Reversibility and Asymmetric Conflict in Event Structures

Iain Phillips

Department of Computing, Imperial College London, England

Irek Ulidowski

Department of Computer Science, University of Leicester, England

Abstract

Reversible computation has attracted increasing interest in recent years, with applications in hardware, software and biochemistry. We introduce reversible forms of prime event structures and asymmetric event structures. In order to control the manner in which events are reversed, we use asymmetric conflict on events. We prove a number of results about reachable configurations; for instance, we show under what conditions reachable configurations which are finite are reachable by purely finite means. We discuss, with examples, reversing in causal order, where an event is only reversed once all events it caused have been reversed, as well as forms of non-causal reversing.

Keywords: reversible computation, event structure, asymmetric conflict

1. Introduction

Causal reversibility in concurrent systems means that events that cause other events can only be undone after the caused events are undone first, and that events which are independent of each other can be reversed in an arbitrary order. The last decade has produced a good understanding of how causal reversibility can be achieved in the settings of operational semantics and process calculi. Research on reversing process calculi can be traced back perhaps to Berry and Boudol's Chemical Abstract Machine [3]. Danos and Krivine reversed CCS [6, 7] and then together with Sobociński took a more abstract approach with an application to Petri nets [8]. A general method for reversing process calculi was proposed in [16], and reversible structures that compute forwards and backwards asynchronously were developed by Cardelli and Laneve [5]. Mechanisms for controlling reversibility based on a rollback construct were devised by Lanese,

Email addresses: `i.phillips@imperial.ac.uk` (Iain Phillips), `iu3@leicester.ac.uk` (Irek Ulidowski)

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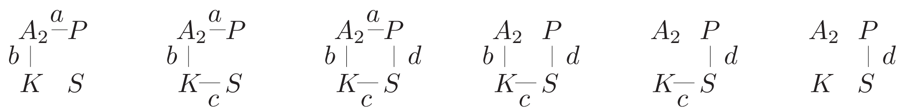

Figure 1: Basic catalytic cycle for substrate phosphorylation by a kinase.

Mezzina, Schmitt and Stefani [12] for a reversible higher-order π calculus [13], and an alternative mechanism based on the execution control operator was proposed in [20].

Perhaps with the exception of [20, 21, 24], other common forms of reversibility, such as *inverse causal* reversibility, have not been studied yet. In [21] we presented an initial study of a form of reversible event structure based on a generalisation of Winskel's enabling relation [25]. In this paper we propose reversible event structures which are strongly contrasted to those of [21], as we here focus on analysing conflict and causation as first-class notions in the setting of reversible computation, rather than maximising expressive power.

We here take the view that reversing an event a means that a is removed from the current configuration (a set of events which have occurred and have not been reversed), and it is as if a had never occurred, apart possibly from indirect effects, such as a having caused another event b before a was reversed.

Our motivating example is the basic catalytic cycle for protein substrate phosphorylation by a kinase. We describe how bonds are created and dissolved in the cycle as presented in [23, Figure 1a]. A kinase K aims to transfer a phosphate group P from a nucleotide Adenosine TriPhospate (ATP), which has three phosphate groups, to a protein substrate S. After the transfer ATP will become Adenosine DiPhosphate (ADP), and so we denote ATP as $A_2 - P$, where the bond between A_2 and P is a, and ADP as A_2. Firstly, $A_2 - P$ and then S bind to the active site of K. We denote the bonds thus created as b and c respectively; see Figure 1, which should be read from left to right. Then phosphorylation takes place: P is transferred from $A_2 - P$ to a Ser, Thr or Tyr residue of S by creating the bond d and then dissolving a. Finally A_2 and then S is released from the active site of K, so b and then c is broken. We note that the order in which bonds are created and broken differs for different kinases in such catalytic cycles [23]; hence we seek a general method for reversing events in an arbitrary order.

Let events a, b, c, d represent the bonds a, b, c, d. The order in which bonds are created can be defined by the *causality* relation $<$ of *prime event structures* (PES) [15, 25]: $a < b < c < d$. To express undoing of events we shall add to PES a new *reverse causality* relation $\prec$: here $a \prec \underline{a}$, $b \prec \underline{b}$ and $c \prec \underline{c}$ mean that a, b, c can be reversed (notation $\underline{a}, \underline{b}, \underline{c}$) as long as they have happened, and $d \prec \underline{a}$, $d \prec \underline{b}$, $d \prec \underline{c}$ force undoing of a, b, c only after d. We do not include $d \prec \underline{d}$, since d is irreversible here. We force that a is undone before b is undone by extending PES further with a *prevention* relation $\triangleright$: $a \triangleright \underline{b}$ prevents undoing of b while a is present; similarly $b \triangleright \underline{c}$. Thus, we obtain a *reversible* PES (RPES). The resulting forward transitions between configurations are $(\emptyset \to)\{a\} \to \{a,b\} \to \{a,b,c\} \to \{a,b,c,d\}$ and reverse transitions are $\{a,b,c,d\} \to \{b,c,d\} \to \{c,d\} \to \{d\}$. This is an example of *inverse* causal reversibility: a is reversed before undoing b even though a causes b, similarly for b and c. See [20, 21] for other examples of non-causal reversibility.

There is a deficiency in the RPES solution in that, for example, a can occur again (so to speak) in configurations $\{b,c,d\}, \{c,d\}, \{d\}$. A general remedy is to add forwards prevention $e \triangleright e'$ to the reverse prevention $e \triangleright \underline{e}'$ already present in RPESs to obtain *reversible* asymmetric event structures (RAES). These are a reversible version of the *asymmetric event structures* (AES) of Baldan, Corradini and Montanari [1]. Prevention $e \triangleright e'$ is *asymmetric* conflict, where both e and e' can happen, but only if e' occurs before e. This generalises the symmetric conflict relation $e \sharp e'$ of PESs. If we add $d \triangleright a$ (d prevents a from taking place) to our example then this disallows a in $\{b,c,d\}, \{c,d\}$ and $\{d\}$.

The second example illustrates another limitation of using PESs in the reversible setting. Consider a long running transaction with a compensation. The full solution is given in Example 4.31; we are only concerned here with the three particular stages of a transaction. The events start, error and comp denote the start of the transaction, an error taking place, which is followed by reversing the computation all the way to the beginning including the start step, and the compensation stage of the transaction respectively. Events $\mathsf{start}, \mathsf{error}$ are reversible and comp is not. We have $\mathsf{start} < \mathsf{error} < \mathsf{comp}$ but we do not intend $\mathsf{start} < \mathsf{comp}$ since start and comp are in conflict. This makes sense in the reversible setting as the computation goes through these configurations: $\emptyset$, $\{\mathsf{start}\}$, $\{\mathsf{start}, \mathsf{error}\}$, $\{\mathsf{error}\}$, $\{\mathsf{error}, \mathsf{comp}\}$ and $\{\mathsf{comp}\}$.

There are two standard ways of explaining causation. Event a *causes* event b ($a < b$) means either (1) in any run (computation), if b occurs then a occurs earlier or (2) if b is enabled at configuration X then we must have $a \in X$. The two views are equivalent if there is no reversing. Suppose that we have three events with $a < b < c$. On view (1) we deduce that $a < c$. On view (2) we also deduce that $a < c$, provided that X is left-closed (downwards closed under $<$), which will be the case for forward-only computation. Thus causation is transitive, as is the case in PESs and AESs.

In the context of reversible computation the second view of causation is simpler, and that is the one that we adopt. If all reversing is causal then all configurations will still be left-closed, and so it is still natural to require $<$ to be transitive. However once we admit the possibility of non-causal reversing, which leads to non-left-closed configurations (such as $\{b,c,d\}$ and $\{c,d\}$ in our example), it is no longer reasonable to insist on $<$ being transitive; if $a < b < c$ then a may have been reversed after b occurs, and before c occurs. Therefore, when defining RAESs we allow causation to be non-transitive. This extension is somewhat orthogonal to the move from symmetric to asymmetric conflict. We introduce the concept of *sustained causation*, where $a \ll b$ means that a causes b and a cannot reverse until b reverses. This is the analogue of standard causation for forwards computation, and we take sustained causation to be transitive.

We also consider the issue of conflict inheritance (if $a < b$ and $a \sharp c$ then

$b \sharp c$) in the reversible setting. If $a < b$ and $a \sharp c$ and a is reversible, then we can undo a in $\{a, b\}$ to reach $\{b\}$. Now there is nothing in $\{b\}$ to prevent c from taking place, and so we expect that $\{b, c\}$ is a configuration, and b and c are not in conflict. Hence, there is no conflict inheritance with $<$. However, we still have conflict inheritance with respect to sustained causality $a \ll b$.

We assign meaning to the structures we consider by defining *configuration systems*, which are transition systems with configurations as states and sets of concurrent events as labels. It is natural to allow *mixed* transitions, which perform both forward and reverse moves. We are not aware of models with mixed transitions having been considered previously.

Our main contributions are as follows. We present an account of conflict and causation in the reversible, and not necessarily causal, setting. We define RPESs and RAESs, and relate them to their respective forward-only counterparts. We prove a number of results about reachable configurations, and, more generally, secured configurations, which are limits of non-monotone sequences. We show under what conditions reachable configurations which are finite are reachable by purely finite means (Theorems 3.32 and 4.51). We show that under causal reversing any reachable configuration is forwards reachable (Theorems 3.40 and 4.54), and we propose conditions for configurations to be reachable under inverse causal reversing (Theorems 3.43 and 4.62). We define mappings between our event structures and show that they preserve configuration systems or reachable configurations (see Figure 3 for a summary).

The paper is structured as follows. In Section 2 we introduce configuration systems. Next we look at reversible PESs (Section 3) and reversible AESs (Section 4), before finishing with some conclusions.

Remark 1.1. An extended abstract of this paper appeared as [18]. The current paper adds full proofs and further results and examples. It also explores the mappings between the various kinds of event structure in more detail. The notion of a 'secured' configuration (part of Definition 2.6) is new, as are Propositions 3.28 and 4.49.

2. Configuration Systems

In this section we describe the model of concurrency we shall use for assigning meaning to the event structures considered in this paper. An event structure will be interpreted as a *configuration system*. Configuration systems are closely related to another model of concurrency, namely *configuration structures*, which have a notion of *configuration* and a notion of concurrent or *step* transition. These were introduced by van Glabbeek and Goltz, and later generalised by van Glabbeek and Plotkin.

Let $\mathfrak{P}(E)$ denote the powerset of a set E.

Definition 2.1 ([10, 9])**.** A *configuration structure* is a pair $\mathcal{C} = (E, \mathsf{C})$ where E is a set of events and $\mathsf{C} \subseteq \mathfrak{P}(E)$. For $X, Y \in \mathsf{C}$, we let $X \to Y$ if $X \subseteq Y$ and for every Z, if $X \subseteq Z \subseteq Y$ then $Z \in \mathsf{C}$.

The idea is that all the (possibly infinitely many) events in $Y \setminus X$ are independent, and so can happen as a single step. Instead of $X \to Y$, we can write $X \xrightarrow{A} Y$ where $A = Y \setminus X$. Note that if $Y = X \cup \{a\}$ and $X, Y \in \mathsf{C}$ then $X \to Y$. This may no longer hold in the reversible setting. As an example, let $E = \{a, b\}$. Suppose that a causes b, so that b cannot occur unless a has already occurred. Then $\{b\}$ is not a possible configuration using forwards computation. However if a is reversible, we can do a followed by b, followed by reversing a, and we reach $\{b\}$. Thus both $\emptyset$ and $\{b\}$ are configurations, but we do not have $\emptyset \xrightarrow{b} \{b\}$.

We propose a new definition appropriate for the reversible setting. We first establish our notation. We let $e, a, b, c, \ldots$ range over events, and $A, B, X, Y, Z, \ldots$ range over sets of events. If an event e is reversible, we have a corresponding reverse event $\underline{e}$. We write $\underline{B}$ for $\{\underline{e} : e \in B\}$. We let $\alpha, \ldots$ range over events or reverse events, and $\Delta, \ldots$ range over sets of events or reverse events.

Definition 2.2. A *configuration system* is a quadruple $\mathcal{C} = (E, F, \mathsf{C}, \to)$ where E is a set of events, $F \subseteq E$ are the reversible events, $\mathsf{C} \subseteq \mathfrak{P}(E)$ is the set of *configurations* and $\to \subseteq \mathsf{C} \times \mathfrak{P}(E \cup \underline{F}) \times \mathsf{C}$ is a labelled transition relation such that if $X \xrightarrow{A \cup \underline{B}} Y$ then:

- $A \cap X = \emptyset$ and $B \subseteq X \cap F$ and $Y = (X \setminus B) \cup A$;
- for every $A' \subseteq A$ and $B' \subseteq B$ we have $X \xrightarrow{A' \cup \underline{B'}} Z \xrightarrow{(A \setminus A') \cup \underline{(B \setminus B')}} Y$ (where $Z = (X \setminus B') \cup A' \in \mathsf{C}$).

We say that $A \cup \underline{B}$ is *enabled* at X if there is Y such that $X \xrightarrow{A \cup \underline{B}} Y$. We say that a transition $X \xrightarrow{A \cup \underline{B}} Y$ is *mixed* if both A and B are non-empty. If $B = \emptyset$ we say the transition is *forwards*, and if $A = \emptyset$ the transition is *reverse*.

The labels on the transitions are optional since they can be deduced from the configurations: if $X \xrightarrow{\Delta} Y$ then $\Delta = (Y \setminus X) \cup \underline{(X \setminus Y)}$.

Remark 2.3. If we let $F = \emptyset$ we get something similar to a configuration structure, but more general. Consider the mutual exclusion example of [10, page 4123]. We have $\emptyset \xrightarrow{d} \{d\} \xrightarrow{e} \{d, e\}$ and symmetrically $\emptyset \xrightarrow{e} \{e\} \xrightarrow{d} \{d, e\}$. However $\emptyset \xrightarrow{\{d,e\}} \{d, e\}$ does not hold, and there is no configuration structure for this example, since it would require that $\emptyset \to \{d, e\}$ holds. However we do get a configuration system.

Recall that a subset $S \subseteq \mathbb{N}$ is *cofinite* if $\mathbb{N} \setminus S$ is finite. The following definition is taken from [21]:

Definition 2.4. Let E be a set. Let $X_0, \ldots$ be an infinite sequence of subsets of E. We say that $X = \lim_{i \to \infty} X_i$ if for every $e \in E$:

1. $\{i \in \mathbb{N} : e \in X_i\}$ is either finite or cofinite;
2. $e \in X$ iff $\{i : e \in X_i\}$ is cofinite.

This is equivalent to saying that that $\lim_{i\to\infty} X_i$ exists if $\liminf_{i\to\infty} X_i = \limsup_{i\to\infty} X_i$ with respect to the discrete metric on events, and if so then $\lim_{i\to\infty} X_i = \liminf_{i\to\infty} X_i = \limsup_{i\to\infty} X_i$.

As a simple example, the sequence

$$\emptyset, \{a_0\}, \{a_0, b_0\}, \{b_0, a_1\}, \{b_0, a_1, b_1\}, \{b_0, b_1, a_2\}, \ldots$$

has limit $\{b_0, b_1, b_2, \ldots\}$, since the a_is are present finitely often, and the b_is cofinitely often.

Definition 2.5. Let $X = \lim_{i\to\infty} X_i$. For $a \in X$, let $\mathsf{last}(a)$ be such that $a \in X_i$ for all $i \geq \mathsf{last}(a)$ and $a \notin X_{\mathsf{last}(a)-1}$.

Thus $\mathsf{last}(a)$ is the index of when a is added to X for the last time. Clearly $\mathsf{last}(a)$ is well-defined and ≥ 1. Note that $\mathsf{last}(a)$ implicitly depends on the sequence $\{X_i : i \geq 0\}$.

We define various kinds of configuration (cf. [10, Definition 3.5]):

Definition 2.6. Let $\mathcal{C} = (E, F, \mathsf{C}, \rightarrow)$ be a configuration system and let $X \in \mathsf{C}$.

- X is a *forwards secured* configuration if there is an infinite sequence of configurations $X_i \in \mathsf{C}$ $(i = 0, \ldots)$ with $X = \bigcup_{i=0}^{\infty} X_i$ and $X_0 = \emptyset$ and $X_i \stackrel{A_{i+1}}{\rightarrow} X_{i+1}$ with $A_{i+1} \subseteq E$;
- X is a *secured* configuration if there is an infinite sequence of configurations $X_i \in \mathsf{C}$ $(i = 0, \ldots)$ with $X = \lim_{i\to\infty} X_i$ and $X_0 = \emptyset$ and $X_i \stackrel{A_{i+1}\cup \underline{B}_{i+1}}{\rightarrow} X_{i+1}$ with $A_{i+1} \subseteq E$ and $B_{i+1} \subseteq F$;
- X is a *reachable* configuration if there is some sequence $\emptyset \stackrel{A_1\cup \underline{B}_1}{\rightarrow} \cdots \stackrel{A_n\cup \underline{B}_n}{\rightarrow} X$ where $A_i \subseteq E$ and $B_i \subseteq F$ for each $i = 1, \ldots, n$;
- X is a *forwards reachable* configuration if there is some sequence $\emptyset \stackrel{A_1}{\rightarrow} \cdots \stackrel{A_n}{\rightarrow} X$ where $A_i \subseteq E$ for each $i = 1, \ldots, n$;
- X is a *finitely reachable* configuration if there is some sequence $\emptyset \stackrel{\alpha_1}{\rightarrow} \cdots \stackrel{\alpha_n}{\rightarrow} X$ where $\alpha_i \in E \cup \underline{F}$ for each $i = 1, \ldots, n$.

See Figure 2 for inclusions between the various notions of configuration. If X is a forwards secured configuration then it is also a secured configuration, since the limit of a monotonically increasing sequence always exists and is just the union. Also if X is reachable then it is secured, since we can extend a finite sequence with empty transitions.

Note that mixed transitions $X \stackrel{A\cup \underline{B}}{\rightarrow} Y$ do not yield new reachable sets compared to forward and reverse transitions, since if $X \stackrel{A\cup \underline{B}}{\rightarrow} Y$ then there is Z such that $X \stackrel{A}{\rightarrow} Z \stackrel{\underline{B}}{\rightarrow} Y$. However mixed transitions allow us to express the independence of forward and reverse events.

It is clear that the finitely reachable configurations are finite and are reachable configurations. Furthermore, finite forwards reachable configurations are

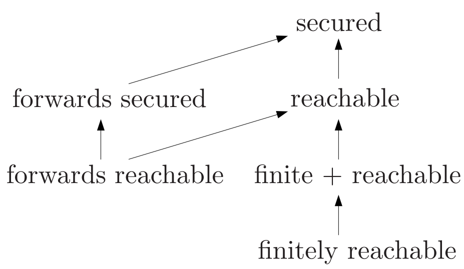

Figure 2: Inclusions between the types of configuration in Definition 2.6.

finitely forwards reachable. However we shall see that it is not necessarily the case that finite, reachable configurations are finitely reachable (Example 3.30), since configurations can grow and shrink during the computation.

3. Reversing in Prime Event Structures

In this section we recall the definition of prime event structure, and formulate the slightly weaker notion of *pre-prime event structure*, which is more suitable for reversing events, since it does not require conflict to be hereditary. These pre-PESs will form the forward component of reversible PESs. We then introduce reversible prime event structures and study their properties.

We shall see that pre-PESs and PESs can be used interchangeably in forward-only computation, since they yield the same forwards secured configurations. On the other hand, when reverse computation is considered, then pre-PESs allow us to reach configurations that are not reachable with PESs.

3.1. Prime Event Structures

We start by recalling the definition of unlabelled prime event structures with binary conflict:

Definition 3.1 ([15])**.** A *prime event structure* (PES) is a triple $\mathcal{E} = (E, <, \sharp)$ where E is a set of events and

1. $< \subseteq E \times E$ is the causality relation, which is an irreflexive partial order such that for every $e \in E$, $\{e' \in E : e' < e\}$ is finite;
2. $\sharp \subseteq E \times E$ is the conflict relation, which is irreflexive, symmetric and *hereditary* with respect to $<$: if $a < b$ and $a \sharp c$ then $b \sharp c$ (all $a, b, c \in E$).

When we generalise this definition to the reversible setting, we shall see that conflict heredity with respect to $<$ no longer necessarily holds. We therefore formulate a weaker form of prime event structure, as follows.

Definition 3.2. A *pre-prime event structure* (pre-PES) is a triple $\mathcal{E} = (E, <, \sharp)$ where E is a set of events and

1. $\sharp \subseteq E \times E$ is irreflexive and symmetric;
2. $< \subseteq E \times E$ is an irreflexive partial order such that for every $e \in E$, $\{e' \in E : e' < e\}$ is finite and conflict-free;
3. if $a < b$ then not $a \sharp b$ (all $a, b \in E$).

Here X is *conflict-free* means that for all $a, b \in X$, it is not the case that $a \sharp b$.

It is straightforward to check that any PES is also a pre-PES. Thus any concepts which are defined for pre-PESs also apply to PESs. Note that if X is conflict-free and $Y \subseteq X$ then Y is also conflict-free.

Definition 3.3. Let $\mathcal{E} = (E, <, \sharp)$ be a pre-PES. We define the associated configuration system $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$ as follows. Let $\mathsf{C} = \{X \subseteq E : X$ is conflict-free$\}$. For $X \in \mathsf{C}$ and $A \subseteq E$, we say that A is *enabled* at X if $A \cap X = \emptyset$, $X \cup A$ is conflict-free, and for every $a \in A$, $\{b \in E : b < a\} \subseteq X$. We define $X \xrightarrow{A} Y$ iff $X, Y \in \mathsf{C}$ and $Y = X \cup A$ and A is enabled at X.

Proposition 3.4. *Let $\mathcal{E} = (E, <, \sharp)$ be a pre-PES. Then $C(\mathcal{E})$ is a configuration system.*

Proof. Straightforward. □

Definition 3.5. Let $\mathcal{E} = (E, <, \sharp)$ be a pre-PES. We define the *causal depth* of an event $e \in E$ by $\mathsf{cdepth}(e) = \max\{\mathsf{cdepth}(e') + 1 : e' < e\}$, where we conventionally let $\max(\emptyset) = 0$.

Causal depth is always finite, since each event has only finitely many causes. Events with no causes have depth zero.

Let $\mathcal{E} = (E, <, \sharp)$ be a pre-PES and let $X \subseteq E$. We say that X is *left-closed (under $<$)* if for any $a \in X$, if $b < a$ then $b \in X$. (We shall also talk about left closure under other orderings in what follows.)

Lemma 3.6. *Let $\mathcal{E}$ be a pre-PES with $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$. If $X \in \mathsf{C}$ is left-closed under $<$ and $X \xrightarrow{A} Y$ then Y is also left-closed.* □

Proposition 3.7. *Let $\mathcal{E} = (E, <, \sharp)$ be a pre-PES and let $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$.*

1. *The forwards secured configurations in C are precisely those which are left-closed.*
2. *$X \in \mathsf{C}$ is (forwards) reachable iff X is left-closed and there is $k \in \mathbb{N}$ such that for all $e \in X$, $\mathsf{cdepth}(e) < k$.*

Proof sketch. 1. If $X \in \mathsf{C}$ is forwards secured with $X = \bigcup_{i=0}^{\infty} X_i$ then we can show by induction using Lemma 3.6 that each X_i is left-closed, so that X is left-closed.
Conversely, given a left-closed $X \in \mathsf{C}$, we can define a forwards securing $X = \bigcup_{i=0}^{\infty} X_i$ by letting $X_n = \{e \in X : \mathsf{cdepth}(e) < n\}$.

2. Suppose that X is forwards reachable. Then $\emptyset \xrightarrow{A_1} X_1 \ldots \xrightarrow{A_n} X_n = X$. Then X is left-closed using Lemma 3.6. We can show by induction that if $e \in X_i$ then $\mathsf{cdepth}(e) < i$.
   Conversely, suppose that X is left-closed and k is such that for all $e \in X$, $\mathsf{cdepth}(e) < k$. Let $X_i = \{e \in X : \mathsf{cdepth}(e) < i\}$ and $A_i = \{e \in X : \mathsf{cdepth}(e) = i - 1\}$. We show that $\emptyset \xrightarrow{A_1} X_1 \ldots \xrightarrow{A_k} X_k = X$. ☐

Of course the term 'forwards' is redundant here, as there are no reverse transitions.

Any pre-PES can be converted into a corresponding PES by taking the hereditary closure.

Definition 3.8. Let $\mathcal{E} = (E, <, \sharp)$ be a pre-PES. The *hereditary closure* of $\mathcal{E}$ is defined by $\mathsf{hc}(\mathcal{E}) = (E, <, \sharp')$ where $\sharp'$ is obtained by closing $\sharp$ under conflict heredity and symmetry using the rules

$$\frac{a \sharp b}{a \sharp' b} \qquad \frac{a \sharp' b < c}{a \sharp' c} \qquad \frac{a \sharp' b}{b \sharp' a}$$

Proposition 3.9. *Let $\mathcal{E} = (E, <, \sharp)$ be a pre-PES.*

1. $\mathsf{hc}(\mathcal{E}) = (E, <, \sharp')$ *is a PES.*
2. *If $\mathcal{E}$ is a PES then* $\mathsf{hc}(\mathcal{E}) = \mathcal{E}$.
3. *Let $X \subseteq E$ be left-closed. Then X is $\sharp$-conflict-free iff X is $\sharp'$-conflict-free.*

Proof. 1. We start by showing that $\sharp'$ satisfies the following conditions for a pre-PES:

   - for every $e \in E$, $\{e' \in E : e' < e\}$ is $\sharp'$-conflict-free
   - if $a < b$ then not $a \sharp' b$

   The above two properties are true for $\sharp$ and are preserved by each application of the three rules to obtain $\sharp'$. Hence they hold for $\sharp'$.
   For $\mathsf{hc}(\mathcal{E})$ to be a PES, we only need to check that $\sharp'$ is irreflexive, symmetric and hereditary with respect to $<$. Using the facts that $\sharp$ is irreflexive and that if $a < b$ then not $a \sharp' b$, we can see that it is impossible to deduce $a \sharp' a$ from the rules for $\sharp'$. Hence $\sharp'$ is irreflexive. It is clear by the third rule that $\sharp'$ is symmetric. It is clear by the second rule that $\sharp'$ is hereditary with respect to $<$.
2. Immediate.
3. Let $X \subseteq E$ be left-closed. It is clear that if X is $\sharp'$-conflict-free then X is $\sharp$-conflict-free, since $\sharp \subseteq \sharp'$. Conversely, suppose X is $\sharp$-conflict-free. We see that the property of X being conflict-free is preserved by each application of the three rules. We use the left-closed condition for the second rule. ☐

The next proposition shows that pre-PESs are no more expressive than PESs as far as configuration systems are concerned. The configuration systems of a pre-PES and its hereditary closure have the same forwards secured configurations and the same transitions on the reachable portion.

Proposition 3.10. *Let $\mathcal{E} = (E, <, \sharp)$ be a pre-PES. Let $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \to)$ and $C(\mathsf{hc}(\mathcal{E})) = (E, \emptyset, \mathsf{C}', \to')$. Then:*

1. $\mathsf{C}' \subseteq \mathsf{C}$.
2. $\to' = \to \cap (\mathsf{C}' \times \mathsf{C}')$.
3. *if $X \subseteq E$ then X is forwards secured in $C(\mathcal{E})$ iff X is forwards secured in $C(\mathsf{hc}(\mathcal{E}))$.*

Proof.

1. $\mathsf{C}' \subseteq \mathsf{C}$ holds because $\sharp \subseteq \sharp'$.
2. This easily follows from the definition of enabling.
3. ($\Rightarrow$) Suppose X is forwards secured in $C(\mathcal{E})$. Then X is $\sharp$-conflict-free and left-closed by Proposition 3.7. So X is $\sharp'$-conflict-free by Proposition 3.9, and X is forwards secured in $C(\mathsf{hc}(\mathcal{E}))$, again by Proposition 3.7.
   ($\Leftarrow$) Suppose X is forwards secured in $C(\mathsf{hc}(\mathcal{E}))$. Then X is $\sharp'$-conflict-free and left-closed by Proposition 3.7. So clearly X is $\sharp$-conflict-free, and X is forwards secured in $C(\mathcal{E})$, again by Proposition 3.7. □

We illustrate the hereditary closure procedure with an example.

Example 3.11. Let $\mathcal{E} = (E, <, \sharp)$ where $E = \{a, b, c\}$ and $a < b$, $a \sharp c$. Then $\mathcal{E}$ is a pre-PES with configurations $\emptyset, \{a\}, \{c\}, \{a, b\}, \{b, c\}$. The corresponding PES $\mathsf{hc}(\mathcal{E})$ is the same, except that we have $b \sharp c$ by conflict heredity, and therefore $\{b, c\}$ is not a configuration. However $\mathcal{E}$ and $\mathsf{hc}(\mathcal{E})$ have the same reachable configurations, as $\{b, c\}$ is not reachable in $\mathcal{E}$.

Note that in Example 3.11, if a were to become reversible we could reach $\{b, c\}$ in $\mathcal{E}$ (but not in $\mathsf{hc}(\mathcal{E})$) by performing $a, b, \underline{a}, c$, and the two structures would no longer be equivalent. We shall not require conflict heredity when defining reversible PESs.

3.2. Reversible Prime Event Structures

We now introduce reversible PESs.

Definition 3.12. A *reversible prime event structure* (RPES) is a sextuple $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ where $(E, <, \sharp)$ is a pre-PES, $F \subseteq E$ are those events of E which are reversible, with reverse events being denoted by $\underline{F} = \{\underline{e} : e \in F\}$ and

1. $\triangleright \subseteq E \times \underline{F}$ is the *prevention* relation;
2. $\prec \subseteq E \times \underline{F}$ is the *reverse causality* relation, where we require $a \prec \underline{a}$ for each $a \in F$, and also that $\{a : a \prec \underline{b}\}$ is finite and conflict-free for every $b \in F$;
3. if $a \prec \underline{b}$ then not $a \triangleright \underline{b}$;
4. $\sharp$ is hereditary with respect to *sustained causation* $\ll$: if $a \ll b$ and $a \sharp c$ then $b \sharp c$, where we define $a \ll b$ to mean that $a < b$ and if $a \in F$ then $b \triangleright \underline{a}$;
5. $\ll$ is transitive.

The intended meaning of $a \prec \underline{b}$ is that for b to be reversed, a must be present in the current configuration. This is a similar concept to forward causation. The intended meaning of $a \triangleright \underline{b}$ is that $\underline{b}$ cannot occur while a is in the current configuration. We shall see later that this has similarities with asymmetric conflict [14, 22, 1].

Note that $a \ll b$, which prevents a from being reversed once b has occurred (and until b is reversed), has something of the force of normal irreversible causation. Indeed, items (4) and (5) of Definition 3.12 could be replaced by stating that $(E, \ll, \sharp)$ is a PES.

Definition 3.13. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES. Let $X \subseteq E$ be conflict-free. For $A \subseteq E$, $B \subseteq F$, we say that $A \cup \underline{B}$ is *enabled* at X if

- $A \cap X = \emptyset$, $B \subseteq X$ and $X \cup A$ is conflict-free;
- for every $a \in A$, if $c < a$ then $c \in X \setminus B$;
- for every $b \in B$, if $d \prec \underline{b}$ then $d \in X \setminus (B \setminus \{b\})$;
- for every $b \in B$, if $d \triangleright \underline{b}$ then $d \notin X \cup A$.

Definition 3.14. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES. We define the associated configuration system $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$ as follows. Let $\mathsf{C} = \{X \subseteq E : X \text{ is conflict-free}\}$. For $X \in \mathsf{C}$ and $A \subseteq E$, $B \subseteq F$, we define $X \xrightarrow{A \cup \underline{B}} Y$ iff $X, Y \in \mathsf{C}$ and $Y = (X \setminus B) \cup A$ and $A \cup \underline{B}$ is enabled at X.

Proposition 3.15. *Let $\mathcal{E}$ be an RPES. Then $C(\mathcal{E})$ is a configuration system.*

Proof. Straightforward. □

Example 3.16. Consider $\mathcal{E}$ with $E = F = \{a, b, c\}$ and $a \ll b \ll c$ (where $\ll$ is sustained causation), and $a \prec \underline{a}$, $b \prec \underline{b}$ and $c \prec \underline{c}$. Note that we can deduce $a \ll c$ by transitivity of $\ll$. When we are in a configuration that contains b we cannot undo a, and we cannot undo a and b when c is present. All subsets of E are the configurations of $C(\mathcal{E})$; the reachable ones are $\emptyset, \{a\}, \{a, b\}, \{a, b, c\}$. On reachable configurations, the forwards transitions are $\emptyset \xrightarrow{a} \{a\} \xrightarrow{b} \{a, b\} \xrightarrow{c} \{a, b, c\}$ and the reverse transitions are $\{a, b, c\} \xrightarrow{\underline{c}} \{a, b\} \xrightarrow{\underline{b}} \{a\} \xrightarrow{\underline{a}} \emptyset$. Hence, the events are reversed in causal order.

3.2.1. Mappings

We now turn to the relationship between RPESs and PESs. We shall see that PESs are embedded in RPESs as those RPESs with no reversible events.

Definition 3.17. For $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ an RPES, we define $\phi_{\mathrm{p}}(\mathcal{E}) = (E, <, \sharp)$. For $\mathcal{E} = (E, <, \sharp)$ a PES, we define $\rho_{\mathrm{p}}(\mathcal{E}) = (E, \emptyset, <, \sharp, \emptyset, \emptyset)$.

Let $\mathcal{E}$ be an RPES. Then clearly $\phi_{\mathrm{p}}(\mathcal{E})$ is a pre-PES. Let $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$. Then $C(\phi_{\mathrm{p}}(\mathcal{E})) = (E, \emptyset, \mathsf{C}, \rightarrow')$ with $\rightarrow' = \rightarrow \cap (\mathsf{C} \times \mathfrak{P}(E) \times \mathsf{C})$. Thus $C(\mathcal{E})$ and $C(\phi_{\mathrm{p}}(\mathcal{E}))$ have the same forwards reachable and forwards secured configurations. The next proposition shows that PESs are embedded in RPESs as precisely those RPESs with no reversible events.

Proposition 3.18. 1. *Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES. If $F = \emptyset$ then $\phi_{\mathrm{p}}(\mathcal{E})$ is a PES and $\rho_{\mathrm{p}}(\phi_{\mathrm{p}}(\mathcal{E})) = \mathcal{E}$. Moreover, $C(\phi_{\mathrm{p}}(\mathcal{E})) = C(\mathcal{E})$.*

2. *Let $\mathcal{E} = (E, <, \sharp)$ be a PES. Then $\rho_{\mathrm{p}}(\mathcal{E})$ is an RPES and $\phi_{\mathrm{p}}(\rho_{\mathrm{p}}(\mathcal{E})) = \mathcal{E}$. Moreover, $C(\rho_{\mathrm{p}}(\mathcal{E})) = C(\mathcal{E})$.*

Proof. Straightforward, noting that if $F = \emptyset$ then sustained causation is just standard causation. □

Example 3.19. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be defined by $E = \{a, b, c\}$, $F = \{a\}$, $a < b$, $a \sharp c$ and $a \prec \underline{a}$. Then $\mathcal{E}$ is an RPES and $\phi_{\mathrm{p}}(\mathcal{E})$ is a pre-PES. However $\phi_{\mathrm{p}}(\mathcal{E})$ is not a PES, since we have $a < b$ and $a \sharp c$ but not $b \sharp c$. Both $C(\mathcal{E})$ and $C(\phi(\mathcal{E}))$ have the same configurations, namely $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}$. The transitions of $C(\phi_{\mathrm{p}}(\mathcal{E}))$ are $\emptyset \xrightarrow{a} \{a\} \xrightarrow{b} \{a, b\}$ and $\emptyset \xrightarrow{c} \{c\}$, and configurations $\{b\}$ and $\{b, c\}$ are not reachable. However in $C(\mathcal{E})$, in addition to the previously described transitions, we have $\{a\} \xrightarrow{\underline{a}} \emptyset$ and $\{a, b\} \xrightarrow{\underline{a}} \{b\} \xrightarrow{c} \{b, c\}$. So configurations $\{b\}$ and $\{b, c\}$ are reachable. However both $C(\mathcal{E})$ and $C(\phi_{\mathrm{p}}(\mathcal{E}))$ have the same forwards reachable configurations. For the hereditary closure $\mathsf{hc}(\phi_{\mathrm{p}}(\mathcal{E}))$ we add conflict between b and c, and therefore eliminate $\{b, c\}$ as a configuration. We still have $\{b\}$ as a configuration, though it is not reachable in $C(\mathsf{hc}(\phi_{\mathrm{p}}(\mathcal{E})))$.

3.3. Reachable Configurations

We now explore how adding reversibility changes what configurations are reachable.

It is sometimes useful to see the reverse causation relation $\prec$ as between pairs of events, rather than events and reverse events, as this reveals chains of causality. The same applies to the reverse prevention relation $\triangleright$ and chains of prevention.

Definition 3.20. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES. For $a \in E, b \in F$, let $a \prec^{\bullet} b$ iff $a \prec \underline{b}$ and $a \neq b$. We also write $b \succ^{\bullet} a$ iff $a \prec^{\bullet} b$. Similarly, for $a \in E, b \in F$, let $a \triangleright^{\bullet} b$ iff $a \triangleright \underline{b}$ and $a \neq b$.

Cycles of reverse causation are possible, and can lead to a form of conflict:

Example 3.21. Consider $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ with $E = F = \{a, b\}$, with $a \prec \underline{b}$ and $b \prec \underline{a}$ (plus $a \prec \underline{a}$ and $b \prec \underline{b}$) and with $<$, $\sharp$ and $\triangleright$ all empty. We have a cycle $a \prec^{\bullet} b \prec^{\bullet} a$. There is a certain aspect of reverse conflict here, in that when $\{a, b\}$ is reached, we can reverse either a or b but not both. Thus we can never reach $\emptyset$ once we have started computation.

Cycles of reverse prevention lead to a form of reverse deadlock: if $a \triangleright^{\bullet} b \triangleright^{\bullet} a$ then once both a and b have occurred, neither can be reversed.

We next define a generalisation of sustained causation $\ll$:

Definition 3.22. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES. For $a, b \in E$ we define $a \ll^{\circ} b$ iff for some $n \geq 1$ there are $a_1, \ldots, a_n$ such that $a = a_1$ and $a_i < b$ $(i = 1, \ldots, n-1)$ and $a_n \ll b$ and $a_n \triangleright^{\bullet} \cdots \triangleright^{\bullet} a_1$.

Note that $a \ll^\circ b$ is just $a \ll b$ in the case that $n = 1$, so that $a \ll b$ implies $a \ll^\circ b$.

The next lemma is the analogue of Lemma 3.6.

Lemma 3.23. *Let $\mathcal{E}$ be an RPES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$.*

1. *If $X \in \mathsf{C}$ is left-closed under $<$ and $X \xrightarrow{A} Y$ then Y is also left-closed under $<$.*
2. *If $X \in \mathsf{C}$ is left-closed under $\ll^\circ$ and $X \xrightarrow{A \cup \underline{B}} Y$ then Y is also left-closed under $\ll^\circ$.*
3. *Suppose that $X \in \mathsf{C}$ and $k \in \mathbb{N}$ are such that for all $e \in X$, $\mathsf{cdepth}(e) < k$. If $X \xrightarrow{A \cup \underline{B}} Y$ then for all $e \in Y$, $\mathsf{cdepth}(e) < k + 1$.*

Proof.
1. By Lemma 3.6, noting that if $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ then $(E, <, \sharp)$ is a pre-PES.
2. Suppose that $X \in \mathsf{C}$ is left-closed under $\ll^\circ$ and $X \xrightarrow{A \cup \underline{B}} Y$. Suppose further that $b \in Y$ and $a_1 \ll^\circ b$. So for some $n \geq 1$ there are $a_2, \ldots, a_n$ such that $a_i < b$ $(i = 1, \ldots, n-1)$ and $a_n \ll b$ and $a_n \triangleright^\bullet \cdots \triangleright^\bullet a_1$. Note that $a_i \ll^\circ b$ $(i = 1, \ldots, n)$. We have $Y = (X \setminus B) \cup A$. There are two cases: (1) If $b \in X \setminus B$ then $a_i \in X$ $(i = 1, \ldots, n)$ since X is left-closed under $\ll^\circ$. Moreover $a_1 \notin B$, since if $n = 1$, $a_1 \in F$ or $b_1 \triangleright \underline{a}_1$, and if $n > 1$ then $a_2 \triangleright \underline{a}_1$. So $a_1 \in X \setminus B$. (2) If $b \in A$ then $a_1 \in X \setminus B$ by the definition of enabling and the fact that $a_1 < b$. So in either case $a_1 \in A \setminus B$ and $a_1 \in Y$ as required.
3. Straightforward. ☐

Sustained causation $\ll$, and the more general $\ll^\circ$, behave in the reversible setting somewhat like standard causation $<$ in the forwards-only setting.

Proposition 3.24. *Let $\mathcal{E}$ be an RPES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$.*

1. *The forwards secured configurations in C are precisely those which are left-closed under $<$.*
2. *If $X \in \mathsf{C}$ is secured then X is left-closed under $\ll^\circ$.*
3. *$X \in \mathsf{C}$ is forwards reachable iff X is left-closed under $<$ and there is $k \in \mathbb{N}$ such that for all $e \in X$, $\mathsf{cdepth}(e) < k$.*
4. *If $X \in \mathsf{C}$ is reachable then X is left-closed under $\ll^\circ$ and there is $k \in \mathbb{N}$ such that for all $e \in X$, $\mathsf{cdepth}(e) < k$.*

Proof.
1. Much the same as Proposition 3.7, using Lemma 3.23 instead of Lemma 3.6.
2. Let $X \in \mathsf{C}$ be $\lim_{i \to \infty} X_i$ with $X_0 = \emptyset$ and $X_i \xrightarrow{A_{i+1} \cup \underline{B}_{i+1}} X_{i+1}$. By Lemma 3.23 we see that each X_i is left-closed under $\ll^\circ$. Suppose that $a \ll^\circ b \in X$. Then $b \in X_i$ for cofinitely many i. Hence $a \in X_i$ for cofinitely many i and so $a \in X$ as required.
3. Much the same as Proposition 3.7, using Lemma 3.23 instead of Lemma 3.6.
4. Immediate from Lemma 3.23. ☐

The next example shows how configurations can be left-closed under $\ll^\circ$ but not secured:

Example 3.25. 1. Let $\mathcal{E}$ be the RPES given by $E = F = \{a', a, b', b\}$ with $a' < a$, $b' < b$ and $a' \sharp b$, $b' \sharp a$ (and $\triangleright$ empty). Note that $\ll^\circ$ is empty. Then $\{a, b\}$ is a configuration of $\mathcal{E}$ which is left-closed under $\ll^\circ$. However, it is not secured.

2. Modify $\mathcal{E}$ to get $\mathcal{E}'$ by removing the conflict and adding $a \triangleright \underline{b}'$ and $b \triangleright \underline{a}'$. Again $\{a, b\}$ is a configuration of $\mathcal{E}'$ which is left-closed under $\ll^\circ$ but not secured.

This motivates us to find further conditions that secured configurations must satisfy.

Definition 3.26. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$. For $a, b \in X$ we define $a \blacktriangleleft^{\mathsf{p}}_X b$ iff at least one of the following holds:

1. $a \ll^\circ b$
2. $\exists a' < a.\ b \sharp a'$
3. $\exists a_1 < a$ such that $a_1 \notin X$ and for some $n \geq 1$ there are $a_2, \ldots, a_n$ such that $a_i < a$ $(i = 2, \ldots, n)$ and $b \triangleright^\bullet a_n \triangleright^\bullet \cdots \triangleright^\bullet a_1$

The idea is that $a \blacktriangleleft^{\mathsf{p}}_X b$ will imply that in a securing $X = \lim_{i\to\infty} X_i$, event b must be added for the last time strictly later than a. Note that in Example 3.25 both $\mathcal{E}$ and $\mathcal{E}'$ have a cycle $a \blacktriangleleft^{\mathsf{p}}_X b \blacktriangleleft^{\mathsf{p}}_X a$ (via clauses (2) and (3) of Definition 3.26 respectively); this will mean that $\{a, b\}$ cannot be secured.

Recall that if $a \in X = \lim_{i\to\infty} X_i$ then $\mathsf{last}(a)$ is the index of the set to which a is added for the last time (Definition 2.5).

Lemma 3.27. *Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$ be secured, with $X = \lim_{i\to\infty} X_i$. Suppose that $a \blacktriangleleft^{\mathsf{p}}_X b$. Then $\mathsf{last}(a) < \mathsf{last}(b)$.*

Proof. There are three cases:

1. Suppose $a \ll^\circ b$. We have $a \notin X_{\mathsf{last}(a)-1}$ by definition of $\mathsf{last}(a)$. Since $X_{\mathsf{last}(a)-1}$ is reachable, we cannot have $b \in X_{\mathsf{last}(a)-1}$, by Proposition 3.24(4). Hence $b \notin X_{\mathsf{last}(a)}$, since $a < b$ and $a \notin X_{\mathsf{last}(a)-1}$. Then $\mathsf{last}(a) < \mathsf{last}(b)$.

2. Suppose $\exists a' < a.\ b \sharp a'$. We have $X_{\mathsf{last}(a)-1} \xrightarrow{A \cup \underline{B}} X_{\mathsf{last}(a)}$ with $a \in A$. Since $a' < a$, we have $a' \in X_{\mathsf{last}(a)-1}$ and $a' \notin B$. Hence $a' \in X_{\mathsf{last}(a)}$. Since $b \sharp a'$, we have $b \notin X_{\mathsf{last}(a)}$. Hence $\mathsf{last}(a) < \mathsf{last}(b)$.

3. Suppose $\exists a_1 < a$ such that $a_1 \notin X$ and for some $n \geq 1$ there are $a_2, \ldots, a_n$ such that $a_i < a$ $(i = 2, \ldots, n)$ and $b \triangleright^\bullet a_n \triangleright^\bullet \cdots \triangleright^\bullet a_1$. As in part (2), we have $a_i \in X_{\mathsf{last}(a)}$ $(i = 1, \ldots, n)$. Since $a_1 \notin X$, we have $a_1 \notin X_j$ for some $j > \mathsf{last}(a)$. But since $a_n \triangleright^\bullet \cdots \triangleright^\bullet a_1$, each of $a_2, \ldots, a_n$ must have been reversed (starting with a_n) after being present in $X_{\mathsf{last}(a)}$. Since $b \triangleright \underline{a}_n$, we must have $b \notin X_k$ for some $k \geq \mathsf{last}(a)$. Hence $\mathsf{last}(a) < \mathsf{last}(b)$. □

Proposition 3.28. *Let $\mathcal{E}$ be an RPES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$ be secured. Then $\blacktriangleleft^{\mathsf{p}}_X$ is well-founded on X.*

Proof. Immediate from Lemma 3.27. □

In Example 3.25 we used conflict $\sharp$ and reverse prevention $\triangleright$. The following example uses reverse causation $\prec$ and shows that a configuration can be left-closed under $\lll^{\circ}$ and have well-founded $\blacktriangleleft^{\mathsf{p}}_X$ but still not be secured (cf. Propositions 3.24(2) and 3.28).

Example 3.29. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be given by $E = \{a, b, c\}$ and $F = \{a\}$, with $a < b < c$ and $a \prec \underline{a}$, $c \prec \underline{a}$ (and empty $\sharp$ and $\triangleright$). Then $\{b\}$ is not a reachable configuration (and therefore not secured, since E is finite), although it is left-closed under $\lll^{\circ}$ (which is the empty relation) and $\blacktriangleleft^{\mathsf{p}}_X$ is the empty relation.

The next example shows that, unlike in the forwards-only setting, we can have reachable configurations which are finite but not finitely reachable.

Example 3.30. Let $E = F = \{a_i : i \in \mathbb{N}\}$. Suppose also that $a_i \prec \underline{a}_i$, $a_{2i+1} < a_{2i}$ and $a_{2i+2} \prec \underline{a}_{2i+1}$ $(i \in \mathbb{N})$. There is no $\sharp$ or $\triangleright$. Then $\mathcal{E} = (E, F, <, \emptyset, \prec, \emptyset)$ is an RPES. The configuration $\{a_0\}$ is reachable in four steps as follows:

$$\emptyset \stackrel{\{a_1,a_3,\ldots\}}{\rightarrow} \{a_1, a_3, \ldots\} \stackrel{\{a_0,a_2,\ldots\}}{\rightarrow} E \xrightarrow{\{\underline{a_1},\underline{a_3},\ldots\}} \{a_0, a_2, \ldots\} \xrightarrow{\{\underline{a_2},\underline{a_4},\ldots\}} \{a_0\}$$

However with single-event transitions we can reach $\{a_0, a_{2i+1}\}$ for any $i \in \mathbb{N}$, but not $\{a_0\}$. Note that there is an infinite descending causal chain $a_0 > a_1 \succ^{\bullet} a_2 > a_3 \succ^{\bullet} \cdots$.

To ensure that every finite, reachable configuration is finitely reachable, we need to impose an extra condition on RPESs.

Lemma 3.31. *Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES and let $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$.*

1. *Suppose that $X \stackrel{A}{\rightarrow} Y$ and Z is such that if $a' < a \in A \cap Z$ then $a' \in Z$. Then $X \cap Z \stackrel{A \cap Z}{\rightarrow} Y \cap Z$.*
2. *Suppose that $X \stackrel{\underline{B}}{\rightarrow} Y$ and Z is such that if $b' \prec^{\bullet} b \in B \cap Z$ then $b' \in Z$. Then $X \cap Z \xrightarrow{\underline{B \cap Z}} Y \cap Z$.*

Proof. Straightforward. □

Theorem 3.32. *Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES. Suppose that for every $e \in E$, $\{e' \in E : e'(< \cup \prec^{\bullet})^* e\}$ is finite. Then every finite, reachable configuration in $C(\mathcal{E})$ is finitely reachable.*

Proof sketch. Suppose that X is finite and reachable in $C(\mathcal{E})$. We can assume that there are no mixed transitions, by converting them into two transitions (one forward and one reverse) as necessary. So we have a computation $\emptyset \xrightarrow{\Delta_1} X_1 \cdots \xrightarrow{\Delta_n} X_n = X$, where $\Delta_i \subseteq E$ or $\Delta_i \subseteq \underline{F}$ $(i = 1, \dots, n)$.

Now let $Z = \{e \in E : \exists a \in X. e(< \cup \prec^{\bullet})^* a\}$. Since X is finite, by the hypothesis Z is also finite. Now if $a < b \in Z$ then $a \in Z$, and if $a \prec^{\bullet} b \in Z$ then $a \in Z$. Hence we can use Lemma 3.31 to deduce that $\emptyset \cap Z \xrightarrow{\Delta_1 \cap Z'} X_1 \cap Z \cdots \xrightarrow{\Delta_n \cap Z'} X_n \cap Z = X$, where Z' is either Z or $\underline{Z}$ as appropriate. Since each transition now has a finite label it is clear that X is finitely reachable—we simply need to sequentialise the step transitions into single-event transitions. $\square$

The next result shows that RPESs do not give us sufficient control to prevent infinite alternations of forwards and reverse moves. In some cases that might be consistent with what we wish to model, but if we wish to prevent infinite alternation from happening we shall need more control over forwards moves, motivating the structures with asymmetric conflict introduced in Section 4.

Proposition 3.33. *Let $\mathcal{E}$ be a RPES such that $C(\mathcal{E})$ has a reachable configuration X with $X \xrightarrow{\underline{b}}$ (some $b \in F$). Then $C(\mathcal{E})$ has a non-terminating computation.*

Proof. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ and $C(\mathcal{E})$ be as stated. Consider a computation with n minimal such that $\emptyset = X_0 \xrightarrow{\Delta_1} X_1 \dots \xrightarrow{\Delta_n} X_n \xrightarrow{\underline{b}} Y$, where $X_n = X$. By minimality of n we know that for $i = 1, \dots, n$ we have $\Delta_i \cap \underline{F} = \emptyset$. Then $\Delta_i = A_i$ $(i = 1, \dots, n)$ and $X = \bigcup_{i=1}^{n} A_i$. Let $b \in A_i$. Then b is enabled at $X_{i-1} \subseteq Y$. Hence b is enabled at Y (there is no forward prevention in RPESs). We have a non-terminating computation $\emptyset \xrightarrow{A_1} \cdots \xrightarrow{A_n} X \xrightarrow{\underline{b}} Y \xrightarrow{b} X \cdots$. If desired we can ensure that the computation uses purely finite means by letting $Z = \{e \in E : e \leq b \text{ or } \exists a \in E. e \leq a \prec \underline{b}\}$. Then Z is finite and left-closed and we have $\emptyset \xrightarrow{A_1 \cap Z} \cdots \xrightarrow{A_n \cap Z} X \cap Z \xrightarrow{\underline{b}} Y \cap Z \xrightarrow{b} X \cap Z \cdots$. $\square$

3.4. Reversing Disciplines

Many patterns of common biochemical reactions involve breaking of previously established bonds out-of-causal order [23]. We now consider several particular disciplines for reversing events, out of many possible disciplines. The most usual is where we require that an event cannot be reversed until all events it has caused have also been reversed; we call this *cause-respecting*. A stronger notion is *causal*, where in addition to cause-respecting we stipulate that a reversible event can be reversed freely if all events it has caused have been reversed.

Definition 3.34. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES. We say that $\mathcal{E}$ is *cause-respecting* if for any $a, b \in E$, if $a < b$ then $a \ll b$, so that all causation is sustained causation. We say that $\mathcal{E}$ is *causal* if for any $a \in E$, $b \in F$, we have (1) $a \prec \underline{b}$ iff $a = b$ and (2) $a \triangleright \underline{b}$ iff $b < a$.

We have already seen a causal RPES in Example 3.16.

Example 3.35. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be defined by $E = F = \{a, b, c\}$, $a < b$ and $a \prec \underline{a}$, $b \prec \underline{b}$, $c \prec \underline{c}$, $c \prec \underline{a}$ with $b \triangleright \underline{a}$ and no conflict. Then $a \ll b$, so that $\mathcal{E}$ is a cause-respecting RPES. However $\mathcal{E}$ is not causal, since c has to be present for a to be reversed ($c \prec \underline{a}$). If we removed $c \prec \underline{a}$ from the definition of $\mathcal{E}$ then we would have a causal RPES.

Any PES can be converted into a causal RPES, once we decide which events are to be reversible.

Proposition 3.36. *Let $\mathcal{E} = (E, <, \sharp)$ be a PES and let $F \subseteq E$. Define $\kappa(\mathcal{E}, F) = (E, F, <, \sharp, \prec, \triangleright)$, where $a \prec \underline{a}$ (all $a \in F$) and $a \triangleright \underline{b}$ for every $a \in E$, $b \in F$ such that $b < a$. Then $\kappa(\mathcal{E}, F)$ is a causal RPES. Also $\phi_{\mathrm{p}}(\kappa(\mathcal{E}, F)) = \mathcal{E}$.*

Proof. Immediate from the definitions. □

Proposition 3.37. *Let $\mathcal{E}$ be a cause-respecting RPES. Then $\phi_{\mathrm{p}}(\mathcal{E})$ is a PES.*

Proof. Much the same as Proposition 3.18. □

Next we investigate secured configurations in cause-respecting RPESs.

Proposition 3.38. *Let $\mathcal{E}$ be a cause-respecting RPES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$.*

1. *If $X \in \mathsf{C}$ is left-closed and $X \xrightarrow{A \cup \underline{B}} Y$ then Y is also left-closed.*
2. *If $X \in \mathsf{C}$ is secured then X is left-closed.*

Proof. Straightforward. □

We now show that if an RPES is causal then any mixed transition can be inverted on left-closed configurations, provided that the events in the transition are reversible.

Proposition 3.39. *Let $\mathcal{E}$ be an RPES and let $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$ be left-closed and let $A, B \subseteq F$. Then:*

1. *If $\mathcal{E}$ is cause-respecting and $X \xrightarrow{\underline{B}} X'$ then $X' \xrightarrow{B} X$.*
2. *If $\mathcal{E}$ is causal and $X \xrightarrow{A \cup \underline{B}} X'$ then $X' \xrightarrow{B \cup \underline{A}} X$.*

Proof.

1. Suppose $\mathcal{E}$ is cause-respecting and $X \xrightarrow{\underline{B}} X'$. We check that B is enabled at $X' = X \setminus B$. Take $b \in B$. Suppose that $a < b$. Since $B \subseteq X$ and X is left-closed, we have $a \in X$. Also $b \triangleright \underline{a}$. This means that $a \notin B$, since $\underline{B}$ is enabled at X. So $a \in X \setminus B$ as required.
2. Suppose $\mathcal{E}$ is causal and $X \xrightarrow{A \cup \underline{B}} X'$. We check that $B \cup \underline{A}$ is enabled at $X' = (X \setminus B) \cup A$. Take $b \in B$. Suppose that $c < b$. Since $B \subseteq X$ and X is left-closed, we have $c \in X$. Also $b \triangleright \underline{c}$. This means that $c \notin B$, since $A \cup \underline{B}$ is enabled at X. So $c \in X \setminus B$ as required.
   Now take $a \in A$. Suppose that $c \prec \underline{a}$. Then $c = a$. So $c \in X' \setminus (A \setminus \{a\})$ as required. Suppose that $c \triangleright \underline{a}$. Then $c > a$. Since $a \notin X$ and X is left-closed, we have $c \notin X$. Also $c \notin A$ since $A \cup \underline{B}$ is enabled at X. Hence $c \notin X \cup A = X' \cup B$, as required. □

The second statement of Proposition 3.39 is related to the Loop Lemma for RCCS [6, Lemma 6], which states that every forward transition has a corresponding reverse transition, and conversely.

Theorem 3.40. *Let $\mathcal{E}$ be a cause-respecting RPES and let $C(\mathcal{E}) = (E, F, \mathsf{C}, \to)$.*

1. *If $X \in \mathsf{C}$ is secured then X is forwards secured.*
2. *If $X \in \mathsf{C}$ is reachable then X is forwards reachable.*

Proof.
1. Let $X_i \in \mathsf{C}$ $(i = 0, \ldots)$ with $X = \lim_{i\to\infty} X_i$ and $X_0 = \emptyset$ and $X_i \overset{A_{i+1}\cup\underline{B}_{i+1}}{\to} X_{i+1}$ with $A_{i+1} \subseteq E$ and $B_{i+1} \subseteq F$.
   By abuse of notation, let $\mathsf{last}(A_i) = \{a \in A_i : \mathsf{last}(a) = i\}$ $(i \geq 1)$. These are the members of X which are added for the last time at stage i. Let $X'_i = \bigcup_{j=1}^{i} \mathsf{last}(A_j)$. We have $X'_{i+1} = X'_i \cup \mathsf{last}(A_{i+1})$ with $X'_i \cap \mathsf{last}(A_{i+1}) = \emptyset$. It is easy to check that $X = \bigcup_{i=0}^{\infty} X'_i$.
   It remains to check that $X'_i \overset{\mathsf{last}(A_{i+1})}{\to} X'_{i+1}$ for all $i \geq 0$. Clearly $X'_{i+1} = X'_i \cup \mathsf{last}(A_{i+1})$ is conflict-free, since X is conflict-free. Suppose that $a \in \mathsf{last}(A_{i+1})$ and $a' < a$. Since $\mathcal{E}$ is cause-respecting, we have $a' \ll a$. So $a' \in X$ by Proposition 3.24. Therefore $a' \blacktriangleleft^{\mathsf{p}}_X a$ and $\mathsf{last}(a') < \mathsf{last}(a)$ by Lemma 3.27. Hence $a' \in X'_i$. Therefore $\mathsf{last}(A_{i+1})$ is enabled at X'_i, and $X'_i \overset{\mathsf{last}(A_{i+1})}{\to} X'_{i+1}$ as required.
2. This follows easily from the proof of part (1). □

Theorem 3.40 is related to a result of Danos and Krivine for RCCS [6, Cor. 1].

We now consider a second reversing discipline.

Definition 3.41. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES. We say that $\mathcal{E}$ is *inverse cause-respecting* if for any $a \in E$, $b \in F$, if $a < b$ then $a \triangleright \underline{b}$. We say that $\mathcal{E}$ is *inverse causal* if for any $a \in E$, $b \in F$, we have (1) $a \prec \underline{b}$ iff $a = b$ and (2) $a \triangleright \underline{b}$ iff $a < b$.

We can get an example of an inverse cause-respecting RPES which is not inverse causal by modifying Example 3.35 by changing $b \triangleright \underline{a}$ to $a \triangleright \underline{b}$.

In an inverse causal RPES reversing can start at any time with a $<$-minimal element of a configuration belonging to F. Plainly we can reach new configurations which are not forwards reachable.

Example 3.42. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an inverse causal RPES with $E = F = \{a, b, c\}$ and $a < b < c$ and no conflict. We also have $a \prec \underline{a}$, $b \prec \underline{b}$ and $c \prec \underline{c}$, and $a \triangleright \underline{b}$, $a \triangleright \underline{c}$ and $b \triangleright \underline{c}$ since $\mathcal{E}$ is inverse causal. The forwards reachable configurations are $\emptyset$, $\{a\}$, $\{a, b\}$ and $\{a, b, c\}$. Reversing from $\{a, b\}$ we can reach $\{b\}$. Reversing from $\{a, b, c\}$ we can reach $\{b, c\}$ followed by $\{c\}$, from which we can reach $\{a, c\}$. Thus every subset of E is a reachable configuration. The empty configuration is reachable from all configurations.

Theorem 3.43. *Let $\mathcal{E}$ be an inverse causal RPES with all events reversible and let $C(\mathcal{E}) = (E, E, \mathsf{C}, \to)$. Let $X \in \mathsf{C}$ be such that there is a finite bound k such that for any $e \in X$, $\mathsf{cdepth}(e) < k$. Then X is reachable.*

Proof sketch. Let $\mathcal{E}$ and X be as stated. The idea is that we build X by first including the events of greatest depth (plus others of lower depth that cause these events), then reversing all the events that caused this; we then build the next level down by forwards computation, followed by further reversing. Finally we build the lowest level (the minimal events).

Let X' be the left closure of X, i.e. $X' = \{e' \in E : \exists e \in X. e' \leq e\}$. Note that since X is conflict-free, then so is X'. Let $k = \max\{\mathsf{cdepth}(e) : e \in X\}$. For $i = 0, \dots, k$, let $A_i = \{e \in X : \mathsf{cdepth}(e) = i\}$ and $X_i = \{e \in X : \mathsf{cdepth}(e) < i\}$. Furthermore let $A'_i = \{e \in X' : \mathsf{cdepth}(e) = i\}$ and $X'_i = \{e \in X' : \mathsf{cdepth}(e) < i\}$. Then $X = \bigcup_{i=0}^{k} A_k$ and $X' = \bigcup_{i=0}^{k} A'_k$. Also $A_i \subseteq A'_i$ for $i = 0, \dots, k$.

We successively reach $Y_i = X'_i \cup \bigcup_{j=i}^{k} A_j$ for $i = k, k-1, \dots, 0$, by alternating forwards and reverse sequences of transitions. Starting from $\emptyset$ we reach $Y_k = X'_k \cup A_k$ by the computation $\emptyset \xrightarrow{A'_0} X'_1 \cdots \xrightarrow{A'_{k-1}} X'_k \xrightarrow{A_k} X'_k \cup A_k$. Assume that we have reached Y_i for $0 < i \leq k$. We reach Y_{i-1} by first using inverse causal reversing $Y_i \xrightarrow{\underline{A'_0}} \cdots \xrightarrow{\underline{A'_{i-1}}} \bigcup_{j=i}^{k} A_j$ and then using forwards computation $\bigcup_{j=i}^{k} A_j \xrightarrow{A'_0} \cdots \xrightarrow{A'_{i-2}} \xrightarrow{A_{i-1}} Y_{i-1}$. Thus eventually we reach $Y_0 = X$ as required. ☐

Remark 3.44. Inspection of the proof of Theorem 3.43 shows that the result still holds with a slightly weaker criterion than $\mathcal{E}$ being inverse causal; we can replace '$a \triangleright \underline{b}$ iff $a < b$' by '$a \triangleright \underline{b}$ implies $a < b$'.

Using Proposition 3.24 we can see that the condition of Theorem 3.43 that $X \in \mathsf{C}$ is such that there is a finite bound k such that for any $e \in X$, $\mathsf{cdepth}(e) < k$, is equivalent to X being included in some forwards reachable configuration Y.

The bounded depth condition is necessary, as the following example shows:

Example 3.45. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be the inverse causal RPES given by $E = F = \{a_i : i \in \mathbb{N}\}$ with $a_i < a_j$ (all $i < j$), no conflict and $a_i \prec \underline{a}_j$ iff $i = j$ and $a_i \triangleright \underline{a}_j$ iff $i < j$. The set $X = E \setminus \{a_1\}$ is a configuration, but it is not secured, let alone reachable. This is because a_1 has to be present to permit a_i to happen for each $i \geq 2$.

We next return to the example of a long-running transaction mentioned in the Introduction. Transactions have been modelled in the context of reversible process calculi in [7, 20, 21, 11].

Example 3.46. Suppose we wish to model a long-running transaction consisting of a series of n events $a_1 < \cdots < a_n$ and an error event error with $a_1 < \mathsf{error}$. We want causal reversing, but only once error has occurred. We can achieve this with $a_i \prec \underline{a}_i$, $\mathsf{error} \prec \underline{a}_i$ and $a_j \triangleright \underline{a}_i$ for $1 \leq i < j \leq n$.

If and when error occurs, the events $a_1, \dots, a_i$ that have occurred so far can be reversed starting with a_i. The problem is that there is nothing to prevent forwards computation continuing once the error has happened. We cannot use conflict here, since $a_i \sharp \mathsf{error}$ prevents error from occurring after a_i, which is not what we want. However if we have asymmetric conflict [14, 22, 1] we can achieve the desired result, which is that a_i is allowed until such time as error has occurred.

Furthermore, we may wish to have a compensation event comp which occurs after the transaction has been reversed. We then have $a_1 < \mathsf{error}$ and $\mathsf{error} < \mathsf{comp}$ so that $a_1 < \mathsf{comp}$, which is strange, since comp should only occur once a_1 has been reversed. The remedy is to adopt direct (immediate) causation, which is not transitive.

4. Reversing in Asymmetric Event Structures

In this section we increase the expressiveness of RPESs by modifying conflict to be asymmetric rather than symmetric, and also dropping the requirement that causation is transitive. The extra expressive power allows us to model more faithfully different forms of reversing events as exemplified in the Introduction and Example 3.46.

4.1. Asymmetric Event Structures

We recall the definition of asymmetric event structures:

Definition 4.1 ([1, Definition 2.4])**.** An *asymmetric event structure* (AES) is a triple $\mathcal{E} = (E, <, \lhd)$ where E is a set of events and for all $a, b \in E$

1. $\lhd \subseteq E \times E$ is the precedence relation (where we write $b \rhd a$ iff $a \lhd b$);
2. $< \subseteq E \times E$ is the causality relation, which is an irreflexive partial order, such that $\{e \in E : e < a\}$ is finite and $\lhd$ is acyclic on $\{e \in E : e \leq a\}$;
3. if $a < b$ then $a \lhd b$;
4. if $a \sharp c$ and $a < b$ then $b \sharp c$, where $\sharp$ is defined to be $\lhd \cap \rhd$.

What we write as $a \lhd b$ was $a \nearrow b$ in [1]. The precedence relation $a \lhd b$ says that event a *weakly causes*, or *precedes* event b, meaning that if both a and b occur then a occurred first. We can also write $a \lhd b$ the other way round as $b \rhd a$. In this case $b \rhd a$ says that b *prevents* a, meaning that if b is present in a configuration then a cannot occur. The meanings of $a \lhd b$ and $b \rhd a$ are of course equivalent; they represent different ways of looking at the same concept. We have already used prevention $b \rhd \underline{a}$ on reverse events with RPESs.

Lemma 4.2. *$\mathcal{E} = (E, <, \lhd)$ is an AES iff $\mathcal{E}$ is an AES according to Definition 4.1 with '$\lhd$ is acyclic on $\{e \in E : e \leq a\}$' replaced by '$\lhd$ is acyclic on $\{e \in E : e < a\}$ and $\lhd$ is irreflexive'.* ☐

In view of Lemma 4.2 we shall use the modified definition when convenient.

Definition 4.3. Let $\mathcal{E} = (E, <, \lhd)$ be an AES. We define the associated configuration system $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$ as follows. Let C consist of those $X \subseteq E$ such that $\lhd$ is well-founded on X. For $X \in \mathsf{C}$ and $A \subseteq E$, we say that A is *enabled* at X if $A \cap X = \emptyset$, and for every $a \in A$, both $\{b \in E : b < a\} \subseteq X$ and $\{b \in E : b \rhd a\} \cap (X \cup A) = \emptyset$. We define $X \xrightarrow{A} Y$ iff $X, Y \in \mathsf{C}$ and $Y = X \cup A$ and A is enabled at X.

Remark 4.4. We here adopt a different definition of configuration from [1, Definition 3.13], where there are two further conditions: left-closed and $\{e' \in X : e' \lhd e\}$ is finite for all $e \in X$. Clearly, left-closed is no longer appropriate in the (non-causal) reversible setting. We take the view in the present work that it is meaningful to have $\{e' \in X : e' \lhd e\}$ infinite. For instance if $A = \{a_i : i \in \mathbb{N}\}$ and $X = A \cup \{e\}$ with $a_i \lhd e$ for all $i \in \mathbb{N}$, we can have $\emptyset \xrightarrow{A} A \xrightarrow{e} X$.

The second part of the next lemma tells us that if X is a configuration and A is enabled at X, then $X \cup A$ is also a configuration.

Lemma 4.5. *Let $\mathcal{E} = (E, <, \lhd)$ be an AES. Let $X \subseteq E$, $A \subseteq E$, and suppose that A is enabled at X. Then:*

1. *If X is left-closed then $X \cup A$ is left-closed;*
2. *If $\lhd$ is well-founded on X then $\lhd$ is well-founded on $X \cup A$;*
3. *If $X = \bigcup_{i=0}^{\infty} X_i$ with $X_0 = \emptyset$ and $X_i \xrightarrow{A_{i+1}} X_{i+1}$ for $i \in \mathbb{N}$ then $\lhd$ is well-founded on X.*

Proof. Let $\mathcal{E} = (E, <, \lhd)$ be an AES. Let $X \subseteq E$, $A \subseteq E$, and suppose that A is enabled at X.

1. Suppose X is left-closed. Let $a \in X \cup A$ and $b < a$. If $a \in A$ then $b \in X$ by the definition of enabling. If $a \in X$ then $b \in X$ since X is left-closed.
2. Suppose $\lhd$ is well-founded on X. Take any $a, b \in X \cup A$. If $a \rhd b$ then $b \in X$, by the definition of enabling. Hence any descending sequence $a_0 \rhd a_1 \rhd \ldots$ in $X \cup A$ must be wholly within X apart from possibly the first element.
3. Let X be as stated. Suppose that X has an infinite descending sequence $a_0 \rhd a_1 \rhd \ldots$. Let $a_i \in A_{k_i}$ $(i \in \mathbb{N})$. Then clearly if $i < j$ then $k_i > k_j$ by the definition of enabling. This contradicts the well-foundedness of $\mathbb{N}$. □

Proposition 4.6. *Let $\mathcal{E} = (E, <, \lhd)$ be an AES. Then $C(\mathcal{E})$ is a configuration system.*

Proof. Straightforward using Lemma 4.5. □

Moving from symmetric to asymmetric conflict increases expressive power:

Example 4.7. Consider the AES $\mathcal{E} = (E, <, \lhd)$ with $E = \{a, b\}$ and $a \lhd b$. Then $C(\mathcal{E})$ consists of all subsets of E and we have $\emptyset \xrightarrow{a} \{a\} \xrightarrow{b} \{a, b\}$ and $\emptyset \xrightarrow{b} \{b\}$. There is no pre-PES for this configuration system.

Definition 4.8. Let $\mathcal{E} = (E, <, \lhd)$ be an AES with $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$. We define the *precedence depth* of events in X by a mapping from X to the ordinals given by $\mathsf{pdepth}_X(e) = \sup\{\mathsf{pdepth}_X(e') + 1 : e' \in X \text{ and } e' \lhd e\}$.

Note that $\mathsf{pdepth}_X(e)$ will be a (not necessarily finite) ordinal number by well-foundedness of $\lhd$ on X in C.

Proposition 4.9. *Let $\mathcal{E}$ be an AES with $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$. Then X is a forwards secured configuration iff X is left-closed and for all $e \in X$, $\mathsf{pdepth}_X(e)$ is finite.*

Proof sketch. ($\Rightarrow$) Suppose that X is forwards secured, with $X = \bigcup_{n=0}^{\infty} X_n$ and $X_i \stackrel{A_i}{\rightarrow} X_{i+1}$ ($i \geq 0$). We show that if $a, b \in X$ and $a \lhd b$ then $a \in A_i$ and $b \in A_j$ for some i, j with $0 \leq i < j$. We then show that if $e \in X_n$ then $\mathsf{pdepth}_X(e) < n$. It then follows that for all $e \in X$, $\mathsf{pdepth}_X(e)$ is finite. We can deduce that X is left-closed using Lemma 4.5.

($\Leftarrow$) Suppose that $X \in \mathsf{C}$ is left-closed and for all $e \in X$, $\mathsf{pdepth}_X(e)$ is finite. Let $X_n = \{e \in X : \mathsf{pdepth}_X(e) < n\}$ (all $n \geq 0$). Then $X_0 = \emptyset$ and $X = \bigcup_{n=0}^{\infty} X_n$. Let $A_n = \{e \in X : \mathsf{pdepth}_X(e) = n\}$ (all $n \geq 0$). We show $X_n \stackrel{A_n}{\rightarrow} X_{n+1}$ (all $n \geq 0$), using the fact that if $a < b$ then $a \lhd b$. It then follows that X is forwards secured. □

We now introduce a wider class of event structures with asymmetric conflict. This will be more useful for reversing than AESs. We weaken the definition of AES in two ways: we no longer require conflict to be hereditary (much as when going from PESs to pre-PESs) and we no longer require causation to be transitive.

Definition 4.10. A *proto-asymmetric event structure* (proto-AES) is a triple $\mathcal{E} = (E, \prec, \lhd)$ where E is a set of events and for any $a, b, e \in E$:

1. $\lhd \subseteq E \times E$ is the precedence relation (with $a \lhd b$ iff $b \rhd a$), which is irreflexive;
2. $\prec \subseteq E \times E$ is the *(direct) causation* relation, which is irreflexive and well-founded; and such that $\{e \in E : e \prec a\}$ is finite and $\lhd$ is acyclic on $\{e \in E : e \prec a\}$;
3. if $a \prec b$ then not $a \rhd b$.

We use the term 'proto-AES' rather than 'pre-AES', since pre-AESs are already defined in [1], as AESs without the conflict heredity condition.

Lemma 4.11. *If $\mathcal{E}$ is an AES then $\mathcal{E}$ is a proto-AES.* □

Definition 4.12. Let $\mathcal{E} = (E, \prec, \lhd)$ be a proto-AES. We define the associated configuration system $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$ as follows. Let C consist of those $X \subseteq E$ such that $\lhd$ is well-founded on X. For $X \in \mathsf{C}$ and $A \subseteq E$, we say that A is *enabled* at X if $A \cap X = \emptyset$, and for every $a \in A$, both $\{b \in E : b \prec a\} \subseteq X$ and $\{b \in E : b \rhd a\} \cap (X \cup A) = \emptyset$. We define $X \stackrel{A}{\rightarrow} Y$ iff $X, Y \in \mathsf{C}$ and $Y = X \cup A$ and A is enabled at X.

The next lemma is the analogue for proto-AESs of Lemma 4.5 for AESs.

Lemma 4.13. *Let $\mathcal{E} = (E, <, \lhd)$ be a proto-AES. Let $X \subseteq E$, $A \subseteq E$, and suppose that A is enabled at X.*

1. *If X is left-closed under $\prec$ then $X \cup A$ is left-closed under $\prec$.*

2. *If X is left-closed under $\prec$ and $\lhd \cup \prec$ is well-founded on X, then $X \cup A$ is left-closed under $\prec$ and $\lhd \cup \prec$ is well-founded on $X \cup A$.*

Proof. Straightforward. □

Definition 4.14. Let $\mathcal{E} = (E, <, \lhd)$ be a proto-AES with $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$, and suppose that $\lhd \cup \prec$ is well-founded on X. We define the *precedence causal depth* of events in X by a mapping from X to the ordinals given by $\mathsf{pcdepth}_X(e) = \sup\{\mathsf{pcdepth}_X(e') + 1 : e' \in X \text{ and } e' \lhd e \text{ or } e' \prec e\}$.

The next result is the analogue for proto-AESs of Proposition 4.9 for AESs.

Proposition 4.15. *Let $\mathcal{E}$ be a proto-AES with $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$. Then X is a forwards secured configuration iff X is left-closed under $\prec$ and $\lhd \cup \prec$ is well-founded on X and for all $e \in X$, $\mathsf{pcdepth}_X(e)$ is finite.*

Proof. The proof is much the same as that of Proposition 4.9, using Lemma 4.13 instead of Lemma 4.5. □

We now define a mapping htc (short for 'hereditary transitive closure') from proto-AESs to AESs.

Definition 4.16. For $\mathcal{E} = (E, \prec, \lhd)$ a proto-AES, we define $\mathsf{htc}(\mathcal{E}) = (E', <, \lhd')$ where

- E' is got from E by excluding any events a such that there is a $\lhd \cup \prec$-cycle in $\{e \in E : e = a \text{ or } e \prec^+ a\}$ (note that E' is left-closed under $\prec$)
- $< \; = \; \prec^+ \cap (E' \times E')$
- $\lhd'$ is obtained by augmenting $\lhd$ with $<$ and closing under conflict heredity using the rules

$$\frac{a \lhd b}{a \lhd' b} \qquad \frac{a < b}{a \lhd' b} \qquad \frac{a \sharp' b < c}{a \sharp' c}$$

  where $a, b, c \in E'$ and we let $\sharp' = \lhd' \cap \rhd'$.

We give an example of how a proto-AES can be converted via the mapping htc into an AES. Let $\mathcal{E} = (E, \prec, \lhd)$ with $E = \{a, b, c, d\}$ and $a \prec b \prec c$, $d \prec c$ and $a \lhd d \lhd a$. Then $\mathcal{E} = (E, \prec, \lhd)$ is a proto-AES. Note that a and d are in conflict ($a \sharp d$) and they are both (direct or indirect) causes of c. The configuration system $C(\mathcal{E})$ has $\emptyset \xrightarrow{a} \{a\} \xrightarrow{b} \{a, b\}$ and $\emptyset \xrightarrow{d} \{d\}$, together with various unreachable configurations. Of course c cannot ever occur. To get a corresponding AES $(E', <, \lhd')$, we must eliminate c, as it has conflicting causes. This gives $E' = \{a, b, d\}$. We then let $a < b$ and $a \lhd' b$ (in a more elaborate example we would have to take the transitive closure of $\prec$). Finally we set $a \lhd' d \lhd' a$, $b \lhd' d \lhd' b$ so that conflict is inherited. This gives an AES $\mathsf{htc}(\mathcal{E}')$. Its configuration system has the same forward secured configurations as $C(\mathcal{E}')$, with some unreachable configurations eliminated.

The next result is the analogue of Proposition 3.9 for pre-PESs and PESs.

Proposition 4.17. *Let $\mathcal{E} = (E, \prec, \lhd)$ be a proto-AES.*

1. $\mathsf{htc}(\mathcal{E}) = (E', <, \lhd')$ *is an AES.*
2. *If $\mathcal{E}$ is an AES then $\mathsf{htc}(\mathcal{E}) = \mathcal{E}$.*
3. *Let $X \subseteq E'$ be left-closed. Then $\lhd \cup \prec$ is well-founded on X iff $\lhd'$ is well-founded on X.*

Proof. 1. We have that $<=\prec^+$ is an irreflexive partial order since $\prec$ is acyclic. Also $\{e : e < a\}$ is finite using the well-foundedness of $\prec$ and König's Infinity Lemma. It is not hard to see that $\{e \in E' : e \leq a\}$ is $\sharp'$-conflict-free, by construction of E'. Suppose that we have a $\lhd'$-cycle in $\{e \in E' : e \leq a\}$. Since $\{e \in E' : e \leq a\}$ is $\sharp'$-conflict-free, the cycle must also be a $\lhd \cup \prec$-cycle, which is impossible by construction of E'. Hence $\lhd'$ is acyclic on $\{e \in E' : e \leq a\}$. Next if $a < b$ then $a \lhd' b$ using the second rule. Finally if $a \sharp' c$ and $a < b$ then $b \sharp' c$ by the third rule.

2. Straightforward.
3. Let $X \subseteq E'$ be left-closed under $<$. It is clear that if $\lhd'$ is well-founded on X then $\lhd \cup \prec$ is well-founded on X, since $\lhd \cup \prec \subseteq \lhd'$. Conversely, suppose X is $\lhd \cup \prec$ is well-founded on X. We see that X must be $\sharp'$-conflict-free using the fact that X is left-closed. Hence any infinite descending chain in X with $\lhd'$ must also be an infinite descending chain with $\lhd \cup \prec$. We conclude that $\lhd'$ is well-founded on X. □

The next result is the analogue of Proposition 3.10 for pre-PESs and PESs.

Proposition 4.18. *Let $\mathcal{E} = (E, \prec, \lhd)$ be a proto-AES. Let $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \to)$ and $C(\mathsf{htc}(\mathcal{E})) = (E', \emptyset, \mathsf{C}', \to')$. Then:*

1. $\mathsf{C}' \subseteq \mathsf{C}$.
2. $\to' \subseteq \to \cap (\mathsf{C}' \times \mathsf{C}')$.
3. *Let $X, Y \in \mathsf{C}'$. Suppose that X is left-closed and $\lhd \cup \prec$ is well-founded on X. If $X \xrightarrow{A} Y$ then $X \xrightarrow{A}' Y$.*
4. *If $X \subseteq E$ then X is forwards secured in $C(\mathcal{E})$ iff X is forwards secured in $C(\mathsf{htc}(\mathcal{E}))$.*

Proof. 1. By Proposition 4.17(3).

2. Immediate from the definitions.
3. Suppose that X is left-closed and $\lhd \cup \prec$ is well-founded on X. Suppose further $X \xrightarrow{A} Y$. Note that $Y = X \cup A$ is left-closed and $\lhd \cup \prec$ is well-founded on X by Lemma 4.13. Hence $\sharp'$ is well-founded on Y by Proposition 4.17. We check that A is enabled at X in $C(\mathsf{htc}(\mathcal{E}))$. Suppose that $b < a \in A$. Then $b \leq c \prec a$ for some c. Since $X \xrightarrow{A} Y$, we have $c \in X$, and so $b \in X$ since X is left-closed. Suppose now that $c \rhd' a \in A$. Suppose for a contradiction that $c \in X \cup A$. Then we cannot have $c \rhd a$, using $X \xrightarrow{A} Y$. Neither can we have $c > a$, since then we would have $c \geq d \succ a$ for some d, and this is impossible by $X \xrightarrow{A} Y$ and X being left-closed. Finally, $c \sharp' a$ is impossible since $\sharp'$ is well-founded on Y. We conclude that $c \notin X \cup A$ as required.

4. Suppose that X is forwards secured in $C(\mathcal{E})$. Then we have $X = \bigcup_{i=0}^{\infty} X_i$ with $X_0 = \emptyset$ and $X_i \in \mathsf{C}$ with $X_i \stackrel{A_{i+1}}{\rightarrow'} X_{i+1}$ $(i = 0, \ldots)$. Each X_i is left-closed and such that $\lhd \cup \prec$ is well-founded on X_i by Lemma 4.13. We next establish that each $X_i \in \mathsf{C}'$. Assume that $X_i \in \mathsf{C}'$. Take $a \in A_{i+1}$. Suppose that $a \notin E'$. Then there is a $\lhd \cup \prec$-cycle in $\{e \in E : e = a \text{ or } e \prec^+ a\} \subseteq X_{i+1}$. This is impossible since $\lhd \cup \prec$ is well-founded on X_{i+1}. Hence $a \in E'$ and $X_{i+1} \subseteq E'$. Now $\lhd'$ is well-founded on X_{i+1} by Proposition 4.17. So $X_{i+1} \in \mathsf{C}'$. Since $X_i \in \mathsf{C}'$ for each i, we can use part (3) to see that the X_i form a forwards securing for X in $C(\mathsf{htc}(\mathcal{E}))$.
Conversely, if X is forwards secured in $C(\mathsf{htc}(\mathcal{E}))$ then X is forwards secured in $C(\mathcal{E})$ using part (2). □

We conclude this section by looking at the relationships between (pre-)PESs and (proto-)AESs.

Definition 4.19. For $\mathcal{E} = (E, <, \sharp)$ a pre-PES, we define $\alpha(\mathcal{E}) = (E, \prec, \lhd)$ where $\prec = <$ and $\lhd = < \cup \sharp$. For $\mathcal{E} = (E, \prec, \lhd)$ a proto-AES, we define $\sigma(\mathcal{E}) = (E, <, \sharp)$ where $< = \prec^+$ and $\sharp = \lhd \cap \rhd$.

Proposition 4.20. 1. *If $\mathcal{E}$ is a PES then $\alpha(\mathcal{E})$ is an AES.*
2. *If $\mathcal{E}$ is an AES then $\sigma(\mathcal{E})$ is a PES.*

Proof. 1. This is [1, Lemma 2.2].
2. Immediate from the definitions. □

Proposition 4.21. *Let $\mathcal{E} = (E, <, \sharp)$ be a PES. Let $C(\mathcal{E}) = (E, \emptyset, \mathsf{C}, \rightarrow)$ and $C(\alpha(\mathcal{E})) = (E, \emptyset, \mathsf{C}', \rightarrow')$. Then*

1. $\mathsf{C} = \mathsf{C}'$
2. $\rightarrow' \subseteq \rightarrow$
3. *if X is left-closed and $X \stackrel{A}{\rightarrow} Y$ then $X \stackrel{A}{\rightarrow'} Y$*
4. *X is forwards secured in $C(\mathcal{E})$ iff X is forwards secured in $C(\alpha(\mathcal{E}))$.*

Proof. 1. We see that for $X \subseteq E$, X is $\sharp$-conflict-free iff $\lhd = < \cup \sharp$ is well-founded on X, using the fact that for any $e \in E$, $\{e' \in E : e' < e\}$ is finite. So $\mathsf{C} = \mathsf{C}'$.
2. Straightforward.
3. Straightforward.
4. By (2) and (3), noting that forwards secured configurations in PESs are left-closed (Proposition 3.7). □

Proposition 4.22. *If $\mathcal{E}$ is a pre-PES then $\alpha(\mathcal{E})$ is a proto-AES and $\sigma(\alpha(\mathcal{E})) = \mathcal{E}$.*

Proof. Straightforward. □

It is not necessarily the case that if $\mathcal{E}$ is a proto-AES then $\sigma(\mathcal{E})$ is a pre-PES, as the following example shows:

Example 4.23. Let $\mathcal{E} = (E, \prec, \lhd)$ be given by $E = \{a, b, c, d\}$, $a \prec b \prec c$ and $d \prec c$ and $a \lhd d$, $a \rhd d$, $a \lhd c$, $a \rhd c$. Then $\mathcal{E}$ is a proto-AES. However in $\sigma(\mathcal{E})$ we have $a < c$, $d < c$ and $a \sharp d$. Hence $\{e : e < c\}$ fails to be conflict-free. Also $a \sharp c$ and $a < c$, which violates condition (3) of Definition 3.2. Note that in a reversible setting a possible run is $a, b, \underline{a}, d, c$, reaching the conflict-free set $\{b, c, d\}$.

4.2. Reversible Asymmetric Event Structures

We now introduce the generalisation of RPESs to the setting of asymmetric conflict and not necessarily transitive causation.

Definition 4.24. A reversible asymmetric event structure (RAES) is a quadruple $\mathcal{E} = (E, F, \prec, \lhd)$ where E is a set of events and $F \subseteq E$ are those events of E which are reversible, and for any $a, b, c, e \in E$ and $\alpha \in E \cup \underline{F}$:

1. $\lhd \subseteq (E \cup \underline{F}) \times E$ is the precedence relation (with $a \lhd b$ iff $b \rhd a$), which is irreflexive;
2. $\prec \subseteq E \times (E \cup \underline{F})$ is the direct causation relation, which is irreflexive and well-founded, and such that $\{e \in E : e \prec \alpha\}$ is finite and $\lhd$ is acyclic on $\{e \in E : e \prec \alpha\}$;
3. $a \prec \underline{a}$ for all $a \in F$;
4. if $a \prec \alpha$ then not $a \rhd \alpha$;
5. $a \prec\!\!\prec b$ implies $a \lhd b$, where *sustained direct causation* $a \prec\!\!\prec b$ means that $a \prec b$ and if $a \in F$ then $b \rhd \underline{a}$;
6. $\prec\!\!\prec$ is transitive;
7. if $a \sharp c$ and $a \prec\!\!\prec b$ then $b \sharp c$, where $\sharp$ is defined to be $\lhd \cap \rhd$.

We have combined the forwards causation $<$ of (R)PESs and reverse causation $\prec$ of RPESs into a single direct causation relation $\prec$; similarly we have combined the forwards precedence $\lhd$ of AESs and the reverse prevention $\rhd$ of RPESs into a single precedence relation $\lhd$. We remark that direct (or immediate) causation $\prec$ was used in flow event structures [4] (with symmetric conflict $\sharp$).

If we set $F = \emptyset$ in Definition 4.24 we get an AES, since all causation is sustained causation. However if $F \neq \emptyset$ then the forwards-only part of an RAES is a proto-AES (see Section 4.2.1), since causation is not required to be transitive and conflict is not required to be hereditary. We discussed the reasons for these design choices in the Introduction.

In Definition 4.24 we also drop the requirement of AESs (Definition 4.1) that if $a < b$ then $a \lhd b$ (though that appears in its sustained causation form in item 5). This does not hold in general in the reversible context. Let $E = \{a, b\}$ and $F = \{a\}$, with $a \prec b$ and $a \prec \underline{a}$. Then we can perform $a, b, \underline{a}$ to reach $\{b\}$. At this point a is enabled. Thus it is not the case that $a \lhd b$, since that means a is disabled when b is present.

Definition 4.25. Let $\mathcal{E} = (E, F, \prec, \lhd)$ be an RAES. Let $X \subseteq E$ be such that $\lhd$ is well-founded on X. For $A \subseteq E$, $B \subseteq F$, we say that $A \cup \underline{B}$ is *enabled* at X if

- $A \cap X = \emptyset$, $B \subseteq X$;
- for every $a \in A$, if $c \prec a$ then $c \in X \setminus B$;
- for every $a \in A$, if $c \triangleright a$ then $c \notin X \cup A$;
- for every $b \in B$, if $d \prec \underline{b}$ then $d \in X \setminus (B \setminus \{b\})$;
- for every $b \in B$, if $d \triangleright \underline{b}$ then $d \notin X \cup A$.

Lemma 4.26. *Let $\mathcal{E} = (E, F, \prec, \triangleleft)$ be an RAES. Let $X \subseteq E$ be such that $\triangleleft$ is well-founded on X. Let $A \subseteq E$, $B \subseteq F$, and suppose that $A \cup \underline{B}$ is enabled at X. Then $\triangleleft$ is well-founded on $(X \setminus B) \cup A$.*

Proof. Let $\mathcal{E}, X, A, B$ be as stated. Suppose that we have an infinite descending chain $a_0 \triangleright a_1 \triangleright, \ldots$ in $(X \setminus B) \cup A$. Then we cannot have $a_i \in A$ with $i > 0$ by the definition of enabling. Thus we have an infinite descending chain in $X \setminus B \subseteq X$, which is impossible. Hence $\triangleleft$ is well-founded on $(X \setminus B) \cup A$. □

We now define a configuration to be a set of events on which $\triangleleft$ is well-founded (and therefore acyclic). The set of configurations is closed under transitions, in view of Lemma 4.26.

Definition 4.27. Let $\mathcal{E} = (E, F, \prec, \triangleleft)$ be an RAES. We define the associated configuration system $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$ as follows. Let C consist of those $X \subseteq E$ such that $\triangleleft$ is well-founded on X. For $X \in \mathsf{C}$ and $A \subseteq E$, $B \subseteq F$, we define $X \xrightarrow{A \cup \underline{B}} Y$ iff $X, Y \in \mathsf{C}$ and $Y = (X \setminus B) \cup A$ and $A \cup \underline{B}$ is enabled at X.

Proposition 4.28. *Let $\mathcal{E} = (E, F, \prec, \triangleleft)$ be an RAES. Then $C(\mathcal{E})$ is a configuration system.*

Proof. Similar to that of Proposition 3.15. □

We now give examples involving asymmetric conflict and non-transitive causation.

Example 4.29. We illustrate how asymmetric conflict can be used to control reversing. Let $\mathcal{E} = (E, F, \prec, \triangleleft)$ be defined as follows. Let $E = \{a_1, \ldots, a_n\}$ and $F = \{a_1, \ldots, a_{n-1}\}$. We have $a_i \prec a_{i+1}$ and $a_i \prec \underline{a_i}$ $(1 \leq i \leq n-1)$; also $a_i \triangleright \underline{a_{i+1}}$ $(1 \leq i \leq n-2)$. So far $\mathcal{E}$ is inverse causal (Definition 3.41), and events which have already been reversed can re-occur. We now add asymmetric conflict $a_i \triangleleft a_j$ $(1 \leq i < j \leq n)$, which prevents such re-occurrences, and also $a_{i+1} \prec \underline{a_i}$ $(1 \leq i \leq n-1)$, which ensures that we make progress towards the goal of the final configuration $\{a_n\}$. Non-empty reachable configurations of $\mathcal{E}$ are of the form $\{a_i, a_{i+1}, \ldots, a_j\}$ $(1 \leq i \leq j \leq n)$. At $\{a_i, \ldots, a_j\}$ we see that a_{j+1} is enabled if $j < n$ and $\underline{a_i}$ is enabled if $i < j$; in fact the mixed $\{a_{j+1}, \underline{a_i}\}$ is concurrently enabled if $i < j < n$. Thus we have a kind of FIFO queue which must be non-empty (apart from the initial empty configuration). All computations terminate, showing that Proposition 3.33 does not apply to RAESs.

Example 4.30. Let $\mathcal{E} = (E, \prec, \lhd)$ with $E = \{a, b, c, d\}$ and $a \prec b \prec c$, $d \prec c$ and $a \lhd d \lhd a$. Also let $F = \{a\}$ and $a \prec \underline{a}$. Then $\mathcal{E} = (E, F, \prec, \lhd)$ is an RAES. Note that a and d are in conflict ($a \sharp d$) and they are both (direct or indirect) causes of c. The configuration system $C(\mathcal{E})$ has $\emptyset \xrightarrow{a} \{a\} \xrightarrow{b} \{a, b\} \xrightarrow{\underline{a}} \{b\} \xrightarrow{d} \{b, d\} \xrightarrow{c} \{b, c, d\}$, $\{a\} \xrightarrow{\underline{a}} \emptyset$ and $\emptyset \xrightarrow{d} \{d\}$, together with various unreachable configurations. So the example illustrates how in the reversible setting an event can have conflicting indirect causes and still occur.

We now revisit our long-running transaction example (Example 3.46).

Example 4.31. The transaction consists of steps $a_1, \ldots, a_n$, and is complete once a_n is performed. After the transaction has started it may be interrupted by an error event error at any stage until it is complete. Once error occurs, the transaction is reversed back to the start, commencing with the most recent a_i. Once all a_is have been reversed, the compensation comp takes place, and error is itself reversed. Let $n \geq 2$. We define:

$$
\begin{array}{ll}
E = \{a_1, \ldots, a_n, \mathsf{error}, \mathsf{comp}\} & F = \{a_1, \ldots, a_{n-1}, \mathsf{error}\} \\
a_i \prec\!\!\prec a_{i+1} \quad (1 \leq i \leq n-1) & a_1 \prec \mathsf{error} \\
a_n \rhd \mathsf{error} & \mathsf{error} \rhd a_i \quad (1 \leq i \leq n) \\
\{\mathsf{error}, a_i\} \prec \underline{a_i} \quad (1 \leq i \leq n-1) & \mathsf{error} \prec \mathsf{comp} \\
a_1 \rhd \mathsf{comp} & \{\mathsf{error}, \mathsf{comp}\} \prec \underline{\mathsf{error}}
\end{array}
$$

Note that a_n and error are in conflict due to $a_n \rhd \mathsf{error}$ and $\mathsf{error} \rhd a_n$. We use sustained causation $a_i \prec\!\!\prec a_{i+1}$ to ensure causal reversing. We can deduce that $a_i \lhd a_j$ for $1 \leq i < j \leq n$. We use asymmetric conflict $\mathsf{error} \rhd a_i$ to prevent the transaction from continuing forwards when an error is present. Symmetric conflict would not work here. We need non-transitive causation: $a_1 \prec \mathsf{error} \prec \mathsf{comp}$ but not $a_1 \prec \mathsf{comp}$.

Runs are of two types:

- $a_1, \ldots, a_n$. Here no error occurs. Since a_n is irreversible, the sustained causation ensures that none of $a_1, \ldots, a_{n-1}$ can be reversed.
- $a_1, \ldots, a_i, \mathsf{error}, \underline{a_i}, \ldots, \underline{a_1}, \mathsf{comp}, \underline{\mathsf{error}}$ (some i with $1 \leq i \leq n-1$). The final configuration is $\{\mathsf{comp}\}$.

Reachable configurations are

$$
\begin{array}{ll}
\emptyset & \{a_1, \ldots, a_i\} \quad (1 \leq i \leq n) \\
\{a_1, \ldots, a_i, \mathsf{error}\} \quad (1 \leq i \leq n) & \{\mathsf{error}\} \\
\{\mathsf{error}, \mathsf{comp}\} & \{\mathsf{comp}\}
\end{array}
$$

Note that all of these configurations are forwards reachable, apart from $\{\mathsf{error}\}$, $\{\mathsf{error}, \mathsf{comp}\}$ and $\{\mathsf{comp}\}$. The example uses mostly causal reversing, but we violate this with a_1 and the 'trigger' event error. Although $a_1 < \mathsf{error}$, we reverse a_1 before error. This is necessary to complete reversing back to the empty configuration.

4.2.1. Mappings

We now turn to the relationship between RAESs and AESs. It is convenient to separate out the forward and reverse aspects of causation and precedence in RAESs.

Definition 4.32. For $\mathcal{E} = (E, F, \prec, \lhd)$ an RAES, let

1. $\prec_E = \prec \cap (E \times E)$ and $\prec_F = \prec \cap (E \times \underline{F})$
2. $\lhd_E = \lhd \cap (E \times E)$ and $\lhd_F = \lhd \cap (E \times \underline{F})$
3. $\rhd_F = \lhd_F$

Thus $\mathcal{E} = (E, F, \prec_E \cup \prec_F, \lhd_E \cup \lhd_F)$.

Definition 4.33. For $\mathcal{E} = (E, F, \prec, \lhd)$ an RAES, we define $\phi_{\mathrm{a}}(\mathcal{E}) = (E, \prec_E, \lhd_E)$. For $\mathcal{E} = (E, \prec, \lhd)$ an AES, we define $\rho_{\mathrm{a}}(\mathcal{E}) = (E, \emptyset, \prec, \lhd)$.

As previously stated, the forward-only part of an RAES is a proto-AES:

Proposition 4.34. *Let $\mathcal{E}$ be an RAES. Then $\phi_{\mathrm{a}}(\mathcal{E})$ is a proto-AES.*

Proof. Immediate from the definitions. □

The next result is the analogue of Proposition 3.18.

Proposition 4.35.
1. *Let $\mathcal{E} = (E, F, \prec, \lhd)$ be an RAES. If $F = \emptyset$ then $\phi_{\mathrm{a}}(\mathcal{E}) = (E, \prec, \lhd)$ is an AES and $\rho_{\mathrm{a}}(\phi_{\mathrm{a}}(\mathcal{E})) = \mathcal{E}$. Moreover, $C(\phi_{\mathrm{a}}(\mathcal{E})) = C(\mathcal{E})$.*
2. *Let $\mathcal{E} = (E, \prec, \lhd)$ be an AES, Then $\rho_{\mathrm{a}}(\mathcal{E}) = (E, \emptyset, \prec, \lhd)$ is an RAES and $\phi_{\mathrm{a}}(\rho_{\mathrm{a}}(\mathcal{E})) = \mathcal{E}$. Moreover, $C(\rho_{\mathrm{a}}(\mathcal{E})) = C(\mathcal{E})$.*

Proof.
1. Follows immediately from the definitions and Lemma 4.2, noting that if $F = \emptyset$ then $\prec = \prec\!\!\prec$.
2. Follows immediately from the definitions and Lemma 4.2, noting that if $F = \emptyset$ then $\prec = \prec\!\!\prec$, and that an AES is also a proto-AES (Lemma 4.11).

□

Recall that for $\mathcal{E} = (E, F, <, \sharp, \prec, \rhd)$ an RPES, and $a, b \in E$, we define sustained causation $a \ll b$ to mean that $a < b$ and if $a \in F$ then $b \rhd a$.

Definition 4.36. For $\mathcal{E} = (E, F, <, \sharp, \prec, \rhd)$ an RPES, we define $\alpha_{\mathrm{r}}(\mathcal{E}) = (E, F, \prec', \lhd')$ where $\prec' = < \cup \prec$ and $\lhd' = \ll \cup \sharp \cup \lhd$. For $\mathcal{E} = (E, F, \prec, \lhd)$ an RAES, we define $\sigma_{\mathrm{r}}(\mathcal{E}) = (E, F, \prec_E, \sharp, \prec_F, \rhd_F)$ where $\sharp = \lhd_E \cap \rhd_E$.

Proposition 4.37. *If $\mathcal{E}$ is an RPES then $\alpha_{\mathrm{r}}(\mathcal{E})$ is an RAES and $\sigma_{\mathrm{r}}(\alpha_{\mathrm{r}}(\mathcal{E})) = \mathcal{E}$.*

Proof. Immediate from the definitions of RPES (Definition 3.12) and RAES (Definition 4.24). □

If $\mathcal{E}$ is an RPES then the configuration systems of $\mathcal{E}$ and $\alpha_{\mathrm{r}}(\mathcal{E})$ are not precisely identical, because the sustained causation $\ll$ of $\mathcal{E}$ added to the precedence relation in $\alpha_{\mathrm{r}}(\mathcal{E})$ inhibits certain forward events from occurring. However, these discrepancies only apply in unreachable configurations.

Lemma 4.38. *Let $\mathcal{E} = (E, F, <, \sharp, \prec, \triangleright)$ be an RPES and let $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Let $\alpha_{\mathrm{r}}(\mathcal{E}) = (E, F, \prec, \triangleleft)$ and let $C(\alpha_{\mathrm{r}}(\mathcal{E})) = (E, F, \mathsf{C}', \rightarrow')$. Then:*

1. $\mathsf{C}' = \mathsf{C}$
2. *If $X \stackrel{A \cup \underline{B}}{\rightarrow} Y$ and X is left-closed under $\ll$ then $X \stackrel{A \cup \underline{B}}{\rightarrow} Y$*
3. *If $X \stackrel{A \cup \underline{B}}{\rightarrow'} Y$ then $X \stackrel{A \cup \underline{B}}{\rightarrow} Y$*

Proof. Straightforward from the definitions. □

Proposition 4.39. *Let $\mathcal{E}$ be an RPES. Then for $X \in \mathsf{C}$:*

1. *X is reachable in $C(\mathcal{E})$ iff X is reachable in $C(\alpha_{\mathrm{r}}(\mathcal{E}))$.*
2. *X is forwards secured in $C(\mathcal{E})$ iff X is forwards secured in $C(\alpha_{\mathrm{r}}(\mathcal{E}))$.*

Proof. By Lemma 4.38 and Propositions 3.24 and 4.45. □

Proposition 4.40. *If $\mathcal{E} = (E, F, \prec, \triangleleft)$ is an RAES and $\prec_E$ is transitive then $\sigma_{\mathrm{r}}(\mathcal{E})$ is an RPES.*

Proof. Straightforward from the definitions. □

If $\mathcal{E} = (E, F, \prec, \triangleleft)$ is an RAES and $\prec_E$ is not transitive then $\sigma_{\mathrm{r}}(\mathcal{E})$ need not be an RPES, as the next example shows.

Example 4.41. Let $E = \{a, b, c\}$, $F = \{a\}$ with $a \prec b \prec c$, $a \triangleleft c$, $c \triangleleft a$, $a \prec \underline{a}$ and $c \triangleright \underline{a}$. Then $\mathcal{E} = (E, F, \prec, \triangleright)$ is an RAES. However $\sigma_{\mathrm{r}}(\mathcal{E})$ is not an RPES since $<$ is not transitive. Suppose we change the definition of σ_{r} to make $< = \prec_E^+$. Then $<$ is transitive. However we then have $a \ll c$ and $a \sharp c$, from which we could deduce $c \sharp c$ if $\sigma_{\mathrm{r}}(\mathcal{E})$ were an RPES. It seems that there is no plausible RPES to which $\mathcal{E}$ can be mapped, even with the inevitable loss of information that any mapping from RAESs to RPESs entails.

We now have two methods of mapping a PES into an RAES—via an AES or via an RPES. The two methods produce the same result:

Proposition 4.42. *Let $\mathcal{E}$ be a PES. Then $\alpha_{\mathrm{r}}(\rho_{\mathrm{p}}(\mathcal{E})) = \rho_{\mathrm{a}}(\alpha(\mathcal{E}))$.*

Proof. Immediate from the definitions (Definitions 3.17, 4.19, 4.33, 4.36). □

We also get a commuting diagram in the converse direction, though that is of lesser interest as it involves loss of information.

4.3. Reachable Configurations

We investigate reachable and secured configurations for RAESs.

The next definition is the analogue of Definition 3.22:

Definition 4.43. Let $\mathcal{E} = (E, F, \prec, \triangleleft)$ be an RAES. For $a, b \in E$ we define $a \lll^\circ b$ iff for some $n \geq 1$ there are $a_1, \ldots, a_n$ such that $a = a_1$ and $a_i \prec b$ $(i = 1, \ldots, n-1)$ and $a_n \lll b$ and $a_n \triangleright^\bullet \cdots \triangleright^\bullet a_1$.

Here we let $b \triangleright^\bullet a$ mean $b \triangleright \underline{a}$ and $a \neq b$ (cf. Definition 3.20). Clearly $a \prec\!\!\prec^\circ b$ is just $a \prec\!\!\prec b$ in the case that $n = 1$, so that $a \prec\!\!\prec b$ implies $a \prec\!\!\prec^\circ b$.

As with RPESs, sustained causation $\prec\!\!\prec$ (and more generally $\prec\!\!\prec^\circ$) in the reversible setting behaves like standard causation in the forwards-only setting.

The next lemma is the analogue of Lemma 3.23 for RPESs.

Lemma 4.44. *Let $\mathcal{E}$ be an RAES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \to)$.*

1. *If $X \in \mathsf{C}$ is left-closed under $\prec$ and $X \xrightarrow{A} Y$ then Y is also left-closed under $\prec$.*
2. *If $X \in \mathsf{C}$ is left-closed under $\prec\!\!\prec^\circ$ and $X \xrightarrow{A \cup \underline{B}} Y$ then Y is also left-closed under $\prec\!\!\prec^\circ$.*
3. *Suppose that $X \in \mathsf{C}$ and $k \in \mathbb{N}$ are such that for all $e \in X$, $\mathsf{cdepth}(e) < k$. If $X \xrightarrow{A \cup \underline{B}} Y$ then for all $e \in Y$, $\mathsf{cdepth}(e) < k + 1$.*

Proof. 1. By Proposition 4.34 and Lemma 4.13.

2. The proof is very much the same as the proof of the corresponding part of Lemma 3.23, replacing $\ll^\circ$ by $\prec\!\!\prec^\circ$.
3. Straightforward. □

The next result is the analogue of Proposition 3.24 for RPESs. The notion of precedence causal depth (Definition 4.14) formulated for proto-AESs can be applied to RAESs in view of Proposition 4.34.

Proposition 4.45. *Let $\mathcal{E}$ be an RAES, $C(\mathcal{E}) = (E, F, \mathsf{C}, \to)$ and $X \in \mathsf{C}$. Then:*

1. *X is forwards secured iff X is left-closed under $\prec$ and $\triangleleft \cup \prec$ is well-founded on X and for all $e \in X$, $\mathsf{pcdepth}_X(e)$ is finite.*
2. *If X is secured then X is left-closed under $\prec\!\!\prec^\circ$.*
3. *X is forwards reachable iff X is left-closed, $\triangleleft \cup \prec$ is well-founded on X and there is $k \in \mathbb{N}$ such that for all $e \in X$, $\mathsf{pcdepth}_X(e) < k$.*
4. *If X is reachable then X is left-closed under $\prec\!\!\prec^\circ$ and there is $k \in \mathbb{N}$ such that for all $e \in X$, $\mathsf{cdepth}(e) < k$.*

Proof. 1. By Proposition 4.15 and Proposition 4.34.

2. Immediate from Lemma 4.44.
3. By an easy modification of the proof of Proposition 4.15 and Proposition 4.34.
4. Immediate from Lemma 4.44. □

It is not necessarily the case that $\triangleleft \cup \prec$ is well-founded on reachable configurations, as the next example shows.

Example 4.46. Let $E = \{a, b, c\}$, $F = \{a\}$ with $a \prec b \prec c \triangleleft a$ and $a \prec \underline{a}$. Then $(E, F, \prec, \triangleleft)$ is an RAES. By Proposition 4.45 we know that $\{a, b, c\}$ is not forwards reachable, since it contains a $\triangleleft \cup \prec$-cycle. However it is reachable by the computation $\emptyset \xrightarrow{a}\xrightarrow{b} \{a, b\} \xrightarrow{\underline{a}} \{b\} \xrightarrow{c}\xrightarrow{a} \{a, b, c\}$.

The next definition is the analogue of Definition 3.26:

Definition 4.47. Let $\mathcal{E} = (E, F, \prec, \lhd)$ be an RAES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$. For $a, b \in X$ we define $a \blacktriangleleft_X^{\mathsf{a}} b$ iff at least one of the following holds:

1. $a \lhd b$
2. $a \prec\!\!\prec^{\circ} b$
3. $\exists a' \prec a.\ b \sharp a'$
4. $\exists a_1 \prec a$ such that $a_1 \notin X$ and for some $n \geq 1$ there are $a_2, \ldots, a_n$ such that $a_i \prec a$ $(i = 2, \ldots, n)$ and $b \rhd^{\bullet} a_n \rhd^{\bullet} \cdots \rhd^{\bullet} a_1$

Note that the first condition $a \lhd b$ was not present in Definition 3.26; the remaining three are unchanged apart from replacing causation by direct causation.

The next lemma is the analogue of Lemma 3.27:

Lemma 4.48. *Let $\mathcal{E} = (E, F, \prec, \lhd)$ be an RAES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$ be secured, with $X = \lim_{i\to\infty} X_i$. Suppose that $a \blacktriangleleft_X^{\mathsf{a}} b$. Then $\mathsf{last}(a) < \mathsf{last}(b)$.*

Proof. There are four cases, depending on how $a \blacktriangleleft_X^{\mathsf{a}} b$ is derived. We give only the first case, since the remaining three are very much as in the proof of Lemma 3.27, using Proposition 4.45 rather than Proposition 3.24.

Suppose $a \lhd b$. We have $X_{\mathsf{last}(a)-1} \overset{A\cup\underline{B}}{\rightarrow} X_{\mathsf{last}(a)}$ with $a \in A$. Since $b \rhd a$, we have $b \notin X_{\mathsf{last}(a)-1} \cup A$. Hence $b \notin X_{\mathsf{last}(a)}$ and $\mathsf{last}(b) > \mathsf{last}(a)$. □

Proposition 4.49. *Let $\mathcal{E}$ be an RAES and $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$ be secured. Then $\blacktriangleleft_X^{\mathsf{a}}$ is well-founded on X.*

Proof. Immediate from Lemma 4.48. □

As in the case of RPESs, we can have reachable configurations which are finite but not finitely reachable; the RPES $\mathcal{E} = (E, F, <, \emptyset, \prec, \emptyset)$ of Example 3.30 is easily converted into an RAES $\mathcal{E}' = (E, F, < \cup \prec, \emptyset)$ with an empty precedence relation. As with RPESs, to ensure that every finite, reachable configuration is finitely reachable, we shall need to impose extra conditions on RAESs.

For $a \in E, b \in F$, let $b \prec^{\bullet} a$ mean $b \prec \underline{a}$ and $a \neq b$ (cf. Definition 3.20). The next lemma is the analogue of Lemma 3.31.

Lemma 4.50. *Let $\mathcal{E} = (E, F, \prec, \lhd)$ be an RAES and let $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$.*

1. *Suppose that $X \overset{A}{\rightarrow} Y$ and Z is such that if $a' \prec a \in A \cap Z$ then $a' \in Z$. Then $X \cap Z \overset{A\cap Z}{\rightarrow} Y \cap Z$.*
2. *Suppose that $X \overset{\underline{B}}{\rightarrow} Y$ and Z is such that if $b' \prec^{\bullet} b \in B \cap Z$ then $b' \in Z$. Then $X \cap Z \overset{\underline{B\cap Z}}{\rightarrow} Y \cap Z$.*

Proof. The proof is very much like that of Lemma 3.31, the main difference being that we have to consider prevention of forward as well as reverse events. □

The next result is the analogue of Theorem 3.32.

Theorem 4.51. *Let $\mathcal{E} = (E, F, \prec, \lhd)$ be an RAES. Suppose that for every $e \in E$, $\{e' \in E : e'(\prec \cup \prec^\bullet)^* e\}$ is finite. Then every finite, reachable configuration in $C(\mathcal{E})$ is finitely reachable.*

Proof. The proof is much the same as that of Theorem 3.32, using Lemma 4.50 instead of Lemma 3.31. □

4.4. Reversing Disciplines

We can define what it means for an RAES to be cause-respecting or causal by a straightforward adaptation of Definition 3.34. As with RPESs, causal implies cause-respecting.

The next result is the analogue of Proposition 3.37.

Proposition 4.52. *Let $\mathcal{E}$ be a cause-respecting RAES. Then $\phi_{\mathrm{a}}(\mathcal{E})$ is an AES.*

Proof. Immediate from the definitions. □

The next result is the analogue of Proposition 3.38.

Proposition 4.53. *Let $\mathcal{E}$ be a cause-respecting RAES and let $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$.*

1. *If $X \in \mathsf{C}$ is left-closed and $X \overset{A \cup \underline{B}}{\rightarrow} Y$ then Y is also left-closed.*
2. *If $X \in \mathsf{C}$ is secured then X is left-closed.*

Proof. Straightforward. □

The next result is the analogue of Theorem 3.40 for RPESs.

Theorem 4.54. *Let $\mathcal{E}$ be a cause-respecting RAES and let $C(\mathcal{E}) = (E, F, \mathsf{C}, \rightarrow)$.*

1. *If $X \in \mathsf{C}$ is secured then X is forwards secured.*
2. *If $X \in \mathsf{C}$ is reachable then X is forwards reachable.*

Proof. 1. Let $X_i \in \mathsf{C}$ $(i = 0, \ldots)$ with $X = \lim_{i \to \infty} X_i$ and $X_0 = \emptyset$ and $X_i \overset{A_{i+1} \cup \underline{B}_{i+1}}{\rightarrow} X_{i+1}$ with $A_{i+1} \subseteq E$ and $B_{i+1} \subseteq F$.
By abuse of notation, let $\mathsf{last}(A_i) = \{a \in A_i : \mathsf{last}(a) = i\}$ $(i \geq 1)$. These are the members of X which are added for the last time at stage i. Let $X'_i = \bigcup_{j=1}^{i} \mathsf{last}(A_j)$. We have $X'_{i+1} = X'_i \cup \mathsf{last}(A_{i+1})$ with $X'_i \cap \mathsf{last}(A_{i+1}) = \emptyset$. We can show by induction that $X'_i \subseteq X_i$ (all $i \geq 0$). It is easy to check that $X = \bigcup_{i=0}^{\infty} X'_i$.
It remains to check that $X'_i \overset{\mathsf{last}(A_{i+1})}{\rightarrow} X'_{i+1}$ for all $i \geq 0$. Clearly $X'_{i+1} = X'_i \cup \mathsf{last}(A_{i+1})$ is conflict-free, since X is conflict-free. Take any $a \in \mathsf{last}(A_{i+1})$. Suppose that $a' \prec a$. Since $\mathcal{E}$ is cause-respecting, we have $a' \prec\!\!\prec a$. So $a' \in X$ by Proposition 4.45. Therefore $a' \blacktriangleleft^{\mathsf{a}}_X a$ and $\mathsf{last}(a') < \mathsf{last}(a)$ by Lemma 4.48. Hence $a' \in X'_i$. Next suppose $a'' \rhd a$. We know that $a'' \notin X_i$, since $X_i \overset{A_{i+1} \cup \underline{B}_{i+1}}{\rightarrow} X_{i+1}$ and $a \in \mathsf{last}(A_{i+1}) \subseteq A_{i+1}$. Hence $a'' \notin X'_i$, since $X'_i \subseteq X_i$. Therefore $\mathsf{last}(A_{i+1})$ is enabled at X'_i, and $X'_i \overset{\mathsf{last}(A_{i+1})}{\rightarrow} X'_{i+1}$ as required.

2. This follows easily from the proof of part (1). ☐

We would like to prove a version of Proposition 3.39, which states that if an RPES is causal then any mixed transition can be inverted on left-closed configurations, provided that the events of the transition are reversible. However that no longer holds in the setting of RAESs, as the next example shows.

Example 4.55. Let $E = F = \{a, b\}$ and let $a \lhd b$, $a \prec \underline{a}$ and $b \prec \underline{b}$. Then $\mathcal{E} = (E, F, \prec, \rhd)$ is a causal RAES. All configurations are forwards reachable and left-closed. Note that a cannot occur after b going forwards, but we can reverse a and b in either order. In particular, we have $\{a, b\} \xrightarrow{\underline{a}} \{b\}$ but not $\{b\} \xrightarrow{a} \{a, b\}$.

Thus we need a different notion than causal (or cause-respecting).

Definition 4.56. Let $\mathcal{E} = (E, F, \prec, \rhd)$ be an RAES. We say that $\mathcal{E}$ is *precedence-respecting* if for any $a \in F$, $b \in E$, if $a \lhd b$ then $b \rhd \underline{a}$. We say that $\mathcal{E}$ is *precedence/cause-respecting* if $\mathcal{E}$ is cause-respecting and precedence-respecting. We say that $\mathcal{E}$ is *precedence causal* if for any $a \in E$, $b \in F$, both (1) $a \prec \underline{b}$ iff $a = b$ and (2) $a \rhd \underline{b}$ iff $b \prec a$ or $b \lhd a$.

Clearly, if $\mathcal{E}$ is precedence causal then $\mathcal{E}$ is precedence/cause-respecting.

We can now obtain the analogue of Proposition 3.39 for RPESs.

Proposition 4.57. *Let $\mathcal{E}$ be an RAES and let $C(\mathcal{E}) = (E, F, \mathsf{C}, \to)$. Let $X \in \mathsf{C}$ be left-closed and let $A, B \subseteq F$.*

1. *If $\mathcal{E}$ is precedence/cause-respecting and $X \xrightarrow{\underline{B}} X'$ then $X' \xrightarrow{B} X$.*
2. *If $\mathcal{E}$ is precedence causal and $X \xrightarrow{A \cup \underline{B}} X'$ then $X' \xrightarrow{B \cup \underline{A}} X$.*

Proof.

1. Suppose $\mathcal{E}$ is precedence/cause-respecting and $X \xrightarrow{\underline{B}} X'$. We check that B is enabled at $X' = X \setminus B$. Take $b \in B$. Suppose that $a < b$. Since $B \subseteq X$ and X is left-closed, we have $a \in X$. Also $b \rhd \underline{a}$, since $\mathcal{E}$ is cause-respecting. This means that $a \notin B$, since $\underline{B}$ is enabled at X. So $a \in X \setminus B$ as required. Suppose that $c \rhd b$. Then $c \rhd \underline{b}$, since $\mathcal{E}$ is precedence-respecting. Hence $c \notin X$, since $\underline{B}$ is enabled at X.
2. Suppose $\mathcal{E}$ is precedence causal and $X \xrightarrow{A \cup \underline{B}} X'$. We check that $B \cup \underline{A}$ is enabled at $X' = (X \setminus B) \cup A$. Take $b \in B$. Suppose that $c < b$. Since $B \subseteq X$ and X is left-closed, we have $c \in X$. Also $b \rhd \underline{c}$, since $\mathcal{E}$ is cause-respecting. This means that $c \notin B$, since $A \cup \underline{B}$ is enabled at X. So $c \in X \setminus B$ as required. Suppose that $c \rhd b$. Then $c \rhd \underline{b}$, since $\mathcal{E}$ is precedence-respecting. Hence $c \notin X \cup A$, since $A \cup \underline{B}$ is enabled at X.
   Now take $a \in A$. Suppose that $c \prec \underline{a}$. Then $c = a$, since $\mathcal{E}$ is precedence causal. So $c \in X' \setminus (A \setminus \{a\})$ as required. Suppose that $c \rhd \underline{a}$. Then $a \prec c$ or $a \lhd c$, since $\mathcal{E}$ is precedence causal. Suppose first that $a \prec c$. Since $a \notin X$ and X is left-closed, we have $c \notin X$. Also $c \notin A$ since $A \cup \underline{B}$ is enabled at X. Hence $c \notin X \cup A = X' \cup B$, as required. Now suppose that $a \lhd c$. Then again $c \notin X \cup A = X' \cup B$, since $A \cup \underline{B}$ is enabled at X. ☐

Any AES can be converted into a precedence causal RAES, once we decide which events are to be reversible (cf. Proposition 3.36).

Definition 4.58. Let $\mathcal{E} = (E, <, \lhd)$ be an AES and let $F \subseteq E$. Define $\pi(\mathcal{E}, F) = (E, F, \prec, \lhd')$, where $\prec \ = \ < \cup \{(a, \underline{a}) : a \in F\}$ and $\lhd' = \lhd \cup \{(\underline{a}, b) : a \in F, b \in E \text{ and } a \lhd b\}$.

Proposition 4.59. *Let $\mathcal{E} = (E, <, \sharp)$ be an AES. Then $\pi(\mathcal{E}, F)$ is a precedence causal RAES. Also $\phi_{\mathrm{a}}(\pi(\mathcal{E}, F)) = \mathcal{E}$.*

Proof. Immediate from the definitions. □

Finally, we can also adapt inverse causal reversing (Definition 3.41) to RAESs.

Definition 4.60. Let $\mathcal{E} = (E, F, <, \sharp, \prec, \rhd)$ be an RPES. We say that $\mathcal{E}$ is *inverse precedence causal* if for any $a \in E$, $b \in F$, both (1) $a \prec \underline{b}$ iff $a = b$ and (2) $a \rhd \underline{b}$ iff $a \prec b$ or $a \lhd b$.

We showed in Theorem 3.43 that in an inverse causal RPES with all events reversible we can reach all configurations with bounded causal depth. This no longer holds for RAESs, as the next example shows.

Example 4.61. Let $\mathcal{E} = (E, F, \prec, \lhd)$ with $E = F = \{a, b, c, d\}$, $a \prec b \sharp c \prec d \sharp a$, and $a \prec \underline{a}$, $b \prec \underline{b}$, $c \prec \underline{c}$, $d \prec \underline{d}$ and $a \rhd \underline{b}$, $c \rhd \underline{d}$ (where as usual $\sharp = \lhd \cap \rhd$). Then $\mathcal{E}$ is an inverse causal RAES. We can make it inverse precedence causal by adding $b \rhd \underline{c}$, $c \rhd \underline{b}$, $a \rhd \underline{d}$, $d \rhd \underline{a}$. In either case, $\{b, d\}$ is a configuration, but it is not reachable.

Theorem 4.62. *Let $\mathcal{E}$ be an inverse precedence causal RAES with all events reversible and let $C(\mathcal{E}) = (E, E, \mathsf{C}, \rightarrow)$. Let $X \in \mathsf{C}$ be such that there is a forwards reachable $X' \in \mathsf{C}$ with $X \subseteq X'$. Then X is reachable.*

Proof sketch. Let $\mathcal{E}$ and X, X' be as stated. By Proposition 4.45 X' is left-closed, $\lhd \cup \prec$ is well-founded on X' and there is $k' \in \mathbb{N}$ such that for all $e \in X'$, $\mathsf{pcdepth}_{X'}(e) < k'$.

Let $k = \max\{\mathsf{pcdepth}_{X'}(e) : e \in X\}$. For $i = 0, \ldots, k$, let $A_i = \{e \in X : \mathsf{pcdepth}_{X'}(e) = i\}$ and $X_i = \{e \in X : \mathsf{pcdepth}_{X'}(e) < i\}$. Furthermore let $A'_i = \{e \in X' : \mathsf{pcdepth}_{X'}(e) = i\}$ and $X'_i = \{e \in X' : \mathsf{pcdepth}_{X'}(e) < i\}$. Then $X = \bigcup_{i=0}^{k} A_k$ and $X' = \bigcup_{i=0}^{k} A'_k$. Also $A_i \subseteq A'_i$ for $i = 0, \ldots, k$.

We successively reach $Y_i = X'_i \cup \bigcup_{j=i}^{k} A_j$ for $i = k, k-1, \ldots, 0$, by alternating forwards and reverse sequences of transitions. Starting from $\emptyset$ we reach $Y_k = X'_k \cup A_k$ by the computation $\emptyset \xrightarrow{A'_0} X'_1 \cdots \xrightarrow{A'_{k-1}} X'_k \xrightarrow{A_k} X'_k \cup A_k$. Assume that we have reached Y_i for $0 < i \leq k$. We reach Y_{i-1} by first using inverse precedence causal reversing $Y_i \xrightarrow{\underline{A'_0}} \cdots \xrightarrow{\underline{A'_{i-1}}} \bigcup_{j=i}^{k} A_j$ and then using forwards computation $\bigcup_{j=i}^{k} A_j \xrightarrow{A'_0} \cdots \xrightarrow{A'_{i-2}} \xrightarrow{A_{i-1}} Y_{i-1}$. Thus eventually we reach $Y_0 = X$ as required. □

Remark 4.63. Inspection of the proof of Theorem 4.62 shows that the result still holds with a slightly weaker criterion than $\mathcal{E}$ being inverse precedence causal; we can replace '$a \rhd \underline{b}$ iff $a \prec b$ or $a \lhd b$' by '$a \rhd \underline{b}$ implies $a \prec b$ or $a \lhd b$'.

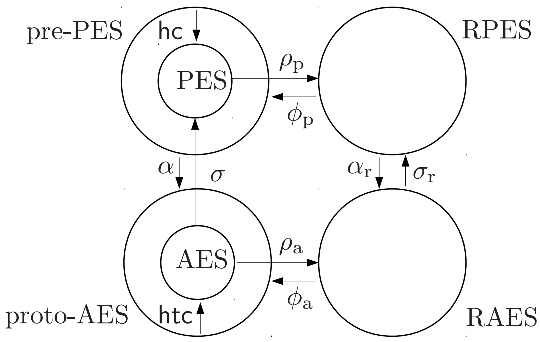

Figure 3: Mappings. Note that σ_r maps RAESs with transitive forwards direct causation to RPESs (Proposition 4.40).

5. Conclusions and Further Work

We have investigated conflict and causation for event structures with reversibility. We started by proposing a reversible form of prime event structure (RPES) where conflict inheritance no longer holds in general. The need for greater expressiveness then led us to two extensions: permitting non-transitive causation, and allowing asymmetric rather than symmetric conflict (useful for controlled reversing, as distinct from processes computing freely either forwards or backwards). These extensions yield our more general model, reversible asynchronous event structures (RAES). The two extensions are somewhat orthogonal and so one could envisage intermediate models.

We have obtained results about which configurations are reachable and, more generally, secured, i.e. limits of non-monotone sequences. For instance we have given conditions under which finite and reachable configurations are guaranteed to be reachable without intermediate infinite configurations. We have presented mappings between the various models (summarised in Figure 3) which show that our notions of RPES and RAES arise naturally from the pre-existing forward-only notions (PES, AES). Our models are general enough to allow several forms of reversibility to be defined and analysed, including the causal and inverse causal disciplines. We believe that RAESs offer the prospect of modelling a wide range of examples in software and biochemistry.

Future work could include formulating labelled versions of reversible event structures and bisimulations and modal logics for them as in [17] and [2, 19], establishing that our RPESs and RAESs are special cases of the reversible event structures in [21], modelling reversible process calculi, and extending existing work on domains and categories for event structures to the present models.

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