Matłoka *Journal of Inequalities and Applications* 2013, **2013**:227
http://www.journalofinequalitiesandapplications.com/content/2013/1/227
Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On some Hadamard-type inequalities for (h_1, h_2)-preinvex functions on the co-ordinates

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Abstract
We introduce the class of (h_1, h_2)-preinvex functions on the co-ordinates, and we prove some new inequalities of Hermite-Hadamard and Fejér type for such mappings.
MSC: Primary 26A15; 26A51; secondary 52A30

Keywords: (h_1, h_2)-preinvex function on the co-ordinates; Hadamard inequalities; Hermite-Hadamard-Fejér inequalities

1 Introduction

A function $f : I \to R$, $I \subseteq R$ is an interval, is said to be a convex function on I if

$$f\big(tx + (1-t)y\big) \le tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the reversed inequality in (1.1) holds, then f is concave.

Many important inequalities have been established for the class of convex functions, but the most famous is the Hermite-Hadamard inequality. This double inequality is stated as follows:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : [a,b] \to R$ is a convex function. The above inequalities are in reversed order if f is a concave function.

In 1978, Breckner introduced an s-convex function as a generalization of a convex function [1].

Such a function is defined in the following way: a function $f : [0,\infty) \to R$ is said to be s-convex in the second sense if

$$f\big(tx + (1-t)y\big) \le t^s f(x) + (1-t)^s f(y) \tag{1.3}$$

holds for all $x, y \in \infty$, $t \in [0,1]$ and for fixed $s \in (0,1]$.

Of course, s-convexity means just convexity when $s = 1$.

In [2], Dragomir and Fitzpatrick proved the following variant of the Hermite-Hadamard inequality, which holds for s-convex functions in the second sense:

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.4}$$

In the paper [3] a large class of non-negative functions, the so-called h-convex functions, is considered. This class contains several well-known classes of functions such as non-negative convex functions and s-convex in the second sense functions. This class is defined in the following way: a non-negative function $f : I \to R$, $I \subseteq R$ is an interval, is called h-convex if

$$f\bigl(tx + (1-t)y\bigr) \leq h(t)f(x) + h(1-t)f(y) \tag{1.5}$$

holds for all $x, y \in I$, $t \in (0,1)$, where $h : J \to R$ is a non-negative function, $h \not\equiv 0$ and J is an interval, $(0,1) \subseteq J$.

In the further text, functions h and f are considered without assumption of non-negativity.

In [4] Sarikaya, Saglam and Yildirim proved that for an h-convex function the following variant of the Hadamard inequality is fulfilled:

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \bigl[f(a)+f(b)\bigr]\cdot\int_0^1 h(t)\,dt. \tag{1.6}$$

In [5] Bombardelli and Varošanec proved that for an h-convex function the following variant of the Hermite-Hadamard-Fejér inequality holds:

$$\begin{aligned}\frac{\int_a^b w(x)\,dx}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) &\leq \int_a^b f(x)w(x)\,dx \\ &\leq (b-a)\bigl(f(a)+f(b)\bigr)\int_0^1 h(t)w\bigl(ta+(1-t)b\bigr)\,dt,\end{aligned} \tag{1.7}$$

where $w : [a,b] \to R$, $w \geq 0$ and symmetric with respect to $\frac{a+b}{2}$.

A modification for convex functions, which is also known as co-ordinated convex functions, was introduced by Dragomir [6] as follows.

Let us consider a bidimensional $\Delta = [a,b] \times [c,d]$ in R^2 with $a < b$ and $c < d$. A mapping $f : \Delta \to R$ is said to be convex on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to R$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to R$, $f_x(v) = f(x,v)$ are convex for all $x \in [a,b]$ and $y \in [c,d]$.

In the same article, Dragomir established the following Hadamard-type inequalities for convex functions on the co-ordinates:

$$\begin{aligned}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy \\ &\leq \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}.\end{aligned} \tag{1.8}$$

The concept of s-convex functions on the co-ordinates was introduced by Alomari and Darus [7]. Such a function is defined in following way: the mapping $f : \Delta \to R$ is s-convex in the second sense if the partial mappings $f_y : [a,b] \to R$ and $f_x : [c,d] \to R$ are s-convex in the second sense.

In the same paper, they proved the following inequality for an s-convex function:

$$4^{s-1}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dx\,dy \\ \le \frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{(s+1)^2}. \tag{1.9}$$

For refinements and counterparts of convex and s-convex functions on the co-ordinates, see [6–10].

The main purpose of this paper is to introduce the class of (h_1,h_2)-preinvex functions on the co-ordinates and establish new inequalities like those given by Dragomir in [6] and Bombardelli and Varošanec in [5].

Throughout this paper, we assume that considered integrals exist.

2 Main results

Let $f : X \to R$ and $\eta : X \times X \to R^n$, where X is a nonempty closed set in R^n, be continuous functions. First, we recall the following well-known results and concepts; see [11–16] and the references therein.

Definition 2.1 Let $u \in X$. Then the set X is said to be invex at u with respect to η if

$$u + t\eta(v,u) \in X$$

for all $v \in X$ and $t \in [0,1]$.

X is said to be an invex set with respect to η if X is invex at each $u \in X$.

Definition 2.2 The function f on the invex set X is said to be preinvex with respect to η if

$$f\bigl(u + t\eta(v,u)\bigr) \le (1-t)f(u) + tf(v)$$

for all $u, v \in X$ and $t \in [0,1]$.

We also need the following assumption regarding the function η which is due to Mohan and Neogy [11].

Condition C Let $X \subseteq R$ be an open invex subset with respect to η. For any $x, y \in X$ and any $t \in [0,1]$,

$$\eta\bigl(y, y + t\eta(x,y)\bigr) = -t\eta(x,y),$$
$$\eta\bigl(x, y + t\eta(x,y)\bigr) = (1-t)\eta(x,y).$$

Note that for every $x, y \in X$ and every $t_1, t_2 \in [0,1]$ from Condition C, we have

$$\eta\bigl(y + t_2\eta(x,y), y + t_1\eta(x,y)\bigr) = (t_2 - t_1)\eta(x,y).$$

In [12], Noor proved the Hermite-Hadamard inequality for preinvex functions

$$f\left(a+\frac{1}{2}\eta(b,a)\right) \le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Definition 2.3 Let $h:[0,1]\to R$ be a non-negative function, $h\not\equiv 0$. The non-negative function f on the invex set X is said to be h-preinvex with respect to η if

$$f\bigl(u+t\eta(v,u)\bigr) \le h(1-t)f(u)+h(t)f(v)$$

for each $u,v\in X$ and $t\in[0,1]$.

Let us note that:

- if $\eta(v,u)=v-u$, then we get the definition of an h-convex function introduced by Varošanec in [3];
- if $h(t)=t$, then our definition reduces to the definition of a preinvex function;
- if $\eta(v,u)=v-u$ and $h(t)=t$, then we obtain the definition of a convex function.

Now let X_1 and X_2 be nonempty subsets of R^n, let $\eta_1 : X_1\times X_1\to R^n$ and $\eta_2 : X_2\times X_2\to R^n$.

Definition 2.4 Let $(u,v)\in X_1\times X_2$. We say $X_1\times X_2$ is invex at (u,v) with respect to η_1 and η_2 if for each $(x,y)\in X_1\times X_2$ and $t_1,t_2\in[0,1]$,

$$\bigl(u+t_1\eta_1(x,u), v+t_2\eta_2(y,v)\bigr)\in X_1\times X_2.$$

$X_1\times X_2$ is said to be an invex set with respect to η_1 and η_2 if $X_1\times X_2$ is invex at each $(u,v)\in X_1\times X_2$.

Definition 2.5 Let h_1 and h_2 be non-negative functions on $[0,1]$, $h_1\not\equiv 0$, $h_2\not\equiv 0$. The non-negative function f on the invex set $X_1\times X_2$ is said to be co-ordinated (h_1,h_2)-preinvex with respect to η_1 and η_2 if the partial mappings $f_y : X_1\to R$, $f_y(x)=f(x,y)$ and $f_x : X_2\to R$, $f_x(y)=f(x,y)$ are h_1-preinvex with respect to η_1 and h_2-preinvex with respect to η_2, respectively, for all $y\in X_2$ and $x\in X_1$.

If $\eta_1(x,u)=x-u$ and $\eta_2(y,v)=y-v$, then the function f is called (h_1,h_2)-convex on the co-ordinates.

Remark 1 From the above definition it follows that if f is a co-ordinated (h_1,h_2)-preinvex function, then

$$\begin{aligned}
&f\bigl(x+t_1\eta_1(b,x), y+t_2\eta_2(d,y)\bigr)\\
&\quad\le h_1(1-t_1)f\bigl(x,y+t_2\eta_2(d,y)\bigr)+h_1(t_1)f\bigl(b,y+t_2\eta_2(d,y)\bigr)\\
&\quad\le h_1(1-t_1)h_2(1-t_2)f(x,y)+h_1(1-t_1)h_2(t_2)f(x,d)\\
&\qquad+h_1(t_1)h_2(1-t_2)f(b,y)+h_1(t_1)h_2(t_2)f(b,d).
\end{aligned}$$

Remark 2 Let us note that if $\eta_1(x,u)=x-u$, $\eta_2(y,v)=y-v$, $t_1=t_2$ and $h_1(t)=h_2(t)=t$, then our definition of a co-ordinated (h_1,h_2)-preinvex function reduces to the definition of a convex function on the co-ordinates proposed by Dragomir [6]. Moreover, if $h_1(t) = h_2(t) = t^s$, then our definition reduces to the definition of an s-convex function on the co-ordinates proposed by Alomari and Darus [7].

Now, we will prove the Hadamard inequality for the new class functions.

Theorem 2.1 *Suppose that* $f : [a, a+\eta(b,a)] \to R$ *is an h-preinvex function, Condition* C *for* η *holds and* $a < a+\eta(b,a)$, $h(\frac{1}{2}) > 0$. *Then the following inequalities hold*:

$$\frac{1}{2h(\frac{1}{2})} f\left(a + \frac{1}{2}\eta(b,a)\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \le \bigl[f(a)+f(b)\bigr] \cdot \int_0^1 h(t)\,dt. \tag{2.2}$$

Proof From the definition of an h-preinvex function, we have that

$$f\bigl(a + t\eta(b,a)\bigr) \le h(1-t)f(a) + h(t)f(b).$$

Thus, by integrating, we obtain

$$\int_0^1 f\bigl(a+t\eta(b,a)\bigr)\,dt \le f(a)\int_0^1 h(1-t)\,dt + f(b)\int_0^1 h(t)\,dt = \bigl[f(a)+f(b)\bigr]\int_0^1 h(t)\,dt.$$

But

$$\int_0^1 f\bigl(a+t\eta(b,a)\bigr)\,dt = \frac{1}{\eta(b,a)} \cdot \int_a^{a+\eta(b,a)} f(x)\,dx.$$

So,

$$\frac{1}{\eta(b,a)} \cdot \int_a^{a+\eta(b,a)} f(x)\,dx \le \bigl[f(a)+f(b)\bigr]\int_0^1 h(t)\,dt.$$

The proof of the second inequality follows by using the definition of an h-preinvex function, Condition C for η and integrating over $[0,1]$.

That is,

$$\begin{aligned} f\left(a+\frac{1}{2}\eta(b,a)\right) &= f(a+t\eta(b,a) + \frac{1}{2}\eta\bigl(a+(1-t)\eta(b,a), a+t\eta(b,a)\bigr) \\ &\le h\left(\frac{1}{2}\right)\bigl[f\bigl(a+t\eta(b,a)\bigr) + f\bigl(a+(1-t)\eta(b,a)\bigr)\bigr], \\ f\left(a+\frac{1}{2}\eta(b,a)\right) &\le h\left(\frac{1}{2}\right)\left[\int_0^1 f\bigl(a+t\eta(b,a)\bigr)\,dt + \int_0^1 f\bigl(a+(1-t)\eta(b,a)\bigr)\right], \\ f\left(a+\frac{1}{2}\eta(b,a)\right) &\le 2\cdot h\left(\frac{1}{2}\right)\frac{1}{\eta(b,a)} \cdot \int_a^{a+\eta(b,a)} f(x)\,dx. \end{aligned}$$

The proof is complete. □

Theorem 2.2 *Suppose that* $f : [a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)] \to R$ *is an* (h_1,h_2)*-preinvex function on the co-ordinates with respect to* η_1 *and* η_2, *Condition* C *for* η_1 *and* η_2 *is fulfilled*, *and* $a < a + \eta_1(b,a)$, $c < c + \eta_2(d,c)$, *and* $h_1(\frac{1}{2}) > 0$, $h_2(\frac{1}{2}) > 0$. *Then one has the following inequalities*:

$$\begin{aligned}
&\frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(a + \frac{1}{2}\eta_1(b,a), c + \frac{1}{2}\eta_2(d,c)\right) \\
&\quad \le \frac{1}{4 \cdot h_1(\frac{1}{2})\eta_2(d,c)} \int_c^{c+\eta_2(d,c)} f\left(a + \frac{1}{2}\eta_1(b,a), y\right) dy \\
&\qquad + \frac{1}{4 \cdot h_2(\frac{1}{2})\eta_1(b,a)} \int_a^{c+\eta_1(b,a)} f\left(x, c + \frac{1}{2}\eta_2(d,c)\right) dx \\
&\quad \le \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dx\,dy \\
&\quad \le \frac{1}{2\eta_1(b,a)} \int_0^1 h_2(t_2)\,dt_2 \left[\int_a^{a+\eta_1(b,a)} f(x,c)\,dx + \int_a^{a+\eta_1(b,a)} f(x,d)\,dx\right] \\
&\qquad + \frac{1}{2\eta_2(d,c)} \int_0^1 h_1(t_1)\,dt_1 \left[\int_c^{c+\eta_2(d,c)} f(a,y)\,dy + \int_c^{c+\eta_2(d,c)} f(b,y)\,dy\right] \\
&\quad \le \left[f(a,c) + f(b,c) + f(a,d) + f(b,d)\right] \int_0^1 h_1(t_1)\,dt_1 \cdot \int_0^1 h_2(t_2)\,dt_2. \qquad (2.3)
\end{aligned}$$

Proof Since f is (h_1, h_2)-preinvex on the co-ordinates, it follows that the mapping f_x is h_2-preinvex and the mapping f_y is h_1-preinvex. Then, by the inequality (2.2), one has

$$\begin{aligned}
\frac{1}{2h_2(\frac{1}{2})} f\left(x, c + \frac{1}{2}\eta_2(d,c)\right) &\le \frac{1}{\eta_2(d,c)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dy \\
&\le \left[f(x,c) + f(x,d)\right] \int_0^1 h_2(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2h_1(\frac{1}{2})} f\left(a + \frac{1}{2}\eta_1(b,a), y\right) &\le \frac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(x,y)\,dx \\
&\le \left[f(a,y) + f(b,y)\right] \int_0^1 h_1(t)\,dt.
\end{aligned}$$

Dividing the above inequalities for $\eta_1(b,a)$ and $\eta_2(d,c)$ and then integrating the resulting inequalities on $[a, a + \eta_1(b,a)]$ and $[c, c + \eta_2(d,c)]$, respectively, we have

$$\begin{aligned}
&\frac{1}{\eta_1(b,a) \cdot 2h_2(\frac{1}{2})} \int_a^{a+\eta_1(b,a)} f\left(x, c + \frac{1}{2}\eta_2(d,c)\right) dx \\
&\quad \le \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dx\,dy \\
&\quad \le \frac{1}{\eta_1(b,a)} \int_0^1 h_2(t)\,dt \left[\int_a^{a+\eta_1(b,a)} f(x,c)\,dx + \int_a^{a+\eta_1(b,a)} f(x,d)\,dx\right]
\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\eta_2(b,a)\cdot 2h_1(\frac{1}{2})}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a),y\right)dy\\&\quad\le \frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dx\,dy\\&\quad\le \frac{1}{\eta_2(d,c)}\int_0^1 h_1(t)\,dt\left[\int_c^{c+\eta_2(c,d)} f(a,y)\,dy+\int_c^{c+\eta_2(c,d)} f(b,y)\,dy\right].\end{aligned}$$

Summing the above inequalities, we get the second and the third inequalities in (2.3).

By the inequality (2.2), we also have

$$\frac{1}{2h_2(\frac{1}{2})}f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)\le \frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f\left(a+\frac{1}{2}\eta_1(b,a),y\right)dy$$

and

$$\frac{1}{2h_1(\frac{1}{2})}f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)\le \frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f\left(x,c+\frac{1}{2}\eta_2(d,c)\right)dx,$$

which give, by addition, the first inequality in (2.3).

Finally, by the same inequality (2.2), we ca also state

$$\begin{aligned}&\frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f(a,y)\,dy\le\left[f(a,c)+f(a,d)\right]\int_0^1 h_2(t)\,dt,\\&\frac{1}{\eta_2(d,c)}\int_c^{c+\eta_2(d,c)} f(b,y)\,dy\le\left[f(b,c)+f(b,d)\right]\int_0^1 h_2(t)\,dt,\\&\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x,c)\,dx\le\left[f(a,c)+f(b,c)\right]\int_0^1 h_1(t)\,dt,\\&\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(x,d)\,dx\le\left[f(a,d)+f(b,d)\right]\int_0^1 h_1(t)\,dt,\end{aligned}$$

which give, by addition, the last inequality in (2.3). □

Remark 3 In particular, for $\eta_1(b,a) = b - a$, $\eta_2(d,c) = d - c$, $h_1(t_1) = h_2(t_2) = t$, we get the inequalities obtained by Dragomir [6] for functions convex on the co-ordinates on the rectangle from the plane R^2.

Remark 4 If $\eta_1(b,a) = b - a$, $\eta_2(d,c) = d - c$, and $h_1(t_1) = h_2(t_2) = t^s$, then we get the inequalities obtained by Alomari and Darus in [7] for s-convex functions on the co-ordinates on the rectangle from the plane R^2.

Theorem 2.3 *Let $f,g : [a, a + \eta_1(b,a)] \times [c, c + \eta_2(d,c)] \to R$ with $a < a + \eta_1(b,a)$, $c < c + \eta_2(d,c)$. If f is (h_1,h_2)-preinvex on the co-ordinates and g is (k_1,k_2)-preinvex on the co-*

ordinates with respect to η_1 *and* η_2, *then*

$$\begin{aligned}
&\frac{1}{\eta_1(b,a)\cdot\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)g(x,y)\,dx\,dy \\
&\quad\le M_1(a,b,c,d)\int_0^1\int_0^1 h_1(t_1)h_2(t_2)k_1(t_1)k_2(t_2)\,dt_1\,dt_2 \\
&\qquad + M_2(a,b,c,d)\int_0^1\int_0^1 h_1(t_1)h_2(t_2)k_1(t_1)k_2(1-t_2)\,dt_1\,dt_2 \\
&\qquad + M_3(a,b,c,d)\int_0^1\int_0^1 h_1(t_1)h_2(t_2)k_1(1-t_1)k_2(t_2)\,dt_1\,dt_2 \\
&\qquad + M_4(a,b,c,d)\int_0^1\int_0^1 h_1(t_1)h_2(t_2)k_1(1-t_1)k_2(1-t_2)\,dt_1\,dt_2,
\end{aligned}$$

where

$$\begin{aligned}
M_1(a,b,c,d) &= f(a,c)g(a,c)+f(a,d)g(a,d)+f(b,c)g(b,c)+f(b,d)g(b,d), \\
M_2(a,b,c,d) &= f(a,c)g(a,d)+f(a,d)g(a,c)+f(b,c)g(b,d)+f(b,d)g(b,c), \\
M_3(a,b,c,d) &= f(a,c)g(b,c)+f(a,d)g(b,d)+f(b,c)g(a,c)+f(b,d)g(a,d), \\
M_4(a,b,c,d) &= f(a,c)g(b,d)+f(a,d)g(b,c)+f(b,c)g(a,d)+f(b,d)g(a,c).
\end{aligned}$$

Proof Since f is (h_1,h_2)-preinvex on the co-ordinates and g is (k_1,k_2)-preinvex on the co-ordinates with respect to η_1 and η_2, it follows that

$$\begin{aligned}
&f\bigl(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\bigr) \\
&\quad\le h_1(1-t_1)h_2(1-t_2)f(a,c)+h_1(1-t_1)h_2(t_2)f(a,d) \\
&\qquad + h_1(t_1)h_2(1-t_2)f(b,c)+h_1(t_1)h_2(t_2)f(b,d)
\end{aligned}$$

and

$$\begin{aligned}
&g\bigl(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\bigr) \\
&\quad\le k_1(1-t_1)k_2(1-t_2)g(a,c)+k_1(1-t_1)k_2(t_2)g(a,d) \\
&\qquad + k_1(t_1)k_2(1-t_2)g(b,c)+k_1(t_1)k_2(t_2)g(b,d).
\end{aligned}$$

Multiplying the above inequalities and integrating over $[0,1]^2$ and using the fact that

$$\begin{aligned}
&\int_0^1\int_0^1 f\bigl(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\bigr)\cdot g\bigl(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\bigr)\,dt_1\,dt_2 \\
&\quad=\frac{1}{\eta_1(b,a)\cdot\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)g(x,y)\,dx\,dy,
\end{aligned}$$

we obtain our inequality. □

In the next two theorems, we will prove the so-called Hermite-Hadamard-Fejér inequalities for an (h_1,h_2)-preinvex function.

Theorem 2.4 *Let $f : [a, a + \eta_1(b,a)] \times [c, c + \eta_2(d,c)] \to R$ be (h_1, h_2)-preinvex on the co-ordinates with respect to η_1 and η_2, $a < a + \eta_1(b,a)$, $c < c + \eta_2(d,c)$, and $w : [a, a + \eta_1(b,a)] \times [c, c + \eta_2(d,c)] \to R$, $w \geq 0$, symmetric with respect to*

$$\left(a + \frac{1}{2}\eta_1(b,a), c + \frac{1}{2}\eta_2(d,c)\right).$$

Then

$$\begin{aligned}
&\frac{1}{\eta_1(b,a) \cdot \eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y) w(x,y)\, dx\, dy \\
&\quad \leq \left[f(a,c) + f(a,d) + f(b,c) + f(b,d)\right] \\
&\qquad \cdot \int_0^1 \int_0^1 h_1(t_1) h_2(t_2) w\left(a + t_1\eta_1(b,a), c + t_2\eta_2(d,c)\right) dt_1\, dt_2.
\end{aligned} \tag{2.4}$$

Proof From the definition of (h_1, h_2)-preinvex on the co-ordinates with respect to η_1 and η_2, we have

(a)

$$\begin{aligned}
&f\left(a + t_1\eta_1(b,a), c + t_2\eta_2(d,c)\right) \\
&\quad \leq h_1(1-t_1)h_2(1-t_2)f(a,c) + h_1(1-t_1)h_2(t_2)f(a,d) \\
&\qquad + h_1(t_1)h_2(1-t_2)f(b,c) + h_1(t_1)h_2(t_2)f(b,d),
\end{aligned}$$

(b)

$$\begin{aligned}
&f\left(a + (1-t_1)\eta_1(b,a), c + (1-t_2)\eta_2(d,c)\right) \\
&\quad \leq h_1(t_1)h_2(t_2)f(a,c) + h_1(t_1)h_2(1-t_2)f(a,d) \\
&\qquad + h_1(1-t_1)h_2(t_2)f(b,c) + h_1(1-t_1)h_2(1-t_2)f(b,d),
\end{aligned}$$

(c)

$$\begin{aligned}
&f\left(a + t_1\eta_1(b,a), c + (1-t_2)\eta_2(d,c)\right) \\
&\quad \leq h_1(1-t_1)h_2(t_2)f(a,c) + h_1(1-t_1)h_2(1-t_2)f(a,d) \\
&\qquad + h_1(t_1)h_2(t_2)f(b,c) + h_1(t_1)h_2(1-t_2)f(b,d),
\end{aligned}$$

(d)

$$\begin{aligned}
&f\left(a + (1-t_1)\eta_1(b,a), c + t_2\eta_2(d,c)\right) \\
&\quad \leq h_1(t_1)h_2(1-t_2)f(a,c) + h_1(t_1)h_2(t_2)f(a,d) \\
&\qquad + h_1(1-t_1)h_2(1-t_2)f(b,c) + h_1(1-t_1)h_2(t_2)f(b,d).
\end{aligned}$$

Multiplying both sides of the above inequalities by $w(a + t_1\eta_1(b,a), c + t_2\eta_2(d,c))$, $w(a + (1-t_1)\eta_1(b,a), c + (1-t_2)\eta_2(d,c))$, $w(a + t_1\eta_1(b,a), c + (1-t_2)\eta_2(d,c))$, $w(a + (1-t_1)\eta_1(b,a), c +$

$t_2\eta_2(d,c))$, respectively, adding and integrating over $[0,1]^2$, we obtain

$$\begin{aligned}
&\frac{4}{\eta_1(b,a)\cdot\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)w(x,y)\,dx\,dy\\
&\quad\le\big[f(a,c)+f(a,d)+f(b,c)+f(b,d)\big]\\
&\qquad\cdot 4\int_0^1\int_0^1 h_1(t_1)h_2(t_2)w\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)\,dt_1\,dt_2,
\end{aligned}$$

where we use the symmetricity of the w with respect to $(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c))$, which completes the proof. □

Theorem 2.5 *Let $f:[a,a+\eta_1(b,a)]\times[c,c+\eta_2(d,c)]\to R$ be (h_1,h_2)-preinvex on the co-ordinates with respect to η_1 and η_2, and $a<a+\eta_1(b,a)$, $c<c+\eta_2(d,c)$, $w:[a,a+\eta_1(b,a)]\times[c,c+\eta_2(d,c)]\to R$, $w\ge 0$, symmetric with respect to $(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c))$. Then, if Condition* C *for η_1 and η_2 is fulfilled, we have*

$$\begin{aligned}
&f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)\cdot\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} w(x,y)\,dx\,dy\\
&\quad\le 4\cdot h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\cdot\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)w(x,y)\,dx\,dy.
\end{aligned}\tag{2.5}$$

Proof Using the definition of an (h_1,h_2)-preinvex function on the co-ordinates and Condition C for η_1 and η_2, we obtain

$$\begin{aligned}
&f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)\\
&\quad\le h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\cdot\big[f\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)\\
&\qquad+f\big(a+t_1\eta_1(b,a),c+(1-t_2)\eta_2(d,c)\big)+f\big(a+(1-t_1)\eta_1(b,a),c+t_2\eta_2(d,c)\big)\\
&\qquad+f\big(a+(1-t_1)\eta_1(b,a),c+(1-t_2)\eta_2(d,c)\big)\big].
\end{aligned}$$

Now, we multiply it by $w(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c))=w(a+t_1\eta_1(b,c),c+(1-t_2)\eta_2(d,c))=w(a+(1-t_1)\eta_1(b,a),c+t_2\eta_2(d,c))=w(a+(1-t_1)\eta_1(b,a),c+(1-t_2)\eta_2(d,c))$ and integrate over $[0,1]^2$ to obtain the inequality

$$\begin{aligned}
&f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)\int_0^1\int_0^1 w\big(a+t_1\eta_1(b,a),c+t_2\eta_2(d,c)\big)\,dt_1\,dt_2\\
&\quad=f\left(a+\frac{1}{2}\eta_1(b,a),c+\frac{1}{2}\eta_2(d,c)\right)\frac{1}{\eta_1(b,a)\cdot\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} w(x,y)\,dx\,dy\\
&\quad\le 4\cdot h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\frac{1}{\eta_1(b,a)\cdot\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)w(x,y)\,dx\,dy,
\end{aligned}$$

which completes the proof. □

Now, for a mapping $f:[a,b]\times[c,d]\to R$, let us define a mapping $H:[0,1]^2\to R$ in the following way:

$$H(t,r)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2},ry+(1-r)\frac{c+d}{2}\right)dx\,dy. \tag{2.6}$$

Some properties of this mapping for a convex on the co-ordinates function and an s-convex on the co-ordinates function are given in [6, 7], respectively. Here we investigate which of these properties can be generalized for (h_1,h_2)-convex on the co-ordinates functions.

Theorem 2.6 *Suppose that* $f:[a,b]\times[c,d]$ *is* (h_1,h_2)*-convex on the co-ordinates. Then*:

(i) *The mapping* H *is* (h_1,h_2)*-convex on the co-ordinates on* $[0,1]^2$,

(ii) $4h_1(\frac{1}{2})h_2(\frac{1}{2})H(t,r)\geq H(0,0)$ *for any* $(t,r)\in[0,1]^2$.

Proof (i) The (h_1,h_2)-convexity on the co-ordinates of the mapping H is a consequence of the (h_1,h_2)-convexity on the co-ordinates of the function f. Namely, for $r\in[0,1]$ and for all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$, we have:

$$\begin{aligned}
&H(\alpha t_1+\beta t_2,r)\\
&\quad=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\biggl((\alpha t_1+\beta t_2,r)x+\bigl(1-(\alpha t_1+\beta t_2)\bigr)\frac{a+b}{2},\\
&\qquad ry+(1-r)\frac{c+d}{2}\biggr)dx\,dy\\
&\quad=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\biggl(\alpha\left(t_1x+(1-t_1)\frac{a+b}{2}\right)+\beta\left(t_2x+(1-t_2)\frac{a+b}{2}\right),\\
&\qquad ry+(1-r)\frac{c+d}{2}\biggr)dx\,dy\\
&\quad\leq h_1(\alpha)\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(t_1x+(1-t_1)\frac{a+b}{2},ry+(1-r)\frac{c+d}{2}\right)dx\,dy\\
&\qquad+h_1(\beta)\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(t_2x+(1-t_2)\frac{a+b}{2},ry+(1-r)\frac{c+d}{2}\right)dx\,dy\\
&\quad=h_1(\alpha)H(t_1,r)+h_1(\beta)H(t_2,r).
\end{aligned}$$

Similarly, if $t\in[0,1]$ is fixed, then for all $r_1,r_2\in[0,1]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$, we also have

$$H(t,\alpha r_1+\beta r_2)\leq h_2(\alpha)H(t,r_1)+h_2(\beta)H(t,r_2),$$

which means that H is (h_1,h_2)-convex on the co-ordinates.

(ii) After changing the variables $u=tx+(1-t)\frac{a+b}{2}$ and $v=ry+(1-r)\frac{c+d}{2}$, we have

$$\begin{aligned}
H(t,r)&=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2},ry+(1-r)\frac{c+d}{2}\right)dx\,dy\\
&=\frac{1}{(b-a)(d-c)}\int_{u_L}^{u_U}\int_{v_L}^{v_U} f(u,v)\frac{b-a}{u_U-u_L}\cdot\frac{d-c}{v_U-v_L}\,du\,dv
\end{aligned}$$

$$= \frac{1}{(u_U - u_L)(v_U - v_L)} \int_{u_L}^{u_U} \int_{v_L}^{v_U} f(u,v)\,du\,dv$$
$$\geq \frac{1}{4h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right),$$

where $u_L = ta + (1-t)\frac{a+b}{2}$, $u_U = tb + (1-t)\frac{a+b}{2}$, $v_L = rc + (1-r)\frac{c+d}{2}$ and $v_U = rd + (1-r)\frac{c+d}{2}$, which completes the proof. □

Remark 5 If f is convex on the co-ordinates, then we get $H(t,r) \geq H(0,0)$. If f is s-convex on the co-ordinates in the second sense, then we have the inequality $H(t,r) \geq 4^{s-1}H(0,0)$.

Competing interests
The author declares that he has no competing interests.

Received: 5 December 2012 Accepted: 18 April 2013 Published: 7 May 2013

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doi:10.1186/1029-242X-2013-227
Cite this article as: Matłoka: On some Hadamard-type inequalities for (h_1, h_2)-preinvex functions on the co-ordinates. *Journal of Inequalities and Applications* 2013 **2013**:227.