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On the optimality of nonlinear fractional disjunctive programming problems

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Abstract

This paper is concerned with the study of necessary and sufficient optimality conditions for convex–concave fractional disjunctive programming problems for which the decision set is the union of a family of convex sets. The Lagrangian function for such problems is defined and the Kuhn–Tucker saddle and stationary points are characterized. In addition, some important theorems related to the Kuhn–Tucker problem for saddle and stationary points are established. Moreover, a general dual problem is formulated, and weak, strong and converse duality theorems are proved. Throughout the presented paper illustrative examples are given to clarify and implement the developed theory.

Keywords: Nonlinear fractional programming; Disjunctive programming; Convexity; Concavity; Optimality; Duality

1. Introduction

Fractional programming [5,12,13,17] models have been a subject of wide interest since they provide a universal apparatus for a wide class of models in corporate planning, agricultural planning, public policy decision making, financial analysis of a firm, marine transportation, health care, educational planning, and bank balance sheet management. However, as is obvious, just considering one criterion at a time usually does not apply to real life problems because almost always two or more objectives are associated with a problem. Generally some of the objectives conflict with each other; therefore, one cannot optimize all objectives simultaneously. Nondifferentiable fractional programming problems play a very important role in formulating the set of most preferred solutions and a decision maker can select the optimal solution.

Disjunctive programs were introduced by Balas [1,2]. Later, Balas in [3] characterized the convex hull of feasible points for a disjunctive program, a class of problems which subsumes pure and mixed integer programs and many other nonconvex programming problems. Grossmann [9] proposed a convex nonlinear relaxation of the nonlinear convex generalized disjunctive programming problem that relies on the convex hull of each of the disjunctions that is obtained by variable desegregation and reformulation of the inequalities. Some topics of optimization disjunctive constraints functions were introduced in [16] by Sherali. Ceria in [4] studied the problem of finding the minimum of a convex function on closure of the convex hull of the union of those sets.

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The duality problem of disjunctive linear fractional programming is studied by Patkar in [15]. Helbig [10,11] develops the optimality and duality theory for families of linear programs with an emphasis on disjunctive linear optimization by proposing a 'vector' optimization problem as a dual problem. The concept of a disjunctive Lagrangian function is introduced and sufficient conditions for optimality are formulated in terms of their saddle points by Eremin [7]. A duality theory for disjunctive linear programming problems of a special kind was suggested by Gonçalves in [8]. Yang in [18] introduced two dual models for a generalized fractional programming problem. Optimality conditions and the duality of non-differentiable multiobjective programming problems were considered in [6,14] and for nondifferentiable nonlinear fractional programming problems considered by Liu in [13]. In this paper, the Lagranian function for this kind of problem will be defined and the Kuhn–Tucker saddle point is characterized. Also the Kuhn–Tucker saddle stationary point is established. A general dual problem is formulated and duality theorems (weak, strongly and converse) are proved.

Let I be an arbitrary (possibly infinite) nonempty index set. For $i \in I$, let $g_r^i : R^n \longrightarrow R$ be a vector map whose components are nonlinear convex functions, $g_r^i(x) \leq 0$, $1 \leq r \leq m$. Assume that $f^i, h^i : R^n \to R$ are convex and concave functions, respectively, and $h^i(x) > 0$ for each $i \in I$. Consider for each $i \in I$, the convex–concave fractional program problem

$$\text{DFP}(i): \quad \min \frac{f^i(x)}{h^i(x)} \quad \text{Subject to} \quad x \in Z_i,\ i \in I,$$

where $Z_i = \{x \in R^n : g_r^i(x) \leq 0\}$.

Assume that $Z_i \neq \emptyset$. Denote

$$M_i = \inf\left\{\frac{f^i(x)}{h^i(x)} : x \in Z_i\right\} \cup \{-\infty, \infty\} \quad \text{is the optimal value of DFP}(i)$$

and let

$$P_i = \left\{x \in Z_i : \frac{f^i(x)}{h^i(x)} = M_i\right\} \quad \text{be the set of optimal solutions of DFP}(i).$$

The disjunctive fractional programming problem is formulated as:

$$\text{DFP} \quad \inf_{i \in I} \inf_{x \in Z} \frac{f^i(x)}{h^i(x)}$$

where $Z = \cup_{i \in I} Z_i$ is the feasible solution set of problem DFP. Denote $M = \inf_{i \in I} M_i$ is the optimal value of DFP.

Let

$$P = \left\{x \in Z : \exists i \in I(x), \inf_i \frac{f^i(x)}{h^i(x)} = M\right\}, \quad \text{the set of optimal solutions of DFP,}$$

where

$$I(x) = \{i \in I' : x \in Z\},$$
$$I' = \{i \in I : Z_i \neq \emptyset\}.$$

The disjunctive objective functions may be taken in the form: $q^i(x, d^i) = f^i(x) - d^i h^i(x)$ where $d^i \geq 0$ for $i \in I'$ are auxiliary parameters. Also the DFP(i) can be reformulated as:

$$\text{DFP}_d \quad \inf_{i \in I} \inf_{x \in Z} (q^i(x, d^i) = f^i(x) - d^i h^i(x)).$$

For $i \in I'$ the Lagrangian function F_i of DFP(i) is defined by

$$F_i(x, \lambda^i) = q^i(x, d^i) + \sum_{r=1}^{m} \lambda_r^i g_r^i(x), \quad 1 \leq r \leq m,\ i \in I'$$

where $\lambda_r^i \in R^m$, $i \in I'$ are the Lagrangian multipliers. Then the Lagrangian function of DFP_d is defined as

$$\Im(x,\lambda) = \inf_{i\in I'} F_i(x,\lambda^i) = \inf_{i\in I'}\left\{q^i(x,d^i) + \sum_{r=1}^{m}\lambda_r^i g_r^i(x)\right\},$$

where $x \in R^n$, $\lambda^i \in R^m$ $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^i, \lambda^{i+1}, \ldots)$, and $h^i(x) > 0$ for $i \in I'$.

2. Kuhn–Tucker saddle point problem

In the following we will consider $I' = \{1, 2, \ldots, s\} \subset I$.

Definition 2.1. For all $x \geq 0$ and $\lambda \geq 0$ a point (x°, λ°) in R^{n+sm}, with $x^\circ \geq 0$ and $\lambda^\circ \geq 0$ is called a saddle point of $\Im(x,\lambda)$ iff

$$\Im(x^\circ,\lambda) \leq \Im(x^\circ,\lambda^\circ) \leq \Im(x,\lambda^\circ). \tag{1}$$

Theorem 2.1 (*Kuhn–Tucker Sufficient Optimality Criteria*). *If for $d^{\circ i} \geq 0$ the point (x°, λ°) is a saddle point of $\Im(x,\lambda)$ and the functions $q^i(x,d^i)$ and $\lambda_r^i g_r^i(x)$ are convex and bounded. Then x° is optimal solution for the problem* DFP_d.

Proof. Let (x°, λ°) be a saddle point of $\Im(x,\lambda)$. Then for all $\lambda \geq 0$ in R^{sm} and all $x \in Z$,

$$\Im(x^\circ,\lambda) \leq \Im(x^\circ,\lambda^\circ) \leq \Im(x,\lambda^\circ)$$

i.e.,

$$\inf_i F_i(x^\circ,\lambda^i) \leq \inf_i F_i(x^\circ,\lambda^{\circ i}) \leq \inf_i F_i(x,\lambda^{\circ i})$$

$$\inf_i\left\{q^i(x^\circ,d^{\circ i}) + \sum_{r=1}^{m}\lambda_r^i g_r^i(x^\circ)\right\} \leq \inf_i\left\{q^i(x^\circ,d^{\circ i}) + \sum_{r=1}^{m}\lambda_r^{\circ i} g_r^i(x^\circ)\right\} \leq \inf_i\left\{q^i(x,d^{\circ i}) + \sum_{r=1}^{m}\lambda_r^{\circ i} g_r^i(x)\right\}. \tag{2}$$

Thus

$$\inf_i q^i(x^\circ,d^{\circ i}) + \inf_i \sum_{r=1}^{m}\lambda_r^i g_r^i(x^\circ) \leq \inf_i q^i(x^\circ,d^{\circ i}) + \inf_i \sum_{r=1}^{m}\lambda_r^{\circ i} g_r^i(x^\circ), \tag{3}$$

i.e.,

$$\inf_i \sum_{r=1}^{m}\lambda_r^i g_r^i(x^\circ) \leq \inf_i \sum_{r=1}^{m}\lambda_r^{\circ i} g_r^i(x^\circ) \leq \sum_{r=1}^{m}\lambda_r^{\circ i} g_r^i(x^\circ) \quad \forall i. \tag{4}$$

Since for each i follows $\inf_i g_r^i(x^\circ) \leq g_r^i(x^\circ)$ $\forall i$, and

$$\sum_{r=1}^{m}\lambda_r^i \inf_i g_r^i(x^\circ) \leq \sum_{r=1}^{m}\lambda_r^i g_r^i(x^\circ) \quad \forall i,$$

then

$$\inf_i \sum_{r=1}^{m}\lambda_r^i \inf_i g_r^i(x^\circ) \leq \inf_i \sum_{r=1}^{m}\lambda_r^i\, g_r^i(x^\circ) \quad \forall i. \tag{5}$$

Let $g_r^k(x^\circ) = \inf_i\ g_r^i(x^\circ)$ and $\lambda_r^s\, g_r^k(x^\circ) = \inf_i\ \lambda_r^i\, g_r^k(x^\circ)$, then from (4) and (5), we get

$$\sum_{r=1}^{m}\lambda_r^s\, g_r^k(x^\circ) \leq \sum_{r=1}^{m}\lambda_r^{\circ i}\, g_r^i(x^\circ) \quad \forall i. \tag{6}$$

Thus

$$\sum_{r=1}^{m} \lambda_r^s g_r^k(x^\circ) \le \sum_{r=1}^{m} \lambda_r^{\circ k} g_r^k(x^\circ),$$

$$\sum_{r=1}^{m} (\lambda_r^s - \lambda_r^{\circ k}) g_r^k(x^\circ) \le 0. \tag{7}$$

Let $\lambda_j^s = \lambda_j^{\circ k}, \lambda_h^s = \lambda_h^{\circ k} + 1, j = 1, 2, \ldots, h, h+1, \ldots, m$. From (7) we get $g_h^k(x^\circ) \le 0$ and for each $h = 1, 2, \ldots, m$, we get $x^\circ \in Z^k$ or $x^\circ \in Z$, i.e., x° is a feasible point of DFP_d, and $\lambda^{\circ i} \ge 0$, then

$$\inf_{i\in I} \sum_{r=1}^{m} \lambda_r^{\circ i} g_r^i(x^\circ) \le 0. \tag{8}$$

By setting $\lambda_r^i = 0$ in the first inequality (2), we get

$$\inf_{i\in I} \sum_{r=1}^{m} \lambda_r^{\circ i} g_r^i(x^\circ) \ge 0. \tag{9}$$

Thus

$$\inf_{i\in I} \sum_{r=1}^{m} \lambda_r^{\circ i} g_r^i(x^\circ) = 0, \tag{10}$$

substituting from (10) in the second inequality of (2), then

$$\inf_{i\in I} q^i(x^\circ, d^{\circ i}) \le \inf_{i\in I} q^i(x, d^{\circ i}) + \inf_{i\in I} \lambda_r^{\circ i} g_r^i(x) \quad \forall x \in Z.$$

Then

$$\inf_{i\in I} q^i(x^\circ, d^{\circ i}) \le \inf_{i\in I} q^i(x, d^{\circ i}) \quad \forall x \in Z$$

i.e. x° is a minimal solution of DFP_d. □

Assumption 2.1. For $q^i(x, d^i), i \in I'$ are convex functions on Conv Z and Conv Z be a convex hull of $Z = \cup_{i\in I} Z_i$, we assume that $\inf_{i\in I'} q^i(x, d^i)$ is a convex function on Conv Z.

To state Kuhn–Tucker saddle point necessary theorem for problem DFP_d, we need the following propostion.

Proposition 2.1. *Under the Assumption 2.1, if the system*

$$\left.\begin{array}{l} \inf_{i\in I'}\{q^i(x, d^{i\circ}) - q^i(x^\circ, d^{i\circ})\} < 0, \\ g_r^i(x) \le 0 \quad \text{for at least one } i \in I' \end{array}\right\} \quad \textit{has no solution } x \in \text{Conv } Z,$$

then there exist $\lambda^i \in R, \lambda^{\circ i} \in R^m, (\lambda^\circ, \lambda^{\circ i}) \ge 0$ *such that*

$$\lambda^\circ \inf_{i\in I'}\{q^i(x, d^{i\circ}) - q(x^\circ, d^{i\circ})\} + \sum_{r=1}^{m} \lambda_r^{\circ i} g_r^i(x) \ge 0 \quad \textit{for all } x \in \text{Conv } Z.$$

Proof. Since Z is convex and $q^i(x, d^{\circ i})$, $g_r^i(x) i \in I'$ are convex on Conv Z, then from Assumption 2.1, we get $\inf_{i\in I'}\{q^i(x, d^{\circ i}) - q^i(x^\circ, d^{\circ i})\}$ is convex. Since the system

$$\left.\begin{array}{l} \inf_{i\in I'}\{q^i(x, d^{\circ i}) - q^i(x^\circ, d^{\circ i})\} < 0, \\ g_r^i(x) \le 0 \quad \text{for at least one } i \in I \end{array}\right\} \quad \text{has no solution on Conv } Z.$$

Then there exist $\lambda^\circ \in R, \lambda^{\circ i} \in R^m,\ (\lambda^\circ, \lambda^{\circ i}) \geq 0$ such that

$$\lambda^\circ \inf_{i\in I'}\{q^i(x, d^{\circ i}) - q^i(x^\circ, d^{\circ i})\} + \sum_{r=1}^{m} \lambda_r^{\circ i}\, g_r^i(x) \geq 0, \quad \text{for all } x \in \text{Conv } Z,\ i \in I'. \quad \square$$

Definition 2.2 (*Constraint Qualification CQ*). For each $i \in I'$, we say $g_r^i(x)$ satisfy Constraint Qualification CQ iff there exists a feasible point $x \in Z$ such that $g_r^i(x) < 0$, for $1 \leq r \leq m$.

Theorem 2.2 (*Kuhn–Tucker Necessary Optimality Criteria*). *If the Assumption 2.1 are satisfied, $g_r^i(x), i \in I'$ satisfy the constraint qualification and for $d^\circ \geq 0$, x° is an optimal solution of the problem* DFP$_d$*, then there exists $\lambda^\circ \geq 0$ such that (x°, μ°) is a saddle point of $\Im(x, \lambda)$.*

Proof. Since x° is a minimal solution of (x°, λ°)DFP$_d$, then the system

$$\left.\begin{array}{l} \inf\limits_{i\in I'} q^i(x, d^{\circ i}) - \inf\limits_{i\in I'} q^i(x^\circ, d^{\circ i}) < 0, \\ g_r^i(x) \leq 0 \quad \text{for at least one } i \in I' = \{1, 2, \ldots, s\} \end{array}\right\}$$

has no solution $x \in$ Conv Z, which implies that the system

$$\left.\begin{array}{l} \inf\limits_{i\in I'}\{q^i(x, d^{\circ i}) - q^i(x^\circ, d^{\circ i})\} < 0, \\ g_r^i(x) \leq 0 \quad \text{for at least one } i \in I' \end{array}\right\}$$

has no solution $x \in$ Conv Z. So, from Proposition 2.1, there exists $\mu^\circ \in R,\ \mu^{\circ i} \in R^{sm}$, $(\mu^\circ, \mu^{\circ i}) \geq 0$, $(\mu^\circ, \mu^{\circ i}) \neq 0$ such that:

$$\mu^\circ \inf_{i\in I'}\{q^i(x, d^{\circ i}) - q^i(x^\circ, d^{\circ i})\} + \sum_{r=1}^{m} \mu_r^{\circ i}\, g_r^i(x) \geq 0, \quad \forall x \in \text{Conv } Z,\ i \in I'. \tag{11}$$

Then for $x = x^\circ$ and $i \in I'$, we get $\sum_{r=1}^{m} \mu_r^{\circ i} g_r^i(x^\circ) \geq 0$, but $\sum_{r=1}^{m} \mu_r^{\circ i} g_r^i(x^\circ) \leq 0$, $i \in I'$. Thus for each $i \in I'$ the inequality (11) will take the form:

$$\mu^\circ \inf_{i\in I'}\{q^i(x, d^{\circ i}) - q^i(x^\circ, d^{\circ i})\} + \sum_{r=1}^{m} \mu_r^{\circ i}\, g_r^i(x) \geq 0, \quad \forall x \in \text{Conv } Z,$$

$$q^i(x, d^{\circ i}) + \sum_{r=1}^{m} \lambda_r^{\circ i}\, g_r^i(x) \geq q^i(x^\circ, d^{\circ i}) + \sum_{r=1}^{m} \lambda_r^{\circ i}\, g_r^i(x^\circ) \tag{12}$$

where $\lambda_r^{\circ i} = \frac{\mu_r^{\circ i}}{\mu^\circ}$.

The inequality (12) implies

$$\inf_{i\in I'}\left\{q^i(x, d^{\circ i}) + \sum_{r=1}^{m} \lambda_r^{\circ i}\, g_r^i(x)\right\} \geq \inf_{i\in I'}\left\{q^i(x^\circ, d^{\circ i}) + \sum_{r=1}^{m} \lambda_r^{\circ i}\, g_r^i(x^\circ)\right\}, \tag{13}$$

i.e., $\Im(x^\circ, \lambda^\circ) \leq \Im(x, \lambda^\circ)$, $\lambda^\circ = (\lambda_r^{\circ 1}, \ldots, \lambda_r^{\circ s})$.

Since $\sum_{r=1}^{m} \mu_r^{\circ i} g_r^i(x^\circ) = 0$ and for $\mu_r^i \geq 0$ we have $\sum_{r=1}^{m} \mu_r^i g_r^i(x^\circ) \leq 0$, and

$$\sum_{r=1}^{m} \mu_r^i g_r^i(x^\circ) \leq \sum_{r=1}^{m} \mu_r^{\circ i}\, g_r^i(x^\circ), \quad i \in I'$$

by adding $\mu^\circ q^i(x^\circ, d^{\circ i})$ to both sides, we get

$$\mu^\circ q^i(x^\circ, d^{i\circ}) + \sum_{r=1}^{m} \mu_r^i g_r^i(x^\circ) \leq \mu^\circ q^i(x^\circ, d^{i\circ}) + \sum_{r=1}^{m} \mu_r^{\circ i}\, g_r^i(x^\circ), \quad i \in I'.$$

If $\mu^\circ = 0$, then from the inequality (11), we get $\sum_{r=1}^{m} \mu_r^{\circ i} g_r^i(x) \geq 0, i \in I', x \in \text{Conv } Z$, which contradicts the Constraint Qualification (CQ) condition. Then $\mu^\circ > 0$ and hence

$$q^i(x^\circ, d^{\circ i}) + \sum_{r=1}^{m} \frac{\mu_r^i}{\mu^\circ} g_r^i(x^\circ) \leq q^i(x^\circ, d^{i\circ}) + \sum_{r=1}^{m} \frac{\mu_r^{\circ i}}{\mu^\circ} g_r^i(x^\circ), \quad i \in I'$$

and

$$\inf_{i \in I'} \left\{ q^i(x^\circ, d^{i\circ}) + \sum_{r=1}^{m} \lambda_r^i g_r^i(x^\circ) \right\} \leq \inf_{i \in I'} \left\{ q^i(x^\circ, d^{i\circ}) + \sum_{r=1}^{m} \lambda_r^{\circ i} g_r^i(x^\circ) \right\}. \tag{14}$$

From (13) and (14) we get

$$\inf_{i \in I'} \left\{ q^i(x^\circ, d^{i\circ}) + \sum_{r=1}^{m} \lambda_r^i g_r^i(x^\circ) \right\} \leq \inf_{i \in I'} \left\{ q^i(x^\circ, d^{i\circ}) + \sum_{r=1}^{m} \lambda_r^{\circ i} g_r^i(x^\circ) \right\}$$
$$\leq \inf_{i \in I} \left\{ q^i(x, d^{i\circ}) + \sum_{r=1}^{m} \lambda_r^{\circ i} g_r^i(x) \right\}$$

i.e.

$$\Im(x^\circ, \lambda) \leq \Im(x^\circ, \lambda^\circ) \leq \Im(x, \lambda^\circ). \quad \square$$

3. Kuhn–Tucker stationary-point problem

Definition 3.1 (*Kuhn–Tuker Stationary Point for Problem DFP$_d$*). Find $x^\circ \in Z$, $d^\circ \geq 0$ and $\lambda^\circ \in R^{sm}$ if they exist, such that

$$\Im_x(x^\circ, \lambda^\circ) \geq 0, \qquad x^\circ \Im_x(x^\circ, \lambda^\circ) = 0 \tag{15}$$
$$\Im_\lambda(x^\circ, \lambda^\circ) \leq 0, \qquad \lambda^\circ \Im_\lambda(x^\circ, \lambda^\circ) = 0. \tag{16}$$

or equivalently

$$\nabla \inf_{i \in I'} q^i(x^\circ, d^{i\circ}) + \sum_{r=1}^{m} \lambda_r^{\circ i} \nabla g_r^i(x^\circ) = 0, \quad d^{i\circ} \geq 0, \ i \in I' \tag{17}$$

$$g_r^i(x^\circ) \leq 0, \quad i \in I' \tag{18}$$

$$\sum_{r=1}^{m} \lambda_r^{\circ i} g_r^i(x^\circ) = 0, \quad i \in I', \ \lambda^\circ \geq 0. \tag{19}$$

Theorem 3.1. *Let $q^i(x, d^i)$, $g_r^i(x)$, $i \in I'$, $r = 1, 2, \ldots, m$ be differentiable convex on* conv Z. *If $q^i(x, d^i)$ and $\lambda_r^i g_r^i(x)$ are bounded functions for each $x \in$* Conv Z *and $g_r^i(x)$, $i \in I'$ satisfy the Constraint Qualification condition CQ. Then for $d^{\circ i} \geq 0, i \in I', x^\circ$ is a optimal solution of* DFP$_d$, *iff there exists $\lambda^\circ \in R^{sm}$, $\lambda^\circ \geq 0$ such that (15) and (16) are satisfied.*

Proof. Since x° is an optimal solution of problem DFP$_d$, then from Theorem 2.2, there exists $\lambda^\circ \in R^{sm}$, $\lambda^\circ \geq 0$, such that (x°, λ°) is a saddle point of $\Im(x, \lambda)$, i.e.,

$$\Im(x^\circ, \lambda) \leq \Im(x^\circ, \lambda^\circ) \leq \Im(x, \lambda^\circ).$$

Suppose there is a negative component of $\Im_x(x^\circ, \lambda^\circ)$, say $\partial \Im(x^\circ, \lambda^\circ)/\partial x_k$, then there exists a vector $x \geq 0$ with components $x_s = x_s^\circ$, $s \neq k$ and $x_k > x_k^\circ$ such that $\Im(x, \lambda^\circ) < \Im(x^\circ, \lambda^\circ)$, which is a contradiction, since (x°, λ°) is a saddle point of $\Im(x, \lambda)$, and hence $\Im_x(x^\circ, \lambda^\circ) \geq 0$.

Since $x^\circ \geq 0$, all of the summands $x_k^\circ \Im_{x_k}(x^\circ, \lambda^\circ)$ in the inner product $x^\circ \Im_x(x^\circ, \lambda^\circ) \geq 0$. Now, if there there exists k such that $x_k^\circ \Im_{x_k}(x^\circ, \lambda^\circ) > 0$ and $x_k^\circ > 0$, there would also exist a vector x with components $x_s = x_s^\circ$, $s \neq k$ and

$0 \leq x_k \leq x_k^\circ$ such that $\Im(x, \lambda^\circ) < \Im(x^\circ, \lambda^\circ)$ contradicts the claim that the saddle point of $\Im(x, \lambda)$ at (x°, λ°). Thus it implies that $x^\circ \Im_x(x^\circ, \lambda^\circ) = 0$. Since $\Im(x^\circ, \lambda)$ is affine linear in λ, then

$$\Im(x^\circ, \lambda) = \Im(x^\circ, \lambda^\circ) + (\lambda - \lambda^\circ)\Im_\lambda(x^\circ, \lambda^\circ) \quad \text{for } \lambda \geq 0.$$

Since a point (x°, λ°) in R^{n+sm}, with $x^\circ \geq 0$ and $\lambda^\circ \geq 0$ is a saddle point of $\Im(x, \lambda)$, we have

$$\Im(x^\circ, \lambda) \leq \Im(x^\circ, \lambda^\circ)$$

i.e.

$$(\lambda - \lambda^\circ)\Im_\lambda(x^\circ, \lambda^\circ) \leq 0 \quad \text{for each } \lambda \in R^m.$$

Thus for certain λ such that $(\lambda - \lambda^\circ) > 0$, we have $\Im_\lambda(x^\circ, \lambda^\circ) \leq 0$, and for other λ, $(\lambda - \lambda^\circ) < 0$ implies $\Im_\lambda(x^\circ, \lambda^\circ) \geq 0$

Then

$$\Im_\lambda(x^\circ, \lambda^\circ) = 0, \quad \text{hence } \lambda^\circ \Im_\lambda(x^\circ, \lambda^\circ) = 0.$$

Conversely, let for $d^{i\circ} \geq 0, i \in I'(x^\circ, \lambda^\circ)$ be a solution of (15), $x^\circ \in Z, \lambda^\circ \in R^{sm}$.

From the convexity and differentiability of $\inf_i q^i(x^\circ, d^\circ)$ for $d^{i\circ} \geq 0, i \in I'$ and Assumption 2.1, we have

$$\begin{aligned}
\inf_i q^i(x, d^{\circ i}) - \inf_i q^i(x^\circ, d^{\circ i}) &\geq \nabla \inf_i q^i(x^\circ, d^{\circ i})(x - x^\circ) \\
&= -\sum_{r=1}^m \lambda_r^{\circ i} \nabla g_r^i(x^\circ)(x - x^\circ) \quad \left(\text{since } \nabla \inf_i q^i(x^\circ) = -\sum_{r=1}^m \lambda_r^{\circ i} \nabla\, g_r^i(x^\circ)\right) \\
&\geq \sum_{r=1}^m \lambda_r^{\circ i}(g_r^i(x^\circ) - g_r^i(x)) \\
&\qquad \text{(by convexity and differentiability of } g_r^i(x^\circ) \text{ and (17), and } \lambda_r^{\circ i} \geq 0) \\
&= -\sum_{r=1}^m \lambda_r^{\circ i} g_r^i(x) \quad \left(\text{since } \sum_{r=1}^m \lambda_r^{\circ i} \nabla\, g_r^i(x^\circ) = 0\right) \\
&\geq 0 \quad (\text{since } \lambda^\circ \geq 0 \text{ and } g_r^i(x) \leq 0).
\end{aligned}$$

Hence

$$\inf_i q^i(x, d^{\circ i}) \geq \inf_i q^i(x^\circ, d^{\circ i}) \quad \text{for any } x \in Z \text{ and } d^{i\circ} \geq 0, i \in I'.$$

Then x° is an optimal solution of problem DFP_d. □

We consider the following example for a DFP problem with two disjunction functions.

Example 3.1. Consider the problem

$$\min_{i \in I} \min_{x \in Z} \left(\frac{f^1(x)}{h^1(x)}, \frac{f^2(x)}{h^2(x)}\right),$$

where $Z = \cup_{i \in I} Z_i$, $Z_i = \{x \in R^2 : g_r^i(x) \leq 0\}$,

$$\frac{f^1(x)}{h^1(x)} = \frac{2x_1 - x_2}{x_1 + x_2}, \qquad \frac{f^2(x)}{h^2(x)} = \frac{x_1 - 3x_2}{2x_1 - x_2}, \qquad g_1^1(x) = x_1 + x_2 - 1 \leq 0, \quad x_1, x_2 \geq 0,$$
$$g_1^2(x) = -x_1 + 2x_2 - 6 \leq 0, \qquad g_2^2(x) = x_1 + x_2 - 5 \leq 0, \qquad g_3^2 = x_1 + x_2 - 1 \geq 0, \quad x_1, x_2 \geq 0,$$

i.e.,

$$Z_1 = \{x \in R^2 / x_1 + x_2 - 1 \leq 0, x_1, x_2 \geq 0\},$$
$$Z_2 = \{x \in R^2 / -x_1 + 2x_2 - 6 \leq 0,\ x_1 + x_2 - 5 \leq 0,\ -x_1 - x_2 + 1 \leq 0, x_1, x_2 \geq 0\}.$$

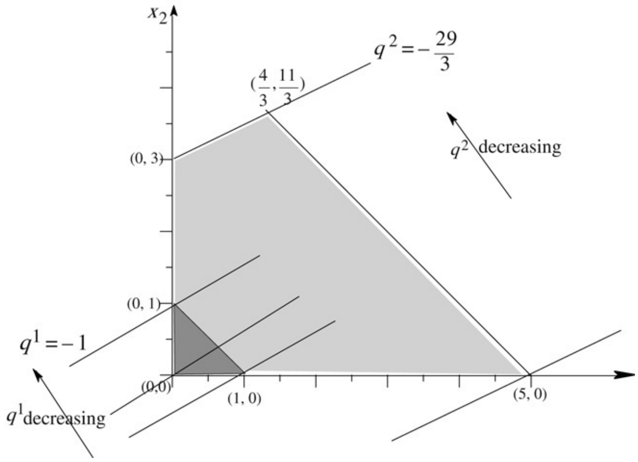

Fig. 1.

It is clear that the optimal solution for DFP(1) is $(x_1, x_2) = (0, 1)$ and the corresponding optimal value is $d^1 = -1$, Also the optimal solution of DFP(2) is $(x_1, x_2) = (\frac{4}{3}, \frac{11}{3})$ and the optimal value is $d^2 = -\frac{29}{3}$. So, $M = \min_i M_i = \min\{-1, -\frac{29}{3}\} = -\frac{29}{3}$ is the minimal value of DFP. Since $Z_1 \neq \emptyset$, $Z_2 \neq \emptyset$. Then $I_P = \{1, 2\}$, $I(x) = \{(1, 2) \in I_P : x \in Z\}$ and (CQ) is valid. The set of solutions of DFP: $P = \{x \in Z : \exists i = 2 \in I(x), q^2(x) = \frac{-29}{3}\}$. See the above Fig. 1. It is clear that the point $(x^\circ, \lambda^\circ) = ((5, 0), (\lambda_1^{\circ 1}, \lambda_1^{\circ 2}, \lambda_2^{\circ 2}, \lambda_3^{\circ 2}))$ is not a saddle point of $\Im(x, \lambda)$ since the Lagrangian function $\Im(x^\circ, \lambda^\circ)$ is

$$\Im(x^\circ, \lambda^\circ) = \inf\left(2 + 4\lambda_1^{\circ 1}, 0.5 - 11\lambda_1^{\circ 2} - 4\lambda_3^{\circ 2}\right).$$

If $2 + 4\lambda_1^{\circ 1} \leq 0.5 - 11\lambda_1^{\circ 2} - 4\lambda_3^{\circ 2}$, then $4\lambda_1^{\circ 1} + 11\lambda_1^{\circ 2} + 4\lambda_3^{\circ 2} \leq -1.5$, which implies that at least one of $\lambda_1^{\circ 1}, \lambda_1^{\circ 2}, \lambda_3^{\circ 2}$ is negative,which contradicts its positivity. So

$$0.5 - 11\lambda_1^{\circ 2} - 4\lambda_3^{\circ 2} = \inf(2 + 4\lambda_1^{\circ 1}, 0.5 - 11\lambda_1^{\circ 2} - 4\lambda_3^{\circ 2})$$

and

$$0.5 - 11\lambda_1^{\circ 2} - 4\lambda_3^{\circ 2} \geq \inf(2 + 4\lambda_1^{1}, 0.5 - 11\lambda_1^{2} - 4\lambda_3^{2}) \quad \forall \lambda_1^1, \lambda_1^2, \lambda_3^2.$$

i.e., for $\lambda_1^2 = \lambda_3^2 = 0$, $0.5 - 11\lambda_1^{\circ 2} - 4\lambda_3^{\circ 2} > 0.5$, which implies at least one of $\lambda_1^{\circ 2}$ and $\lambda_3^{\circ 2}$ is negative, which is a contradiction.

4. Duality using Mond–Weir type

According to optimality Theorems 2.1 and 2.2, we formulate the Mond–Weir type dual (M–WDFD) of the disjunctive fractional problem (DFP_d) as follows:

$$\text{(M–WDFD)} \quad \text{Max}_{(u,\mu)\in B} \left(F(u) = \sup_{i \in I} \left(\frac{f^i(u)}{h^i(u)} \right) \right), \tag{20}$$

where B denotes the set of $(u, \mu) \in R^n \times R_+^m$ satisfying the following conditions:

$$\sup_{i \in I} \nabla_u \left\{ \left(\frac{f^i(u)}{h^i(u)} \right) + \sum_{j=1}^{m} \mu_j^i g_j^i(u) \right\} = 0, \tag{21}$$

$$\sum_{j=1}^{m} \mu_j^i g_j^i(u) = 0, \quad \mu_j^i \geq 0, \ i \in I, \ j = 1, \dots, m, \tag{22}$$

$$\left(\frac{f^i(u)}{h^i(u)}\right) \geq 0, \quad i \in I. \tag{23}$$

Theorem 4.1 (*Weak Duality*). *Let* x *be feasible for* (DFP$_d$)*, and* (u, μ) *be feasible for* (M–WDFD). *If for all feasible* (u, μ)*, the functions* $(\frac{f^i(u)}{h^i(u)})$ *are pseudoconvex and* $\sum_{j=1}^m \mu_j^i g_j^i(u)$ *are quasiconvex for each* $i \in I$*, then* inf(DFP$_d$) $\geq$ sup(M–WDFD).

Proof. Assume that

$$\frac{f^i(x)}{h^i(x)} < \frac{f^i(u)}{h^i(u)} \quad \forall i \in I \tag{24}$$

and, by the pseudoconvexity of $(\frac{f^i(u)}{h^i(u)})$, (24) implies

$$(x-u)^t \nabla_u \left(\frac{f^i(u)}{h^i(u)}\right) < 0. \tag{25}$$

Hence

$$\sup_{i \in I}\left((x-u)^t \nabla_u \left(\frac{f^i(u)}{h^i(u)}\right)\right) < 0. \tag{26}$$

From Eq. (21) and inequality (26), it follows that

$$\sup_{i \in I}\left\{(x-u)^t \nabla_u \sum_{j=1}^m \mu_j^i g_j^i(u)\right\} > 0. \tag{27}$$

By (20), inequality (27) implies that

$$\sup_{i \in I} \sum_{j=1}^m \mu_j^i g_j^i(x) > \sup_{i \in I} \sum_{j=1}^m \mu_j^i g_j^i(u) \geq 0.$$

Then $\sum_{j=1}^m \mu_j^i g_j^i(x) > 0$, contradicting the assumption that x is feasible with respect to (DFP$_d$). □

Theorem 4.2 (*Strong Duality*). *If* x° *is an optimal solution of* (DFP$_d$) *and* CQ *is satisfied. Then there is a feasible* $(u^\circ, \mu^\circ) \in B$ *for* (M–WDFD) *and the corresponding value of* Inf(DFP$_d$) = sup(M–WDFD).

Proof. Since x° is an optimal solution of (DFP$_d$) and $g_j^i(x)$ satisfies the CQ. Then there are $\mu^\circ = \mu_j^i \geq 0, i \in I, j = 1, \ldots, m$ such that the Kuhn–Tucker conditions (20)–(23) are satisfied. It follows that (u°, μ°) is feasible for (M–WDFD). Hence

$$\inf_{i \in I} \frac{f^i(x^\circ)}{h^i(x^\circ)} = \sup_{i \in I} \frac{f^i(u^\circ)}{h^i(u^\circ)}. \quad \square$$

Theorem 4.3 (*Converse Duality*). *Let* x° *be an optimal solution of* (DFP$_d$) *and* CQ *is satisfied. If* (u^*, μ^*) *is an optimal solution of* (M–WDFD) *and* $(\frac{f^i(u)}{h^i(u)})$ *is strictly pseudoconvex at* u^**, then* $u^* = x^\circ$ *is an optimal solution of* (DFP$_d$).

Proof. Let x° be an optimal solution of (DFP$_d$), and assume CQ is satisfied. Assume that $x^\circ \neq u^*$. Then there is an optimal solution (u^*, μ^*) of (M–WDFD). Then

$$\inf_{i \in I}\left(\frac{f^i(x^\circ)}{h^i(x^\circ)}\right) = \sup_{i \in I}\left(\frac{f^i(u^*)}{h^i(u^*)}\right). \tag{28}$$

Because o(u°, μ°) is feasible with respect to (M–WDFD), it follows that:

$$\sum_{j=1}^m \mu_j^{*i} g_j^i(x^\circ) \leq \sum_{j=1}^m \mu_j^{*i} g_j^i(u^*).$$

The quasiconvexity of $\sum_{j=1}^{m} \mu_j^{*i} g_j^i(x)$ implies that

$$\sup_{i\in I}(x^\circ - u^*)\sum_{j=1}^{m} \nabla_u \mu_j^{*i} g_j^i(u^*) \leq 0. \tag{29}$$

From (28) and (29) it follows that

$$\sup_{i\in I}(x^\circ - u^*)\nabla_u \left(\frac{f^i(u^*)}{h^i(u^*)}\right) \geq 0. \tag{30}$$

From (30) and the strict pseudoconvexity of $(\frac{f^i(u)}{h^i(u)})$ at u^*, it follows that

$$\inf_{i\in I} \nabla_x \left(\frac{f^i(x^\circ)}{h^i(x^\circ)}\right) > \sup_{i\in I} \nabla_u \left(\frac{f^i(u^*)}{h^i(u^*)}\right).$$

This contradicts (28). Hence $x^\circ = u^*$ is an optimal solution of (DFP_d). □

Example 4.2. Consider the disjunctive fractional problem:

$$CP_2(\text{i}) \quad \min_i \min_{x\in Z} \left(\frac{x_1^2 - x_2}{x_1 + x_2}, \frac{x_1 + x_2^2}{x_1 - x_2}\right)$$

where Z is as in Example 3.1.

Then the optimal solution for $\text{DFP}_2(1)$ is $(x_1, x_2) = (0, 1)$, and the corresponding optimal value is $d^1 = -1$. Also the optimal solution for $\text{DFP}_2(2)$ is $(x_1, x_2) = (\frac{4}{3}, \frac{11}{3})$, and the corresponding optimal value is $d^2 = -\frac{133}{7}$. So

$$M = \min_i M_i = \min\left(-1, -\frac{133}{7}\right) = -\frac{133}{7}$$

is the minimum value of DPP_2. Since $Z_1 \neq \phi$, $Z_2 \neq \phi$. Then $I_P = \{1, 2\}$, $I(x) = \{(1, 2) \in I_P : x \in Z\}$ and (CQ) is valid. The set of solutions of DPP_2 is

$$P = \left\{x \in Z : \exists i = 2 \in I(x),\ \min_{x\in Z_2} \frac{x_1 + x_2^2}{x_1 - x_2} = -\frac{133}{7}\right\}.$$

5. Conclusion

This paper has addressed the solution of disjunctive programming problems, which corresponds to continuous optimization problems that involve disjunctions with convex–concave nonlinear fractional objective functions. We used Dinkelbach‘s global approach for finding the maximum of this problem. We first described the Kuhn–Tucker saddle point of disjunctive nonlinear fractional programming problems by using the decision set that is the union of a family of convex sets. Also, we discussed necessary and sufficient optimality conditions for disjunctive nonlinear fractional programming problems. For this class of problems, we studied the dual problem; we proposed and proved weak, strong and converse duality theorems.

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