Bob} = {}^{\mathsf{D}}(\mathsf{T}^{(n)}_{Bob}G^{(n)}\mathsf{T}^{(n)\dagger}_{Bob}) = {}^{\mathsf{D}}G^{(n)}_{1_{Alice}}$ (where ${}^{\mathsf{D}}G^{(n)}$ and ${}^{\mathsf{D}}G^{(n)}_{1}$ are canonical gates), we will say that *Alice sees Bob using the gate* ${}^{\mathsf{D}}G^{(n)}_{1_{Alice}}$ *in place of her gate* ${}^{\mathsf{D}}G^{(n)}_{Alice}$.
- When ${}^{\mathsf{D}}G^{(n)}_{Alice} = {}^{\mathsf{D}}G^{(n)}_{Bob}$ we will say that Alice and Bob *see and use the same gate*, which represents (in their truth-perspective) the canonical gate ${}^{\mathsf{D}}G^{(n)}$.

On this basis, one can conclude that, generally, Alice *sees* a kind of “deformation” in Bob’s logical behavior. As an example, suppose that Alice has the canonical truth-perspective (i.e. $\mathsf{T}_{Alice} = \mathtt{I}^{(1)}$), while Bob's truth-perspective is the Hadamard-operator (i.e. $\mathsf{T}_{Bob} = \sqrt{\mathtt{I}}^{(1)}$). Accordingly, the truth-values systems of Alice and of Bob are the following:

- $\{|1_{Alice}\rangle, |0_{Alice}\rangle\} = \{|1\rangle, |0\rangle\}$;
- $\{|1_{Bob}\rangle, |0_{Bob}\rangle\} = \left\{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right\}$,

In such a case, Alice will *see* a quite strange behavior in Bob's use of the logical connective *negation*. Since ${}^{\mathsf{D}}\mathtt{NOT}^{(1)}_{Bob} = {}^{\mathsf{D}}(\sqrt{\mathtt{I}}^{(1)}\,\mathtt{NOT}^{(1)}\sqrt{\mathtt{I}}^{(1)\dagger})$, we will obtain, for instance, that:

$${}^{\mathsf{D}}\mathtt{NOT}^{(1)}_{Bob}\ {}^{\mathtt{I}}P^{(1)}_1 = {}^{\mathtt{I}}P^{(1)}_1 = P^{(1)}_{\frac{1}{\sqrt{2}}(|0_{Bob}\rangle - |1_{Bob}\rangle)}.$$

In other words, Alice *sees* that Bob's negation of her *Truth* is her *Truth* itself, which represents instead an intermediate truth-value for Bob.

We can also consider a third agent Eve whose truth-perspective is the following: $\mathsf{T}_{Eve} = \begin{pmatrix} \cos(\frac{\pi}{8}) & \sin(\frac{\pi}{8}) \\ -\sin(\frac{\pi}{8}) & \cos(\frac{\pi}{8}) \end{pmatrix}$. In such a case, Alice will *see* Eve using the Hadamard-gate in place of her negation, i.e.,

$${}^{\mathsf{D}}\mathtt{NOT}^{(1)}_{Eve} = {}^{\mathsf{D}}\sqrt{\mathtt{I}}^{(1)}_{Alice}.$$

As expected, generally, different agents with different truth-perspectives will see and use different gates. An interesting question is the following: can different agents (with different truth-perspectives) *see and use* the same gate corresponding to a given canonical gate? The following theorem gives a positive answer to this question, in the case of same special gates.

Theorem 6.2. *Let ${}^{\mathsf{D}}G^{(n)}$ be one of the following canonical gates: the negation ${}^{\mathsf{D}}\mathtt{NOT}^{(n)}$, the Hadamard-gate ${}^{\mathsf{D}}\sqrt{\mathtt{I}}^{(n)}$.*

(i) *There is an infinite set of agents such that for any* i *and* j *belonging to this set:*
  (i.1) i *and* j see and use *the same gate corresponding to the canonical gate* ${}^{\mathsf{D}}G^{(n)}$*;*
  (i.2) *if* $\mathsf{i} \neq \mathsf{j}$*, then* T_i *and* T_j *are not probabilistically equivalent (in other words,* $\mathtt{p}_{\mathsf{T}_\mathsf{i}}(\rho) \neq \mathtt{p}_{\mathsf{T}_\mathsf{j}}(\rho)$*, for some qumix* ρ*);*

(ii) *There is an infinite set of agents (with different truth-perspectives* T_i*) who* see and use *different gates* ${}^{\mathsf{D}}G^{(n)}_{\mathsf{T}_\mathsf{i}}$*, all different from the canonical gate* ${}^{\mathsf{D}}G^{(n)}$*. In other words, for any* i *and* j *belonging to this set:*
  (ii.1) *if* $\mathsf{i} \neq \mathsf{j}$*, then* ${}^{\mathsf{D}}G^{(n)}_{\mathsf{T}_\mathsf{i}} \neq {}^{\mathsf{D}}G^{(n)}_{\mathsf{T}_\mathsf{j}}$*;*
  (ii.2) ${}^{\mathsf{D}}G^{(n)}_{\mathsf{T}_\mathsf{i}} \neq {}^{\mathsf{D}}G^{(n)}$.

Proof.

(i) Consider the set of truth-perspectives having the following form:
$\mathsf{T}(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$.
There are infinitely many $\theta \in [0, 2\pi)$ such that:

(i.1) ${}^{\mathsf{D}}G^{(n)}_{\mathsf{T}(\theta)} = {}^{\mathsf{D}}G^{(n)}$.

(i.2) If $\theta \neq \theta'$, then $\mathsf{T}(\theta)$ and $\mathsf{T}(\theta')$ are not probabilistically equivalent.

(ii) Consider the set of truth-perspectives having the following form:

$$\mathsf{T}'(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) - \frac{i}{\sqrt{2}}\sin(\frac{\theta}{2}) & -\frac{i}{\sqrt{2}}\sin(\frac{\theta}{2}) \\ -\frac{i}{\sqrt{2}}\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) + \frac{i}{\sqrt{2}}\sin(\frac{\theta}{2}) \end{pmatrix}.$$

There are infinitely many $\theta \in (0, 2\pi)$ such that:

(ii.1) if $\theta \neq \theta'$, then ${}^{\mathsf{D}}G^{(n)}_{\mathsf{T}(\theta)} \neq {}^{\mathsf{D}}G^{(n)}_{\mathsf{T}(\theta\ )}$;

(ii.2) ${}^{\mathsf{D}}G^{(n)}_{\mathsf{T}(\theta)} \neq {}^{\mathsf{D}}G^{(n)}$.

□

7. Quantifiers as qumix operations

Now we want to extend our semantics to the full first-order language $\mathcal{L}$. As is well known, in most semantic approaches the interpretation of the universal quantifier $\forall$ generally involves an infinitary procedure that cannot be represented as a finite computational step.

What kind of intuitive reasons induce us to assert the truth of a universal sentence (say, "All humans are mortal", "All neutrinos have a non-null mass",....)? In the happiest situations we can base our assertion on a theoretical proof (which generally gives rise to a kind of "by-pass"). In other situations we may refer to an empirical evidence or to an inductive extrapolation. Sometimes we are simply proposing a conjecture or even an act of faith.

Consider the following simple example of a universal sentence:

$$\alpha = \forall x \mathbf{P}^1 x.$$

We have: $AtStr(\alpha) = (3)$; $\mathcal{H}^{\alpha} = \mathcal{H}^{(3)}$

The syntactical tree of α is:

$$Level_2^{\alpha} = (\mathbf{P}^1 x)$$
$$Level_1^{\alpha} = (\forall x \mathbf{P}^1 x)$$

Once chosen a truth-perspective T, is it possible to obtain an appropriate T-gate tree for α?

Any model $\mathtt{Hol}_{\mathsf{T}}$ will assign a qumix to the top level of the syntactical tree of α:

$$\mathtt{Hol}_{\mathsf{T}} : \ (\mathbf{P}^1 x) \ \mapsto \ \rho \in \mathsf{D}(\mathcal{H}^{(3)}).$$

Hence, we shall look for an operation ${}^{\forall}\mathbf{Q}_{\mathsf{T}}$ (which is defined on $\mathcal{H}^{(3)}$ and depends on T) such that:

$$\mathtt{Hol}_{\mathsf{T}}((\forall x \mathbf{P}^1 x)) = {}^{\forall}\mathbf{Q}_{\mathsf{T}}\rho$$

A very reasonable condition that should be required seems to be the following:

$$\mathtt{p}_{\mathsf{T}}({}^{\forall}\mathbf{Q}_{\mathsf{T}}\rho) \ \leq \ \mathtt{p}_{\mathsf{T}}(\rho).$$

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Semantically, this condition is important because it is connected with the validity of the *Dictum de omni-Principle* ($\forall x \mathbf{P}^1 x \models \mathbf{P}^1 x$).

Interestingly enough, one is dealing with a requirement that also characterizes knowledge-operations. As we have seen in Section 5, for any $\rho \in EpD(\mathbf{K}_{\mathsf{T}}^{(n)})$ we have: $\mathtt{p}_{\mathsf{T}}(\mathbf{K}_{\mathsf{T}}^{(n)}\rho) \leq \mathtt{p}_{\mathsf{T}}(\rho)$. And we already know that knowledge-operations cannot be generally represented as unitary quantum operations. As happens in the case of epistemic operators, quantifiers also can be interpreted as special examples of qumix operations that are generally irreversible. Unlike logical connectives, the use of quantifiers seems to involve a kind of theoretic "jump", quite similar to quantum measurement-phenomena.

Of course, not all universal formulas are so simple as $\forall x \mathbf{P}^1 x$. Consider, for instance, the following sentence:

$$\alpha = \forall x(\mathbf{P}^1 x \wedge \mathbf{P}^2 \mathbf{a} x) = \forall x(\mathsf{T}(\mathbf{P}^1 x, \mathbf{P}^2 \mathbf{a} x, \mathbf{f}))$$

(say, *All are nice and Alice likes them*).
We have: $AtStr(\alpha) = (3,4,1)$; $\mathcal{H}^\alpha = \mathcal{H}^{(3)} \otimes \mathcal{H}^{(4)} \otimes \mathcal{H}^{(1)} = \mathcal{H}^{(8)}$.

Here $\forall$ binds the variable x in two different occurrences of x in two different subformulas of α. How can such syntactical features be reflected at a semantical level? Fortunately (unlike classical semantics), the quantum computational semantics has an *intensional* character that allows us to "preserve the memory" of the linguistic complexity of all formulas.

In the case of the sentence $\alpha = \forall x(\mathsf{T}(\mathbf{P}^1 x, \mathbf{P}^2 \mathbf{a} x, \mathbf{f}))$, the behavior of the quantifier $\forall$ can be associated to a syntactical configuration, formally described by the following conventional notation:

$$(1[1], 2[2], (3,4,1)).$$

The interpretation of $(1[1], 2[2], (3,4,1))$ is: $\forall$ binds the first variable of the first atomic subformula occurring in α and the second variable of the second atomic subformula occurring in α, while $(3,4,1)$ is the atomic structure of α.

This notation can be naturally generalized. Any universal formula

$$\alpha = \forall x \delta$$

can be associated to a syntactical configuration (called *quantifier-configuration*) that will be represented as follows:

$$qconf^\alpha = (m_1[d_1^{m_1}, \ldots, d_u^{m_1}], \ldots, m_r[d_1^{m_r}, \ldots, d_v^{m_r}], (n_1, \ldots, n_k)),$$

where: $r \leq At(\alpha) = n_1 + \ldots + n_k$.

The interpretation of $qconf^\alpha$ is the expected one. Of course, different formulas may have the same quantifier configuration. Since any quantum configuration $qconf$ refers to a particular atomic structure, it turns out that $qconf$ determines the semantic space $\mathcal{H}_{qconf}$ of all formulas whose quantifier-configuration is $qconf$.

On this basis, we can now introduce the notions of T-*quantifier map* and of *first-order epistemic realization* for the language $\mathcal{L}$.

Definition 7.1. (T-Quantifier map)
A T-quantifier map *is a a map* $\mathsf{Q_T}$ *that associates to any quantifier-configuration* $qconf$ *a qumix operation* $\mathsf{Q_T}(qconf)$*, defined on the space* $\mathcal{H}_{qconf}$*. The following condition is required for any qumix* ρ *of* $\mathcal{H}_{qconf}$ *:*

$$\mathtt{p_T}([\mathsf{Q_T}(qconf)]\rho) \leq \mathtt{p_T}(\rho).$$

Definition 7.2. (First-order epistemic realization)
A first-order epistemic realization *for* $\mathcal{L}$ *is a triplet* $(\mathtt{Hol_T}, \mathsf{E_T}, \mathsf{Q_T})$*, where* $\mathtt{Hol_T}$ *is a holistic map for the language* $\mathcal{L}$*,* $\mathsf{E_T}$ *is an epistemic map (which associates to any pair* (α, t) *consisting of a formula* α *and of a term* t *occurring in an epistemic connective* Kt *of* α *an epistemic situation* $\mathsf{E_T}(\alpha, t) = (\mathsf{T}_{\mathtt{Hol}^\alpha_\mathsf{T}(t)}, \mathbf{K}_{\mathtt{Hol}^\alpha_\mathsf{T}(t)})$*) and* $\mathsf{Q_T}$ *is a quantifier map.*

As happens for the language $\mathcal{L}^{-Ep}$, any first-order epistemic realization $(\mathtt{Hol_T}, \mathsf{E_T}, \mathsf{Q_T})$ determines for any formula α a special gate tree, called the $(\mathtt{Hol_T}, \mathsf{E_T}, \mathsf{Q_T})$ - *first-order epistemic pseudo gate tree of* α. As an example, consider the sentence:

$$\alpha = \neg\forall x \mathbf{P}^1 x.$$

The syntactical tree of α is:

$$Level^\alpha_3 = (\mathbf{P}^1 x)$$
$$Level^\alpha_2 = (\forall x \mathbf{P}^1 x)$$
$$Level^\alpha_1 = (\neg\forall x \mathbf{P}^1 x)$$

Accordingly, the $(\mathtt{Hol_T}, \mathsf{E_T}, \mathsf{Q_T})$ - first-order epistemic pseudo gate tree of α can be naturally identified with the following pseudo-gate sequence:

$$(\mathsf{Q_T}(qconf^{\forall x \mathbf{P}^1 x}),\ {}^{\mathsf{D}}\mathtt{NOT}^{(3)}_\mathsf{T}).$$

On this basis, we can now define the concept of *holistic model* for the language $\mathcal{L}$.

Definition 7.3. (Holistic model of $\mathcal{L}$)
A holistic model *of* $\mathcal{L}$ *is a first-order epistemic realization* $(\mathtt{Hol_T}, \mathsf{E_T}, \mathsf{Q_T})$ *that satisfies the following conditions for any formula* α*.*

(1) *Let* $({}^{\mathsf{D}}G^{(\mathtt{Hol_T},\mathsf{E_T},\mathsf{Q_T})}_{\mathsf{T}_{(h-1)}}, \ldots, {}^{\mathsf{D}}G^{(\mathtt{Hol_T},\mathsf{E_T},\mathsf{Q_T})}_{\mathsf{T}_{(1)}})$ *be the* $(\mathtt{Hol_T}, \mathsf{E_T}, \mathsf{Q_T})$ *- first-order epistemic pseudo gate tree of* α *and let* $1 \leq i < h$*. Then,*

$$\mathtt{Hol_T}(Level_i(\alpha)) = {}^{\mathsf{D}}G^{(\mathtt{Hol_T},\mathsf{E_T},\mathsf{Q_T})}_{\mathsf{T}_{(i)}}(\mathtt{Hol_T}(Level_{i+1}(\alpha))).$$

The meaning of each level (different from the top level) is obtained by applying the corresponding gate (or pseudo-gate) to the meaning of the level that occurs immediately above.

(2) Contextual Dictum de omni
Suppose that $\forall x\beta(x)$ *and* $\beta(t_1) \wedge \ldots \wedge \beta(t_n)$ *are both subformulas of* α*. Then,*

$$\mathtt{p_T}(\mathtt{Hol}^\alpha_\mathsf{T}(\forall x\beta(x))) \leq \mathtt{p_T}(\mathtt{Hol}^\alpha_\mathsf{T}(\beta(t_1) \wedge \ldots \wedge \beta(t_n))).$$

The concepts of *truth*, *validity* and *logical consequence* for the language $\mathcal{L}$ can be now defined like in the case of $\mathcal{L}^{-Ep}$, *mutatis mutandis*.

It is worth-while noticing that, unlike most first-order semantic approaches, our holistic models do not refer to any *domain of individuals* dealt with as a closed set (in a classical sense). Generally, any context γ contains a finite number of individual terms for which any model provides contextual meanings. At the same time, the interpretation of a universal formula does not require "ideal tests" that should be performed on *all* elements of a hypothetical domain (which might be highly indeterminate). In a sense, we could say that the *universe of discourse* associated to a given holistic model behaves here as a kind of *open set*. This way of thinking seems to be in agreement with a number of concrete semantic phenomena, where the individual-domain cannot be precisely determined in an extensional way. In fact, many universal sentences that are currently asserted either in common-life contexts or in scientific theories (say, "All teenagers like danger", "All photons are bosons") do not generally refer to closed domains. Such situations, however, do not prevent a correct use of the universal quantifier.

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References

[1] D. Aharonov, A. Kitaev, and N. Nisan, "Quantum circuits with mixed states", *STOC '98: Proceedings of the thirtieth annual ACM symposium on Theory of computing*, ACM Press, pp. 20–30, 1998.

[2] A. Baltag and S. Smets, "Quantum Logic as a Dynamic Logic", in: T. Kuipers, J. van Benthem and H. Visser (eds.), *Synthese* **179(2)**, pp. 285–306, 2011.

[3] A. Baltag and S. Smets, "Complete Axiomatizations for Quantum Actions", *International Journal of Theoretical Physics* **44 (12)**, pp.2267–2282, 2005.

[4] A. Baltag and S. Smets, "LQP: The Dynamic Logic of Quantum Information", *Mathematical Structures in Computer Science*, special issue on Quantum Programming Languages, **16 (3)**, pp.491–525, 2006.

[5] E. Beltrametti, M.L. Dalla Chiara, R. Giuntini, R. Leporini, G. Sergioli, "Epistemic Quantum Computational Structures in a Hilbert-Space Environment", *Fundamenta Informaticae* **115**, pp. 1–14, 2012. DOI 10.3233/FI-2012-637.

[6] E. Beltrametti, M.L. Dalla Chiara, R. Giuntini, R. Leporini, G. Sergioli, "A Quantum Computational Semantics for Epistemic Logical Operators. Part I: Epistemic Structures", *International Journal of Theoretical Physics*, 2013. DOI 10.1007/s10773–013–1642-z.

[7] E. Beltrametti, M.L. Dalla Chiara, R. Giuntini, R. Leporini, G. Sergioli, "A Quantum Computational Semantics for Epistemic Logical Operators. Part II: Semantics", *International Journal of Theoretical Physics*, 2013. DOI 10.1007/s10773-013-1696-y.

[8] E. Beltrametti, M.L. Dalla Chiara, R. Giuntini, G. Sergioli, "Quantum teleportation and quantum epistemic semantics", *Mathematica Slovaca* **62**, pp. 1–24, 2012.
[9] M.L. Dalla Chiara, H. Freytes, R. Giuntini, A. Ledda, R. Leporini, G. Sergioli, "Entanglement as a semantic resource", *Foundations of Physics* **40**, pp. 1494–1518, 2011.
[10] M.L. Dalla Chiara, R. Giuntini, R. Leporini, "Logics from quantum computation", *International Journal of Quantum Information* **3**, pp. 293–337, 2005.
[11] M.L. Dalla Chiara, R. Giuntini, R. Leporini, G. Sergioli, "Logical arguments in quantum computation", to appear in *Mathematica Slovaca*.
[12] G. Gudder, "Quantum computational logics", *International Journal of Theoretical Physics* **42**, pp. 39–47, 2003.
[13] K. Kraus, *Effects and Operations*, Springer, Berlin, 1983.
[14] M. Nielsen, I. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.

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