Existence and Uniqueness of solution of Volterra Integrodifferential Equation of Fractional Order via S-Iteration

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Abstract

In this paper, we study the existence, uniqueness and other properties of solutions of Volterra integrodifferential equation of fractional order involving the Caputo fractional derivative. The tool employed in the analysis is based on application of $S-$ iteration method. Since the study of qualitative properties in general required differential and integral inequalities, but here $S-$iteration method itself has equally important contribution to study various properties such as dependence on initial data, closeness of solutions and dependence on parameters and functions involved therein. An example in support of the all-established results is given.

Keywords: Existence and uniqueness; Normal $S-$iterative method; Fractional derivative; Continuous dependence; Closeness; Parameters.

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1 Introduction

We consider the following Volterra integrodifferential equation of fractional order involving the Caputo fractional derivative of the type:

$$\big(D^{\alpha}_{*a}\big)y(t) = \mathcal{F}\Big(t, y(t), \int_a^t h\big(s, y(s)\big)ds\Big), \tag{1}$$

for $t \in I = [a,b],\ n-1 < \alpha \leq n,\ \ n \in \mathbb{N}$, with the given initial conditions

$$y^{(j)}(a) = c_j,\ j = 0, 1, 2, \cdots, n-1, \tag{2}$$

where $\mathcal{F} : I \times X \times X \to X,\ h : I \times X \to X$ are continuous functions and $c_j\ \ (j = 0, 1, 2, \ldots, n-1)$ are given elements in X.

Several researchers have introduced many iteration methods for certain classes of operators in the sense of their convergence, equivalence of convergence and rate of convergence etc. (see [1, 3, 4, 5, 6, 8, 9, 18, 19, 20, 21, 22, 23, 24, 31, 32]). The most of iterations devoted for both analytical and numerical approaches. The $S-$ iteration method, due to simplicity and fastness, has attracted the attention and hence, it is used in this paper.

The problems of existence, uniqueness and other properties of solutions of special forms of IVP (1)-(2) and its variants have been studied by several researchers under variety of hypotheses by using different techniques, [2, 7, 10, 11, 12, 13, 14, 15, 16, 26, 27, 29, 30] and some of references cited therein. In recently, Soltuz and Grosan [33] have studied the special version of equation (1) for different qualitative properties of solutions. Authors are motivated by the work of Sahu [31] and influenced by [5,33].

The main objective of this paper is to use normal $S-$iteration method to establish the existence and uniqueness of solution of the initial value problem (1)-(2) and other qualitative properties of solutions.

2 Preliminaries

Before proceeding to the statement of our main results, we shall setforth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let X be a Banach space with norm $\|\cdot\|$ and $I = [a,b]$ denotes an interval of the real line $\mathbb{R}$. For the fractional order $\alpha,\ n-1 < \alpha \leq n,\ \ n \in \mathbb{N}$, we define $B = C^r(I,X)$, (where $r = n$ for $\alpha \in \mathbb{N}$ and $r = n-1$ for $\alpha \notin \mathbb{N}$), as a Banach space of all r times continuously differentiable functions from I into X, endowed with the norm

$$\|y\|_B = \sup\{\|y(t)\| : y \in B\}, \quad t \in I.$$

Definition 2.1 (28). *The Riemann Liouville fractional integral (left-sided) of a function $h \in C^1[a,b]$ of order $\alpha \in \mathbf{R}_+ = (0,\infty)$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds,\ t \in I$$

where Γ is the Euler gamma function.

Definition 2.2 (28). *Let $n-1 < \alpha \le n,\ n \in \mathbb{N}$. Then the expression*

$$D_a^\alpha h(t) = \frac{d^n}{dt^n}\big[I_a^{n-\alpha} h(t)\big],\ t \in [a,b]$$

is called the (left-sided) Riemann Liouville derivative of h of order α whenever the expression on the right-hand side is defined.

Definition 2.3 (25). *Let $h \in C^n[a,b]$ and $n-1 < \alpha \le n,\ n \in \mathbb{N}$. Then the expression*

$$\big(D_{*a}^\alpha\big) h(t) = I_a^{n-\alpha} h^{(n)}(t),\ t \in [a,b]$$

is called the (left-sided) Caputo derivative of h of order α.

Lemma 2.1 (17). *If the function $f = (f_1, \cdots, f_n) \in C^1[a,b]$, then the initial value problems*

$$\big(D_{*a}^{\alpha_i}\big) y(t) = f_i(t, y_1, \cdots, y_n),\ \ y_i^{(k)}(0) = c_k^i,\ \ i = 1,2,\cdots,n,\ k = 1,2,\cdots,m_i$$

where $m_i < \alpha_i \le m_i + 1$ is equivalent to Volterra integral equations:

$$y_i(t) = \sum_{k=0}^{m_i} c_k^i \frac{t^k}{k!} + I_a^{\alpha_i} f_i(t, y_1, \cdots, y_n),\ 1 \le i \le n.$$

As a consequence of the Lemma 2.1, it is easy to observe that if $y \in B$ and $\mathcal{F} \in C^1[a,b]$, then $y(t)$ satisfies the integral equation

$$y(t) = \sum_{j=0}^{n-1} \frac{c_j}{j!} (t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, y(s), \int_a^s h\big(\sigma, y(\sigma)\big) d\sigma\Big) ds, \tag{3}$$

which is equivalent to (1)-(2).

We need the following pair of known results:

Theorem 2.1. *([31], p.194) Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction operator with contractivity factor $m \in [0,1)$ and fixed point x^*. Let α_k and β_k be two real sequences in $[0,1]$ such that $\alpha \leq \alpha_k \leq 1$ and $\beta \leq \beta_k < 1$ for all $k \in \mathbb{N}$ and for some $\alpha, \beta > 0$. For given $u_1 = v_1 = w_1 \in C$, define sequences u_k, v_k and w_k in C as follows:*

S-iteration process: $\begin{cases} u_{k+1} = (1-\alpha_k)Tu_k + \alpha_k Ty_k, \\ y_k = (1-\beta_k)u_k + \beta_k Tu_k, k \in \mathbb{N}. \end{cases}$

Picard iteration: $v_{k+1} = Tv_k, k \in \mathbb{N}.$

Mann iteration process: $w_{k+1} = (1-\beta_k)w_k + \beta_k Tw_k, k \in \mathbb{N}.$

Then we have the following:

(a) $\|u_{k+1} - x^*\| \leq m^k \Big[1-(1-m)\alpha\beta\Big]^k \|u_1 - x^*\|,$ *for all* $k \in \mathbb{N}.$

(b) $\|v_{k+1} - x^*\| \leq m^k \|v_1 - x^*\|,$ *for all* $k \in \mathbb{N}.$

(c) $\|w_{k+1} - x^*\| \leq \Big[1-(1-m)\beta\Big]^k \|w_1 - x^*\|,$ *for all* $k \in \mathbb{N}.$

Moreover, the S-iteration process is faster than the Picard and Mann iteration processes.

Definition 2.4. *([31], p.194) In particular, for $\alpha_k = 1,\ k \in \mathbb{N} \cup \{0\}$ in the S-iteration process, then it reduces to as follows:*

$$\begin{cases} u_0 \in C, \\ u_{k+1} = Ty_k, \\ y_k = (1-\xi_k)u_k + \xi_k Tu_k,\ k \in \mathbb{N} \cup \{0\}. \end{cases} \tag{4}$$

This is called normal $S-$iteration method.

Note: For our convenience, we replaced β_k in the S-iteration process by ξ_k.

Lemma 2.2. *([33], p.4) Let $\{\beta_k\}_{k=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ one has satisfied the inequality*

$$\beta_{k+1} \leq (1-\mu_k)\beta_k + \mu_k \gamma_k, \tag{5}$$

where $\mu_k \in (0,1)$, for all $k \in \mathbb{N} \cup \{0\}$, $\displaystyle\sum_{k=0}^{\infty} \mu_k = \infty$ and $\gamma_k \geq 0,\ \forall k \in \mathbb{N} \cup \{0\}$. Then the following inequality holds

$$0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k. \tag{6}$$

3 Existence and Uniqueness of Solutions via $S-$iteration

Now, we are able to state and prove the following main theorem which deals with the existence and uniqueness of solutions of the problem (1)-(2).

Theorem 3.1. *Assume that there exist functions $p,\ q \in C(I, \mathbb{R}_+)$ such that*

$$\|\mathcal{F}\Big(t, u_1, u_2\Big) - \mathcal{F}\Big(t, v_1, v_2\Big)\| \leq p(t)\Big[\|u_1 - v_1\| + \|u_2 - v_2\|\Big] \tag{7}$$

and

$$\|h(t, u_1) - h(t, v_1)\| \leq q(t)\|u_1 - v_1\|,$$

for $t \in I$. If $\Theta = {I_a}^{\alpha} p(t)\big(1 + (b-a)Q\big) < 1$ $\big($where $Q = \sup\limits_{a \leq t \leq b} q(t)\big)$, then the iterative sequence $\{y_k\}_{k=0}^{\infty}$ generated by normal $S-$ iteration method (4) *with the real control sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\sum\limits_{k=0}^{\infty} \xi_k = \infty$, converges to a unique point $y \in B$, which is the required solution of the equations* (1)-(2) *with the following estimate:*

$$\|y_{k+1} - y\|_B \leq \frac{\Theta^{k+1}}{e^{\big(1-\Theta\big)\sum_{i=0}^{k} \xi_i}} \|y_0 - y\|_B. \tag{8}$$

Proof. Let $y(t) \in B$ and define the operator

$$\begin{aligned}(Ty)(t) &= \sum_{j=0}^{n-1} \frac{c_j}{j!}\,(t-a)^j \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, y(s), \int_a^s h\big(\sigma, y(\sigma)\big) d\sigma\Big) ds,\ t \in I. \end{aligned} \tag{9}$$

Let $\{y_k\}_{k=0}^{\infty}$ be iterative sequence generated by normal $S-$iteration method (4) for the operator given in (9).
We will show that $y_k \to y$ as $k \to \infty$.
From (4), (9) and assumption, we obtain

$$\begin{aligned}&\|y_{k+1}(t) - y(t)\| \\ &= \|(Tz_k)(t) - (Ty)(t)\| \\ &= \|\sum_{j=0}^{n-1} \frac{c_j}{j!}\,(t-a)^j + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big) d\sigma\Big) ds\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=0}^{n-1} \frac{c_j}{j!}\,(t-a)^j - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, y(s), \int_a^s h\big(\sigma, y(\sigma)\big)d\sigma\Big)ds\| \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|\mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big)d\sigma\Big) \\
&\qquad - \mathcal{F}\Big(s, y(s), \int_a^s h\big(\sigma, y(\sigma)\big)d\sigma\Big)\|ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big[\|z_k(s)-y(s)\| + \int_a^s q(\sigma)\|z_k(\sigma)-y(\sigma)\|d\sigma\Big]ds.
\end{aligned} \tag{10}
$$

Now, we estimate

$$
\begin{aligned}
\|z_k(t)-y(t)\| &= \Big[(1-\xi_k)\|y_k(t)-y(t)\| + \xi_k\|(Ty_k)(t)-(Ty)(t)\|\Big] \\
&\le (1-\xi_k)\|y_k(t)-y(t)\| + \xi_k\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s) \\
&\qquad \times \Big[\|y_k(s)-y(s)\| + \int_a^s q(\sigma)\|y_k(\sigma)-y(\sigma)\|d\sigma\Big]ds.
\end{aligned} \tag{11}
$$

Now, by taking supremum in the inequalities (10) and (11), we obtain

$$
\begin{aligned}
\|y_{k+1}-y\|_B &\le \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big[\|z_k-y\|_B + \int_a^s q(\sigma)\|z_k-y\|_B d\sigma\Big]ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big[\|z_k-y\|_B + (b-a)Q\|z_k-y\|_B\Big]ds \\
&\le {I_a}^{\alpha}p(t)\Big(1+(b-a)Q\Big)\|z_k-y\|_B \\
&= \Theta\|z_k-y\|_B
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
\|z_k-y\|_B &\le \Big[(1-\xi_k)\|y_k-y\|_B + \xi_k\Theta\|y_k-y\|_B\Big] \\
&= \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-y\|_B,
\end{aligned} \tag{13}
$$

respectively.
Therefore, using (13) in (12), we have

$$
\|y_{k+1}-y\|_B \le \Theta\Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-y\|_B. \tag{14}
$$

Thus, by induction, we get

$$
\|y_{k+1}-y\|_B \le \Theta^{k+1}\prod_{j=0}^{k}\Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_0-y\|_B. \tag{15}
$$

Since $\xi_k \in [0,1]$ for all $k \in \mathbb{N} \cup \{0\}$, the definition of Θ and $\xi_k \leq 1$ yields,

$$\Rightarrow \xi_k \Theta < \xi_k$$

$$\Rightarrow \xi_k \Big(1 - \Theta\Big) < 1, \; \forall \, k \in \mathbb{N} \cup \{0\}. \tag{16}$$

From the classical analysis, we know that

$$1 - x \leq e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots, \; x \in [0,1].$$

Hence by utilizing this fact with (16) in (15), we obtain

$$\begin{aligned} \|y_{k+1} - y\|_B &\leq \Theta^{k+1} e^{-\left(1-\Theta\right) \sum_{j=0}^{k} \xi_j} \|y_0 - y\|_B \\ &= \frac{\Theta^{k+1}}{e^{\left(1-\Theta\right) \sum_{i=0}^{k} \xi_i}} \|y_0 - y\|_B. \end{aligned} \tag{17}$$

Since $\sum_{k=0}^{\infty} \xi_k = \infty$,

$$e^{-\left(1-\Theta\right) \sum_{j=0}^{k} \xi_j} \to 0 \quad \text{as} \quad k \to \infty. \tag{18}$$

Hence, using this, the inequality (17) implies $\lim_{k\to\infty} \|y_{k+1} - y\|_B = 0$ and therefore, we have $y_k \to y$ as $k \to \infty$. $\square$

Remark: It is an interesting to note that the inequality (17) gives the bounds in terms of known functions, which majorizes the iterations for solutions of the problem (1)-(2) for $t \in I$.

4 Continuous dependence via $S-$iteration

In this section, we shall deal with continuous dependence of solution of the problem (1) on the initial data, functions involved therein and also on parameters.

4.1 Dependence on initial data

Suppose $y(t)$ and $\overline{y}(t)$ are solutions of (1) with initial data

$$y^{(j)}(a) = c_j, \; j = 0, 1, 2, \cdots, n-1, \tag{19}$$

and

$$\overline{y}^{(j)}(a) = d_j,\ j = 0, 1, 2, \cdots, n-1, \tag{20}$$

respectively, where c_j, d_j are elements of the space X.
Then looking at the steps as in the proof of Theorem 3.1, we define the operator for the equation (1) with the initial conditions (20):

$$\begin{aligned}(\overline{T}\overline{y})(t) &= \sum_{j=0}^{n-1} \frac{d_j}{j!}\,(t-a)^j \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, \overline{y}(s), \int_a^s h\big(\sigma, \overline{y}(\sigma)\big)d\sigma\Big)ds,\ t \in I. \end{aligned} \tag{21}$$

We shall deal with the continuous dependence of solutions of equations (1) on initial data.

Theorem 4.1. *Suppose the function $\mathcal{F}$ in equation* (1) *satisfies the condition* (7)*. Consider the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in* (9) *and $\overline{T}$ in* (21)*, respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$. If the sequence $\{\overline{y}_k\}_{k=0}^{\infty}$ converges to $\overline{y}$, then we have*

$$\|y - \overline{y}\|_B \leq \frac{3M}{\Big(1 - \Theta\Big)}, \tag{22}$$

where

$$M = \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!}\,(b-a)^j.$$

Proof. From iteration (4) and equations (9); (21) and assumptions, we obtain

$$\begin{aligned}&\|y_{k+1}(t) - \overline{y}_{k+1}(t)\| \\ &= \|(Tz_k)(t) - (\overline{T}\overline{z}_k)(t)\| \\ &= \|\sum_{j=0}^{n-1}\frac{c_j}{j!}\,(t-a)^j + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big)d\sigma\Big)ds \\ &\quad - \sum_{j=0}^{n-1}\frac{d_j}{j!}\,(t-a)^j - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma\Big)ds\| \\ &\leq \sum_{j=0}^{n-1}\frac{\|c_j - d_j\|}{j!}\,(b-a)^j \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|\mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big)d\sigma\Big) \\
&\quad -\mathcal{F}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma\Big)\|ds \\
&\le M + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big[\|z_k(s)-\overline{z}_k(s)\| + \int_a^s q(\sigma)\|z_k(\sigma)-\overline{z}_k(\sigma)\|d\sigma\Big]ds.
\end{aligned}
\tag{23}
$$

Recalling the equations (12) and (13), the above inequality becomes

$$
\|y_{k+1}-\overline{y}_{k+1}\|_B \le M + \Theta\|z_k - \overline{z}_k\|_B, \tag{24}
$$

and similarly, it is seen that

$$
\|z_k-\overline{z}_k\|_B \le \xi_k M + \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-\overline{y}_k\|_B. \tag{25}
$$

Therefore, using (25) in (24) and using hypothesis $\Theta < 1$, and $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}\cup\{0\}$, the resulting inequality becomes

$$
\begin{aligned}
\|y_{k+1}-\overline{y}_{k+1}\|_B &\le M + \|z_k-\overline{z}_k\|_B \\
&\le M + \xi_k M + \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-\overline{y}_k\|_B \\
&\le 2\xi_k M + \xi_k M + \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-\overline{y}_k\|_B \\
&\le \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-\overline{y}_k\|_B + \xi_k\Big(1-\Theta\Big)\frac{3M}{\Big(1-\Theta\Big)}.
\end{aligned}
\tag{26}
$$

We denote

$$
\begin{aligned}
&\beta_k = \|y_k-\overline{y}_k\|_B \ge 0, \\
&\mu_k = \xi_k\Big(1-\Theta\Big) \in (0,1), \\
&\gamma_k = \frac{3M}{\Big(1-\Theta\Big)} \ge 0.
\end{aligned}
$$

The assumption $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}\cup\{0\}$ implies $\sum_{k=0}^{\infty}\xi_k = \infty$. Now, it can be easily seen that (26) satisfies all the conditions of Lemma 2.2 and hence, we have

$$
0 \le \limsup_{k\to\infty}\beta_k \le \limsup_{k\to\infty}\gamma_k
$$

$$\Rightarrow 0 \leq \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leq \limsup_{k\to\infty} \frac{3M}{\Big(1-\Theta\Big)}$$

$$\Rightarrow 0 \leq \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leq \frac{3M}{\Big(1-\Theta\Big)}. \tag{27}$$

Using the assumptions, $\lim\limits_{k\to\infty} y_k = y$, $\lim\limits_{k\to\infty} \overline{y}_k = \overline{y}$, we get from (27) that

$$\|y - \overline{y}\|_B \leq \frac{3M}{\Big(1-\Theta\Big)}, \tag{28}$$

which shows that the dependency of solutions of the equations (1)-(2) and (1) with the initial conditions (20) on given initial data. □

4.2 Closeness of solution via $S-$iteration

Consider the problem (1)-(2) and the corresponding problem

$$\big(D^{\alpha}_{*a}\big)\overline{y}(t) = \overline{\mathcal{F}}\Big(t, \overline{y}(t), \int_a^t h\big(s, \overline{y}(s)\big)ds\Big), \tag{29}$$

for $t \in I = [a, b],\ n-1 < \alpha \leq n,\ n \in \mathbb{N}$, with the given initial conditions

$$\overline{y}^{(j)}(a) = d_j,\ j = 0, 1, 2, \cdots, n-1, \tag{30}$$

where $\overline{\mathcal{F}}$ is defined as $\mathcal{F}$ and $d_j \quad (j = 0, 1, 2, \ldots, n-1)$ are given elements in X.

Then looking at the steps as in the proof of Theorem 3.1, we define the operator for the equations (29)- (30)

$$\begin{aligned}(\overline{T}\overline{y})(t) = &\sum_{j=0}^{n-1} \frac{d_j}{j!}\,(t-a)^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{\mathcal{F}}\Big(s, \overline{y}(s), \int_a^s h\big(\sigma, \overline{y}(\sigma)\big)d\sigma\Big)ds,\ t \in I.\end{aligned} \tag{31}$$

The next theorem deals with the closeness of solutions of the problems (1)-(2) and (29)-(30).

Theorem 4.2. *Consider the sequences* $\{y_k\}_{k=0}^{\infty}$ *and* $\{\overline{y}_k\}_{k=0}^{\infty}$ *generated normal* $S-$ *iterative method associated with operators* T *in* (9) *and* $\overline{T}$ *in* (31)*, respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0, 1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N} \cup \{0\}$. *Assume that* *(i) all conditions of Theorem 3.1 hold, and $y(t)$ and $\overline{y}(t)$ are solutions of* (1)-(2) *and* (29)-(30) *respectively,*

(ii) there exist non negative constant ϵ such that

$$\|\mathcal{F}\Big(t, u_1, u_2\Big) - \overline{\mathcal{F}}\Big(t, u_1, u_2\Big)\| \le \epsilon,\ \forall\, t \in I. \tag{32}$$

If the sequence $\{\overline{y}_k\}_{k=0}^{\infty}$ converges to $\overline{y}$, then we have

$$\|y - \overline{y}\|_B \le \frac{3\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)}. \tag{33}$$

Proof. From iteration (4) and equations (9); (31) and hypotheses, we obtain

$$
\begin{aligned}
&\|y_{k+1}(t) - \overline{y}_{k+1}(t)\| \\
&= \|(Tz_k)(t) - (\overline{T}\overline{z}_k)(t)\| \\
&= \|\sum_{j=0}^{n-1} \frac{c_j}{j!}\,(t-a)^j + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big)d\sigma\Big)ds \\
&\quad - \sum_{j=0}^{n-1} \frac{d_j}{j!}\,(t-a)^j - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\overline{\mathcal{F}}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma\Big)ds\| \\
&\le \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!}\,(b-a)^j \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|\mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big)d\sigma\Big) \\
&\qquad - \overline{\mathcal{F}}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma\Big)\|ds \\
&\le M + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|\mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big)d\sigma\Big) \\
&\qquad - \mathcal{F}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma\Big)\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|\mathcal{F}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma\Big) \\
&\qquad - \overline{\mathcal{F}}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma\Big)\|ds \\
&\le M + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\epsilon ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big[\|z_k(s)-\overline{z}_k(s)\| + \int_a^s q(\sigma)\|z_k(\sigma)-\overline{z}_k(\sigma)\|d\sigma\Big]ds \\
&\leq M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big[\|z_k(s)-\overline{z}_k(s)\| + \int_a^s q(\sigma)\|z_k(\sigma)-\overline{z}_k(\sigma)\|d\sigma\Big]ds.
\end{aligned} \tag{34}
$$

Recalling the derivations obtained in equations (12) and (13), the above inequality becomes

$$
\|y_{k+1}-\overline{y}_{k+1}\|_B \leq M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)} + \Theta\|z_k-\overline{z}_k\|_B, \tag{35}
$$

and similarly, it is seen that

$$
\|z_k-\overline{z}_k\|_B \leq \xi_k\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big] + \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-\overline{y}_k\|_B. \tag{36}
$$

Therefore, using (36) in (35) and using hypothesis $\Theta < 1$, and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$, the resulting inequality becomes

$$
\begin{aligned}
&\|y_{k+1}-\overline{y}_{k+1}\|_B \\
&\leq \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big] + \|z_k-\overline{z}_k\|_B \\
&\leq \Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big] + \xi_k\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big] + \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-\overline{y}_k\|_B \\
&\leq 2\xi_k\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big] + \xi_k\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big] + \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-\overline{y}_k\|_B \\
&\leq \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\|y_k-\overline{y}_k\|_B + \xi_k\Big(1-\Theta\Big)\frac{3\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)}.
\end{aligned} \tag{37}
$$

We denote

$$
\begin{aligned}
\beta_k &= \|y_k-\overline{y}_k\|_B \geq 0, \\
\mu_k &= \xi_k\Big(1-\Theta\Big) \in (0,1), \\
\gamma_k &= \frac{3\Big[M+\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)} \geq 0.
\end{aligned}
$$

The assumption $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N} \cup \{0\}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (37) satisfies all the conditions of Lemma 2.2 and hence, we have

$$\begin{aligned} & 0 \leq \limsup_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \gamma_k \\ \Rightarrow\ & 0 \leq \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leq \limsup_{k\to\infty} \frac{3\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)} \\ \Rightarrow\ & 0 \leq \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leq \frac{3\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)}. \end{aligned} \tag{38}$$

Using the assumptions, $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \overline{y}_k = \overline{y}$, we get from (38) that

$$\|y - \overline{y}\|_B \leq \frac{3\Big[M + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}\Big]}{\Big(1-\Theta\Big)}, \tag{39}$$

which shows that the dependency of solutions of IVP (1)-(2) on the function involved on the right hand side of the given equation. □

Remark: The inequality (39) relates the solutions of the problems (1)-(2) and (29)-(30) in the sense that, if $\mathcal{F}$ and $\overline{\mathcal{F}}$ are close as $\epsilon \to 0$, then not only the solutions of the problems (1)-(2) and (29)-(30) are close to each other (i.e. $\|y-\overline{y}\|_B \to 0$), but also depends continuously on the functions involved therein and initial data.

4.3 Dependence on Parameters

We next consider the following problems

$$\big(D^\alpha_{*a}\big)y(t) = \mathcal{F}\Big(t, y(t), \int_a^t h\big(s, y(s)\big)ds, \mu_1\Big), \tag{40}$$

for $t \in I = [a, b],\ n-1 < \alpha \leq n,\ n \in \mathbb{N}$, with the given initial conditions

$$y^{(j)}(a) = c_j,\ j = 0, 1, 2, \cdots, n-1, \tag{41}$$

and

$$\big(D^\alpha_{*a}\big)\overline{y}(t) = \mathcal{F}\Big(t, \overline{y}(t), \int_a^t h\big(s, \overline{y}(s)\big)ds, \mu_2\Big), \tag{42}$$

for $t \in I = [a,b],\ n-1 < \alpha \leq n,\ n \in \mathbb{N}$, with the given initial conditions

$$\overline{y}^{(j)}(a) = d_j,\ j = 0,1,2,\cdots,n-1, \tag{43}$$

where $\mathcal{F} : I \times X \times X \times \mathbb{R} \to X$ is continuous function, $c_j,\ \ d_j\ \ (j = 0,1,2,\ldots,n-1)$ are given elements in X and constants $\mu_1,\ \mu_2$ are real parameters.

Let $y(t),\ \overline{y}(t) \in B$ and following steps from the proof of Theorem 3.1, define the operators for the equations (40) and (42), respectively

$$\begin{aligned}(Ty)(t) &= \sum_{j=0}^{n-1} \frac{c_j}{j!}\,(t-a)^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, y(s), \int_a^s h\big(\sigma, y(\sigma)\big) d\sigma, \mu_1\Big) ds,\ t \in I;\end{aligned} \tag{44}$$

and

$$\begin{aligned}(\overline{T}\overline{y})(t) &= \sum_{j=0}^{n-1} \frac{d_j}{j!}\,(t-a)^j \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{y}(s), \int_a^s h\big(\sigma, \overline{y}(\sigma)\big) d\sigma, \mu_2\Big) ds,\ t \in I.\end{aligned} \tag{45}$$

The following theorem states the continuous dependency of solutions on parameters.

Theorem 4.3. *Consider the sequences* $\{y_k\}_{k=0}^{\infty}$ *and* $\{\overline{y}_k\}_{k=0}^{\infty}$ *generated normal* $S-$ *iterative method associated with operators* T *in* (44) *and* $\overline{T}$ *in* (45)*, respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0,1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N} \cup \{0\}$*. Assume that*

(i) $y(t)$ *and* $\overline{y}(t)$ *are solutions of* (40)-(41) *and* (42)-(43) *respectively,*

(ii) there exist functions $\overline{p},\ r \in C(I, \mathbb{R}_+)$ *such that*

$$\|\mathcal{F}\Big(t, u_1, u_2, \mu_1\Big) - \mathcal{F}\Big(t, v_1, v_2, \mu_1\Big)\| \leq \overline{p}(t)\Big[\|u_1 - v_1\| + \|u_2 - v_2\|\Big],$$

and

$$\|\mathcal{F}\Big(t, u_1, u_2, \mu_1\Big) - \mathcal{F}\Big(t, u_1, u_2, \mu_2\Big)\| \leq r(t)\Big|\mu_1 - \mu_2\Big|.$$

If the sequence $\{\overline{y}_k\}_{k=0}^{\infty}$ *converges to* $\overline{y}$, *then we have*

$$\|y-\overline{y}\|_B \le \frac{3\Big[M+|\mu_1-\mu_2|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Theta}\Big)}, \tag{46}$$

where $\overline{\Theta} = {I_a}^{\alpha}\overline{p}(t)\big(1+(b-a)Q\big) < 1,\ \ t \in I.$

Proof. From iteration (4) and equations (44); (45) and hypotheses, we obtain

$$\begin{aligned}
&\|y_{k+1}(t)-\overline{y}_{k+1}(t)\| \\
&= \|(Tz_k)(t)-(\overline{T}\overline{z}_k)(t)\| \\
&= \|\sum_{j=0}^{n-1}\frac{c_j}{j!}\,(t-a)^j + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big)d\sigma, \mu_1\Big)ds \\
&\quad - \sum_{j=0}^{n-1}\frac{d_j}{j!}\,(t-a)^j - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma, \mu_2\Big)ds\| \\
&\le \sum_{j=0}^{n-1}\frac{\|c_j-d_j\|}{j!}\,(b-a)^j \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|\mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big)d\sigma, \mu_1\Big) \\
&\qquad - \mathcal{F}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma, \mu_2\Big)\|ds \\
&\le M + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|\mathcal{F}\Big(s, z_k(s), \int_a^s h\big(\sigma, z_k(\sigma)\big)d\sigma, \mu_1\Big) \\
&\qquad - \mathcal{F}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma, \mu_1\Big)\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|\mathcal{F}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma, \mu_1\Big) \\
&\qquad - \mathcal{F}\Big(s, \overline{z}_k(s), \int_a^s h\big(\sigma, \overline{z}_k(\sigma)\big)d\sigma, \mu_2\Big)\|ds \\
&\le M + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}r(s)|\mu_1-\mu_2|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\overline{p}(s)\Big[\|z_k(s)-\overline{z}_k(s)\| + \int_a^s q(\sigma)\|z_k(\sigma)-\overline{z}_k(\sigma)\|d\sigma\Big]ds \\
&\le M + |\mu_1-\mu_2|{I_a}^{\alpha}r(t)
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\overline{p}(s)\Big[\|z_k(s)-\overline{z}_k(s)\| + \int_a^s q(\sigma)\|z_k(\sigma)-\overline{z}_k(\sigma)\|d\sigma\Big]ds. \tag{47}$$

Recalling the derivations obtained in equations (12) and (13), the above inequality becomes

$$\|y_{k+1}-\overline{y}_{k+1}\|_B \le M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t) + \overline{\Theta}\|z_k-\overline{z}_k\|_B, \tag{48}$$

and similarly, it is seen that

$$\|z_k-\overline{z}_k\|_B \le \xi_k\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big] + \Big[1-\xi_k\Big(1-\overline{\Theta}\Big)\Big]\|y_k-\overline{y}_k\|_B. \tag{49}$$

Therefore, using (49) in (48) and using hypothesis $\overline{\Theta} < 1$, and $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}\cup\{0\}$, the resulting inequality becomes

$$\begin{aligned}
&\|y_{k+1}-\overline{y}_{k+1}\|_B \\
&\le \Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big] + \|z_k-\overline{z}_k\|_B \\
&\le \Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big] + \xi_k\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big] \\
&\quad + \Big[1-\xi_k\Big(1-\overline{\Theta}\Big)\Big]\|y_k-\overline{y}_k\|_B \\
&\le 2\xi_k\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big] + \xi_k\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big] \\
&\quad + \Big[1-\xi_k\Big(1-\overline{\Theta}\Big)\Big]\|y_k-\overline{y}_k\|_B \\
&\le \Big[1-\xi_k\Big(1-\overline{\Theta}\Big)\Big]\|y_k-\overline{y}_k\|_B + \xi_k\Big(1-\overline{\Theta}\Big)\frac{3\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1-\overline{\Theta}\Big)}.
\end{aligned} \tag{50}$$

We denote

$$\begin{aligned}
\beta_k &= \|y_k-\overline{y}_k\|_B \ge 0, \\
\mu_k &= \xi_k\Big(1-\overline{\Theta}\Big) \in (0,1), \\
\gamma_k &= \frac{3\Big[M + |\mu_1-\mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1-\overline{\Theta}\Big)} \ge 0.
\end{aligned}$$

The assumption $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}\cup\{0\}$ implies $\sum_{k=0}^{\infty}\xi_k = \infty$. Now, it can be easily seen that (50) satisfies all the conditions of Lemma 2.2 and hence we have

$$0 \le \limsup_{k\to\infty}\beta_k \le \limsup_{k\to\infty}\gamma_k$$

$$\Rightarrow 0 \leq \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leq \limsup_{k\to\infty} \frac{3\Big[M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Theta}\Big)}$$

$$\Rightarrow 0 \leq \limsup_{k\to\infty} \|y_k - \overline{y}_k\|_B \leq \frac{3\Big[M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Theta}\Big)}. \tag{51}$$

Using the assumption $\lim_{k\to\infty} y_k = y,\ \lim_{k\to\infty} \overline{y}_k = \overline{y}$, we get from (51) that

$$\|y - \overline{y}\|_B \leq \frac{3\Big[M + |\mu_1 - \mu_2| {I_a}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Theta}\Big)}, \tag{52}$$

which shows the dependence of solutions of the problem (1)-(2) is on parameters μ_1 and μ_2. □

Remark: The result dealing with the property of a solution called “dependence of solutions on parameters”. Here the parameters are scalars. Notice that the initial conditions do not involve parameters. The dependence on parameters is an important aspect in various physical problems.

5 Example

We consider the following problem:

$$\big(D_*^{\alpha}\big) y(t) = \frac{3t}{5}\Big[\frac{t - \sin(y(t))}{2} + \frac{1}{9}\int_0^t \frac{e^{-s}}{(2+s)^2} y(s) ds\Big], \tag{53}$$

for $t \in [0,1],\ n-1 < \alpha \leq n,\ n \in \mathbb{N}$, with the given initial conditions

$$y^{(j)}(0) = c_j,\ j = 0, 1, 2, \cdots, n-1. \tag{54}$$

Comparing this equation with the equation (1), we get $\mathcal{F} \in C(I \times \mathbb{R}^2, \mathbb{R})$, with

$$\mathcal{F}\Big(t, y(t), \int_0^t h(s, y(s)) ds\Big) = \frac{3t}{5}\Big[\frac{t - \sin(y(t))}{2} + \frac{1}{9}\int_0^t \frac{e^{-s}}{(2+s)^2} y(s) ds\Big]$$

and

$$h(t, y(t)) = \frac{3t}{45} \frac{e^{-t}}{(2+t)^2} y(t).$$

Now, one can easily show that

$$\begin{aligned}&\Big|\mathcal{F}(t,y(t),z(t))-\mathcal{F}(t,\overline{y}(t),\overline{z}(t))\Big|\\&\le\frac{3t}{5}\Big[\frac{1}{2}\Big|\sin(y(t))-\sin(\overline{y}(t))\Big|+\frac{1}{9}\Big|z(t)-\overline{z}(t)\Big|\Big]\\&\le\frac{3t}{10}\Big[\Big|y-\overline{y}\Big|+\Big|z-\overline{z}\Big|\Big],\end{aligned}\tag{55}$$

and

$$\Big|h(t,y(t))-h(t,z(t))\Big|\le\frac{3t}{45}\frac{e^{-t}}{(2+t)^2}\Big|y-z\Big|,\tag{56}$$

where $p(t)=\dfrac{3t}{10}$, and $q(t)=\dfrac{3t}{45}\dfrac{e^{-t}}{(2+t)^2}$. Therefore, we have

$$Q=\sup_{t\in[0,1]}\{q(t)\}=\frac{3}{180}=\frac{1}{60}.$$

Thus, we the estimate

$$\begin{aligned}\Theta&=I_a{}^{\alpha}p(t)\big(1+(b-a)Q\big)\\&=I_a{}^{\alpha}\frac{3t}{10}\Big(1+\frac{1}{60}\Big)\\&=\frac{3}{10}\Big(1+\frac{1}{60}\Big)(I_a{}^{\alpha})(t)\\&=\frac{61}{200}(I_a{}^{\alpha})(t)\\&=\frac{61}{200}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\&\le\frac{1}{\Gamma(\alpha+2)},\ \ (t\le 1).\end{aligned}\tag{57}$$

Therefore, the condition $\Theta<1$ is satisfied only if $\dfrac{1}{\Gamma(\alpha+2)}<1$.
We define the operator $T:B\to B$ by

$$\begin{aligned}(Ty)(t)&=\sum_{j=0}^{n-1}\frac{c_j}{j!}\,t^j\\&+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\frac{3s}{5}\Big[\frac{s-\sin(y(s))}{2}+\frac{1}{9}\int_0^s\frac{e^{-\sigma}}{(2+\sigma)^2}y(\sigma)d\sigma\Big]ds,\end{aligned}\tag{58}$$

for $t\in I$. Since all conditions of Theorem 3.1 are satisfied and so by its conclusion, the sequence $\{y_n\}$ associated with the normal $S-$iterative method (4) for the operator T in (58) converges to a unique solution $y\in B$.

6 Conclusions

Firstly, we proved the main result, which address the existence and uniqueness of the solution to the IVP (1)-(2) by the method of normal $S-$iteration. Next, we discussed various properties of solutions like continuous dependence on the initial data, closeness of solutions, and dependence on parameters and functions involve therein. Finally, we provided an appropriate example to support all of the findings.

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References

[1]. R. Agarwal, D. O'Regan, and D. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. Journal of Nonlinear and Convex Analysis, 8(2007), pp. 61-79.

[2]. Y. Atalan and V. Karakaya, Iterative solution of functional Volterra-Fredholm integral equation with deviating argument, Journal of Nonlinear and Convex Analysis, 18(4)(2017), pp. 675-684.

[3]. Y. Atalan, and V. Karakaya, Stability of Nonlinear Volterra-Fredholm Integro Differential Equation: A Fixed Point Approach, Creative Mathematics and Informatics 26 (3)(2017), 247-254.

[4]. Y. Atalan and V. Karakaya, An example of data dependence result for the class of almost contraction mappings, Sahand Communications in Mathematical Analysis(SCMA)Vol.17, No.1(2020),pp. 139-155.

[5]. Yunus Atalan, Faik Gürsoy and Abdul Rahim Khan, Convergence of S-iterative method to a solution of Fredholm integral equation and data dependency, Facta Universitatis, Ser. Math. Inform. Vol. 36, No 4(2021), 685-694.

[6]. V. Berinde, and M. Berinde, The fastest Krasnoselskij iteration for approximating fixed points of strictly pseudo-contractive mappings, Carpathian J. Math, 21(1-2)(2005), pp. 13-20.

[7]. V. Berinde, Existence and approximation of solutions of some first order iterative differential equations, Miskolc Mathematical Notes, Vol. 11(2010), No. 1, pp. 13-26.

[8]. C. E. Chidume, Iterative approximation of fixed points of Lipschitz pseudocontractivemaps, Proceedings of the American Mathematical Society, vol. 129, no. 8, pp. 2245-2251, 2001.

[9]. R. Chugh, V. Kumar and S. Kumar, Strong Convergence of a new three step iterative scheme in Banach spaces, American Journal of Computational Mathematics 2(2012), pp. 345-357.

[10]. M. B. Dhakne and H. L. Tidke, Existence and uniqueness of solutions of nonlinear mixed integrodifferential equations with nonlocal condition in banach spaces, Electronic Journal of Differential Equations, Vol. 2011, No. 31(2011), pp. 1-10.

[11]. M. Dobritoiu, An integral equation with modified argument, Stud. Univ. Babes-Bolyai Math. XLIX (3)2004, pp. 27-33.

[12]. M. Dobritoiu, System of integral equations with modified argument, Carpathian J. Math, 24(2008), No. 2, pp. 26-36.

[13]. M. Dobritoiu, A nonlinear Fredholm integral equation, TJMM, 1(2009), No. 1-2, pp. 25-32.

[14]. M. Dobritoiu, A class of nonlinear integral equations, TJMM, 4(2012), No. 2, pp.117-123.

[15]. M. Dobritoiu, The approximate solution of a Fredholm integral equation, International Journal of Mathematical Models and Methods in Applied Sciences, Vol.8(2014), pp. 173-180.

[16]. M. Dobritoiu, The existence and uniqueness of the solution of a nonlinear fredholmvolterra integral equation with modified argument via geraghty contractions, Mathematics 2020, 9, 29, https://dx.doi.org/ 10.3390/math9010029.

[17]. Varsha Daftardar-Gejji, Hossein Jafari, Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives, J. Math. Anal. Appl. 328(2007) 10261033.

[18]. F. Gürsoy and V. Karakaya, Some Convergence and Stability Results for Two New Kirk Type Hybrid Fixed Point Iterative Algorithms. Journal of Function Spaces doi:10.1155/2014/684191, 2014.

[19]. F. Gürsoy, V. Karakaya and B. E. Rhoades, Some Convergence and Stability Results for the Kirk Multistep and Kirk-SP Fixed Point Iterative Algorithms. Abstract and Applied Analysis doi:10.1155/2014/806537, 2014.

[20]. E. Hacioglu, F. Grsoy, S. Maldar, Y. Atalan, and G.V. Milovanovic, Iterative approximation of fixed points and applications to two-point second-order boundary value problems and to machine learning, Applied Numerical Mathematics, 167(2021), 143-172.

[21]. N. Hussain, A. Rafiq, Damjanović, B. and R. Lazović, On rate of convergence of various iterative schemes. Fixed Point Theory and Applications 2011(1), 1-6.

[22]. S. Ishikawa, Fixed points by a new iteration method, Proceedings of the American Mathematical Society, vol. 4491974), pp. 147-150.

[23]. S. M. Kang, A. Rafiq and Y. C. Kwun, Strong convergence for hybrid $S-$ iteration scheme, Journal of Applied Mathematics, Vol. 2013, Article ID 705814, 4 Pages, http://dx.doi.org/10.1155/2013/705814.

[24]. S. H. Khan, A Picard-Mann hybrid iterative process. Fixed Point Theory and Applications (1)2013, pp. 1-10.

[25]. Kilbas, A. A., Srivastava, H.M., Trujillo, J.J.: *Theory and Applications of Fractional Differential Equations,* Vol 204, Elsevier, Amsterdam, 2006.

[26]. S. Maldar, F. Grsoy, Y. Atalan, and M. Abbas, On a three-step iteration process for multivalued Reich- Suzuki type a-nonexpansive and contractive mappings, Journal of Applied Mathematics and Computing, (2021), 1-21.

[27]. S. Maldar, Iterative algorithms of generalized nonexpansive mappings and monotone operators with application to convex minimization problem, Journal of Applied Mathematics and Computing, (2021), 1-28.

[28]. Podlubny, I.: *Fractional Differential Equations,* Academic Press, New York, 1999.

[29]. B. G. Pachpatte, On Fredholm type integrodifferential equation, Amkang Journal of Mathematics, Vol.39, No. 1(2008), pp. 85-94.

[30]. B. G. Pachpatte, On higher order Volterra-Fredholm integrodifferential equation, Fasciculi Mathematici, No. 47(2007), pp. 35-48.

[31]. D. R. Sahu, Applications of the S-iteration process to constrained minimization problems and split feasibility problems, Fixed Point Theory, vol. 12, no. 1(2011), pp. 187-204.

[32]. D. R. Sahu and A. Petrusel, Strong convergence of iterative methods by strictly pseudocontractive mappings in Banach spaces, NonlinearAnalysis.Theory,Methods & Applications, vol. 74, no. 17(2011), pp. 6012-6023.

[33]. S. Soltuz and T. Grosan, Data dependence for Ishikawa iteration when dealing with contractive-like operators, Fixed Point Theory Appl, 2008, 242916 (2008). https://doi.org/10.1155/2008/242916.