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Some numerical algorithms to evaluate Hadamard finite-part integrals

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Abstract

Some algorithms are described for the numerical evaluation of Hadamard finite part integrals of type $\fint_{-1}^{1}[f(x)/(x-t)^2]v^{\alpha,\beta}\,dx$, $|t|<1$, where $v^{\alpha,\beta}$ is a Jacobi weight. Convergence results and some numerical examples are given.

Keywords: Hilbert transform; Hadamard finite-part integral; Orthogonal polynomials; Spline interpolation

1. Introduction

Several boundary problems of applied mathematics are formulated as singular integral equations involving integrals, called hypersingular, since their kernels have a singularity of order greater than the dimension of the integrals.

Let $f^{(p)}\in \mathrm{Lip}\,\lambda$, $0<\lambda\leqslant 1$, $|t|<1$, $0\leqslant p\in\mathbb{N}$, i.e. $|f^{(p)}(x_1)-f^{(p)}(x_2)|\leqslant \mathscr{C}|x_1-x_2|^{\lambda}$, x_1, $x_2\in[-1,1]$, $\mathscr{C}>0$. The Hadamard finite part integral is defined by

$$H_p(f;t)=\fint_{-1}^{1}\frac{f(x)}{(x-t)^{p+1}}\,dx$$

$$=\int_{-1}^{1}\frac{f(x)-\sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}}\,dx+\sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}\fint_{-1}^{1}\frac{dx}{(x-t)^{p+1-k}},$$

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where the first integral of the right-hand side is a generalized Riemann integral and

$$\int_{-1}^{1} \frac{\mathrm{d}x}{(x-t)^{p+1-k}} = \frac{1}{(p-k)!}\frac{\mathrm{d}^{p-k}}{\mathrm{d}t^{p-k}} \int_{-1}^{1} \frac{\mathrm{d}x}{x-t}.$$

Then we can set

$$H_p(f, t) = \int_{-1}^{1} \frac{f(x)}{(x-t)^{p+1}}\,\mathrm{d}x = \frac{1}{p!}\frac{\mathrm{d}^p}{\mathrm{d}t^p} \int_{-1}^{1} \frac{f(x)}{x-t}\,\mathrm{d}x.$$

The analytical properties of the Hadamard finite part integrals and their occurrences can be found in [9, 16].

In the following we set

$$H_0(f; t) = H(f; t) = \lim_{\varepsilon \to 0} \int_{|x-t| \geqslant \varepsilon} \frac{f(x)}{x-t}\,\mathrm{d}x.$$

We want to approximate the weighted Hadamard integral $H_1(fv^{\alpha,\beta})$, that is

$$\begin{aligned} H_1(fv^{\alpha,\beta}) &= \int_{-1}^{1} \frac{f(x)}{(x-t)^2}\, v^{\alpha,\beta}(x)\,\mathrm{d}x \\ &= \int_{-1}^{1} \frac{f(x)-f(t)-f'(t)(x-t)}{(x-t)^2}\, v^{\alpha,\beta}(x)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left(f(t) \int_{-1}^{1} \frac{v^{\alpha,\beta}(x)}{x-t}\,\mathrm{d}x \right), \end{aligned} \tag{1}$$

where $v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta > -1$, is a Jacobi weight. Setting

$$F(f; t) = \int_{-1}^{1} \frac{f(x)-f(t)}{x-t}\, v^{\alpha,\beta}(x)\,\mathrm{d}x, \tag{2}$$

$H_1(fv^{\alpha,\beta}, t)$ can be rewritten in this way:

$$H_1(fv^{\alpha,\beta}, t) = F(f; t)' + \frac{\mathrm{d}}{\mathrm{d}t}\left(f(t) \int_{-1}^{1} \frac{v^{\alpha,\beta}(x)}{x-t}\,\mathrm{d}x \right). \tag{3}$$

We notice that if $\alpha, \beta > 0$ and we assume $f' \in TD$, where

$$TD := \left\{ f \in C([-1,1]) \,|\, \int_0^1 u^{-1}\omega(f, u)\,\mathrm{d}u < \infty \right\},$$

and

$$\omega(f, \delta) = \sup_{|x_1 - x_2| \leqslant \delta} |f(x_1) - f(x_2)|, \quad x_1, x_2 \in [-1, 1], \ \delta \geqslant 0$$

denotes the modulus of continuity of the function f, then the range of $F(f)'$ is $[-1,1]$. The range of $F(f)'$ is also $[-1,1]$, if $-1 < \alpha, \beta \leqslant 0$, and we assume $f' \in \mathrm{Lip}\,\lambda$, $\max\{-\alpha, -\beta\} < \lambda \leqslant 1$.

The numerical methods usually proposed in literature to evaluate $H_1(fv^{\alpha,\beta}, t)$ are based on the polynomial approximation (global or local) of the functions f and f'. Instead in the numerical methods proposed in this paper we approximate the function $F'(f)$ by polynomials and/or splines.

Since the analytical expression of $\fint_{-1}^{1} v^{\alpha,\beta}(x)/(x-t)\,\mathrm{d}x$ is known [11] and some numerical methods to approximate the second term of the right-hand side of (3) can be constructed, we are interested in approximating $F(f)'$. To this end we construct some polynomial approximations (local or global) of the function $F(f)$ first, then we use the derivative of the above polynomials to approximate $F(f)'$. Computing an approximation of $F(f)'$ in this way, the evaluation of $f'(t)$ is not required.

The paper is divided in six sections. In Sections 2, 3 and 4 we state the above mentioned methods and give some error estimates, while Section 5 contains their proofs. Finally, in Section 6 we give some numerical examples. For each of one we compare the errors due to our algorithms from among them, and with other methods.

2. First algorithm

In the following we denote by $\{p_m(v^{\alpha,\beta})\}_{m=0}^{\infty}$ the sequence of the orthonormal Jacobi polynomials with positive leading coefficient, i.e.

$$\int_{-1}^{1} p_m(v^{\alpha,\beta},x)p_n(v^{\alpha,\beta},x)v^{\alpha,\beta},x)\,\mathrm{d}x = \delta_{m,n},$$

$$p_m(v^{\alpha,\beta},x) = \gamma_m(v^{\alpha,\beta})x^m + \text{terms of lower degree}, \quad \gamma_m(v^{\alpha,\beta}) > 0. \tag{4}$$

In [18] Paget derives a formula to approximate $F(f;t)'$, for $t \in (-1,1)$, based on the ordinary Gaussian rule, i.e.

$$F(f;t)' = \sum_{k=1}^{N} \lambda_{N,k}^{\alpha,\beta} \frac{f(x_{N,k}^{\alpha,\beta}) - f(t) - f'(t)(x_{N,k}^{\alpha,\beta} - t)}{(x_{N,k}^{\alpha,\beta} - t)^2} + e_N(f;t)$$

$$=: F_N'(f;t) + e_N(f;t),$$

where $\{x_{N,k}^{\alpha,\beta}\}_{k=1}^{N}$ are the zeros of the Jacobi polynomial $p_N(v^{\alpha,\beta})$ and $\{\lambda_{N,k}^{\alpha,\beta}\}_{k=1}^{N}$ are the Christoffel numbers defined as

$$\lambda_{N,k}^{\alpha,\beta} = \left[\sum_{j=0}^{N-1} p_j^2(v^{\alpha,\beta}, x_{N,k}^{\alpha,\beta})\right]^{-1}, \quad k = 1, \ldots, N.$$

In general, the sequence $\{F_N(f)'\}_{N\in\mathbb{N}}$ does not converge to $F(f)'$ whenever f' is only a Hölder continuous function (see [4]). Moreover, for a fixed $N \in \mathbb{N}$, severe numerical cancellation happens in $F_N(f;t)'$, whenever t is very close to one of the quadrature nodes $x_{N,k}^{\alpha,\beta}$. To overcome these problems we propose an algorithm that makes use of some ideas contained in [2]. To be more precise in the aforesaid paper the authors are interested in approximating Cauchy principal value integrals $H(fv^{\alpha,\beta})$ by

$$\sum_{k=1}^{N} \lambda_{N,k}^{\alpha,\beta} \frac{f(x_{N,k}^{\alpha,\beta}) - f(t)}{(x_{N,k}^{\alpha,\beta} - t)} + f(t) \fint_{-1}^{1} \frac{v^{\alpha,\beta}(x)}{x-t}\,\mathrm{d}x.$$

They prove that there exists a subsequence $\{F_{N_\nu}(f;t)\}_{\nu\in\mathbb{N}}$ uniformly convergent to $F(f;t)$ in any closed subset $[a,b]\subset(-1,1)$, for $f\in TD$. For this procedure the problem of numerical cancellation is avoided, nevertheless they do not give a method to construct the numerical algorithm to approximate $H(v^{\alpha,\beta}f)$. In order, to introduce the new algorithm, we need the following

Lemma 2.1. *Let $x^{\alpha,\beta}_{m,k}$, $k=1,\dots,m$ and $x^{\alpha,\beta}_{m+1,j}$, $j=1,\dots,m+1$ be the zeros of $p_m(v^{\alpha,\beta})$ and $p_{m+1}(v^{\alpha,\beta})$ respectively. Assume $x^{\alpha,\beta}_{m,k}$, $x^{\alpha,\beta}_{m+1,j}\in[a,b]\subset(-1,1)$. Then*

$$\min_{j,k}|x^{\alpha,\beta}_{m,k}-x^{\alpha,\beta}_{m+1,j}|>\frac{\mathscr{C}}{m},$$

where $\mathscr{C}$ is a positive constant independent of m, but c depends on a and b.

We remark that the previous lemma is substantially equivalent to the Lemma 3.1 in [2], but in Section 5 we give a different proof.

Now we are able to give the algorithm.

Let $t\in[a,b]\subset(-1,1)$ be fixed. Because of the density of the zeros of orthogonal polynomials, there exists N large enough, such that a finite number of zeros of $p_N(v^{\alpha,\beta})p_{N+1}(v^{\alpha,\beta})$ belongs to $[a,b]$. Since $t\in[a,b]$, two cases are possible: either $t\in[x^{\alpha,\beta}_{N,d},x^{\alpha,\beta}_{N+1,d+1}]$ or $t\in[x^{\alpha,\beta}_{N+1,d},x^{\alpha,\beta}_{N,d}]$ for some $d\in\{1,\dots,N\}$.

In the first case, if $x^{\alpha,\beta}_{N,d}-t>x^{\alpha,\beta}_{N+1,d+1}-t$, we choose the quadrature rule

$$F(f;t)'\approx F_N(f;t)'=\sum_{k=1}^{N}\lambda^{\alpha,\beta}_{N,k}\frac{f(x^{\alpha,\beta}_{N,k})-f(t)-f'(t)(x^{\alpha,\beta}_{N,k}-t)}{(x^{\alpha,\beta}_{N,k}-t)^2};$$

otherwise we choose the quadrature rule $F(f;t)'\approx F_{N+1}(f;t)'$.

Notice that, from Lemma 2.1, the denominators of the chosen quadrature rule, are greater than $\mathscr{C}/N^2$ and, define the "*amplification factor*" by

$$K_N(t)=\sum_{k=1}^{N}\frac{\lambda^{\alpha,\beta}_{N,k}}{(x^{\alpha,\beta}_{N,k}-t)^2},\quad t\in[a,b],$$

then $K_N(t)\sim\mathscr{O}(N)$, as all numerical methods for this problem in literature. The second case is treated in a similar way. The computational cost of this algorithm is comparable with the cost of the ordinary gaussian rule.

About the convergence we recall the following result [3, Theorem 3.1]:

Theorem 2.2. *Let $f\in C^{(k+1)}([-1,1])$, $k\geqslant 1$. Then for any fixed $t\in[a,b]\subset(-1,1)$ setting $e_N(f;t)=F(f;t)'-F_N(f;t)'$ or $e_N(f;t)=F(f;t)'-F_{N+1}(f;t)'$ according to the position of t, we have*

$$|e_N(f;t)|\leqslant\mathscr{C}\frac{\log N}{N^k}\,\omega\left(f^{(k+1)};\frac{1}{N}\right),\tag{5}$$

where $\mathscr{C}$ is a positive constant independent of f and N.

3. Second algorithm

As we have seen, to approximate $F(f;t)'$ by the previous algorithm, we need to evaluate the derivative of the function f in t. Now we propose an algorithm not requiring the computation of $f'(t)$.

The underlying idea is to approximate the function $F(f)$ by some "local" Lagrange polynomials $\mathscr{L}_{r+1}(F(f))$. More precisely, for any fixed $t \in [a, b] \subset (-1, 1)$, there exists $k \in \{1, \dots, N-1\}$ such that $t \in [x_{N,k}^{\alpha,\beta}, x_{N,k+1}^{\alpha,\beta}]$.

Now we introduce the points t_j,

$$t_{j+[r/2]} = \frac{x_{N,k+j}^{\alpha,\beta} + x_{N,k+j+1}^{\alpha,\beta}}{2}, \quad j = -\left[\frac{r}{2}\right], \dots, \left[\frac{r+1}{2}\right],^{1} \tag{6}$$

and we compute

$$F_N(f; t_j) = \sum_{k=1}^{N} \lambda_{N,k}^{\alpha,\beta} \frac{f(x_{N,k}^{\alpha,\beta}) - f(t_j)}{x_{N,k}^{\alpha,\beta} - t_j}, \quad j = 0, \dots, r.^{2} \tag{7}$$

Then we construct the Lagrange polynomial interpolating $F_N(f)$ in the points $\{t_j\}_{j=0}^{r}$, i.e.

$$\mathscr{L}_{r+1}(F_N(f); x) = \sum_{k=0}^{r} l_{r,k}(x) F_N(f; t_k),$$

and we approximate $F(f;t)'$ by $\mathscr{L}'_{r+1}(F_N(f); t)$.

We observe that the choice of r depends on the smoothness of the function f. For instance, if $f \in C^{(k)}([-1,1])$, $k \geqslant 1$, then we choose $r = k-1$. In any case, the error due to the approximation of $F(f;t)'$ by the derivative of the Lagrange polynomial on $r+1$ knots must be comparable with the error committed when approximating $F(f)$ by the Gaussian rule. Therefore, we suggest to apply this technique when $r \ll N$. In this case, the computational cost of this algorithm is comparable with the previous one, since it requires only $N + r + 1$ evaluation of f, $2(r+1)N$ additions and $2(r+1)N$ multiplications to compute the interpolating polynomial and $r^2/2$ operations to evaluate its derivative in t.

We have chosen as interpolation knots the points t_j, $j = 0, \dots, r$. Nevertheless, other choices of knots are possible, provided that they are sufficiently far from the quadrature nodes $\{x_{N,k}^{\alpha,\beta}\}_{k=1}^{N}$ to avoid numerical cancellation.

Setting $\Phi_r^{\mathrm{loc}}(f)$ to be the approximation error of $F(f)'$ by the second algorithm, that is

$$\Phi_r^{\mathrm{loc}}(f;t) = F(f;t)' - \mathscr{L}_{r+1}(F_N(f);t)',$$

then the following theorem holds:

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[1] $[a]$ denotes the integer part of $a \in \mathbb{R}$.

[2] Notice that $F_N(f)$ is well defined in $(-1, 1)$, since $f \in C^r$, $r \geqslant 1$. Whenever $t = x_{N,j}^{\alpha,\beta}$, for some $j \in \{1, \dots, N\}$, then (7) becomes: $F_N(f, x_{N,j}^{\alpha,\beta}) = \sum_{k=1, k \neq j}^{N} \lambda_{N,k}^{\alpha,\beta} \dfrac{f(x_{N,k}^{\alpha,\beta}) - f(t_j)}{x_{N,k}^{\alpha,\beta} - t_j} + f'(x_{N,j}^{\alpha,\beta})$.

Theorem 3.1. *Let $f \in C^r([-1,1])$, $r \geqslant 2$, $|t| < 1 - \mathscr{C}N^{-2}$, with $\mathscr{C}$ a fixed positive constant. Then*

$$|\Phi_r^{\mathrm{loc}}(f;t)| \leqslant \frac{\mathscr{C}_r}{N^{r-1}}\,\omega((F(f))^{(r)}, \Delta_N(t)) + e_N(f,t), \tag{8}$$

where

$$\Delta_N(t) = \frac{\sqrt{1-t^2}}{N} + \frac{1}{N^2},$$

and $\mathscr{C}_r$ is a positive constant dependent only on r, and $e_N(f)$ is the error of Gaussian rule (7).

Remark. The error estimate of the previous theorem is a function of $F(f)$. It is possible to prove, using some inverse theorems of the Polynomial Approximation Theory (see [10]), that, if $f^{(r)} \in \mathrm{Lip}_M\lambda$, $0 < \lambda \leqslant 1$, then $(F(f))^{(r)} \in \mathrm{Lip}\,(\lambda - \varepsilon)$, $\varepsilon > 0$. Then the following inequality holds:

$$\omega((F(f))^{(r)}, \Delta_N(t)) \leqslant \frac{\mathscr{C}}{N^{\lambda-\varepsilon}}.$$

Then, assuming $f \in C^{r+\lambda}([-1,1])$, $0 < \lambda \leqslant 1$, and taking into account (see [2]) that

$$e_N(f) \leqslant \mathscr{C}\frac{\log N}{N^{r+\lambda}},$$

(8) becomes

$$|\Phi_r^{\mathrm{loc}}(f;t)| \leqslant \frac{\mathscr{C}}{N^{r-1+\lambda-\varepsilon}}.$$

4. Third algorithm

Unlike the previous algorithm, we now propose a numerical method of global type.

In the following we assume $-1 < \alpha, \beta < 1$, since if $\alpha, \beta \geqslant 1$, we consider $(1-x)^{\alpha-[\alpha]}(1+x)^{\beta-[\beta]}$ as the weight function, and $f(x)(1-x)^{[\alpha]}(1+x)^{[\beta]}$ as the density function.

At first we suppose $-1 < \alpha, \beta \leqslant 0$. In this case we choose

$$\{x_{m,k}^{\alpha+1,\beta+1}\}_{k=0}^{m+1}, \qquad x_{m,0}^{\alpha+1,\beta+1} = -1 = -x_{m,m+1}^{\alpha+1,\beta+1}$$

as interpolation knots and compute $F_{m+1}(f; x_{m,k}^{\alpha+1,\beta+1})$, $k = 0, \dots, m+1$. Then we construct the Lagrange polynomial $\mathscr{L}_{m+2}(F_{m+1}(f))$ and approximate $F(f)'$ by $\mathscr{L}_{m+2}(F_{m+1}(f))'$, that is

$$F(f)' \approx \mathscr{L}_{m+2}(F_{m+1}(f))'. \tag{9}$$

Now we suppose $0 < \alpha, \beta < 1$. In this case we choose

$$\{x_{m+1,k}^{\alpha-1,\beta-1}\}_{k=0}^{m+2}, \qquad x_{m+1,0}^{\alpha-1,\beta-1} = -1 = -x_{m+1,m+2}^{\alpha-1,\beta-1}$$

as interpolation knots and compute $F_m(f; x_{m+1,k}^{\alpha-1,\beta-1})$, $k = 0, \dots, m+2$. Then we construct the Lagrange polynomial $\mathscr{L}_{m+3}(F_m(f))$ and, as in the previous case, we set

$$F(f)' \approx \mathscr{L}_{m+3}(F_m(f))'. \tag{10}$$

For the other possible choice of α and β we can use the same technique, and, for the details, we refer to [15]. The considered procedures use the method of additional knots [14] and are based on some results about the interlacing property of the zeros of orthogonal polynomials [7, 8].

We need some further notation. Let Π_n be the class of polynomials of degree at most n. We set

$$\|f\| = \max_{|x| \leqslant 1} |f(x)|,$$

and for any $g \in C^{(0)}([-1,1])$ we denote by

$$E_n(g) = \min_{P \in \Pi_n} \|g - P\|$$

the best uniform approximation error by algebraic polynomials of degree at most n.

Now we state some results about the convergence of the third method. At first we consider $0 < \alpha, \beta < 1$. Then, recalling (10), we get

$$\Phi_{m+3}(f) = F(f)' - \mathscr{L}_{m+3}(F_m(f))'. \tag{11}$$

Theorem 4.1. *Let $v^{\alpha,\beta}$, $\alpha, \beta > 0$ and assume $f' \in TD$. Then*

$$\|\Phi_{m+3}(f)\| \leqslant \mathscr{C} \log^2 m \, E_{m+2}(f'),$$

where $\mathscr{C}$ is a positive constant independent of f and m.

Whenever the parameter α, β of $v^{\alpha,\beta}$ are not positive, as we have previously observed, for the existence of $F(f)'$ we need $f' \in \operatorname{Lip} \lambda$, and $\max(-\alpha, -\beta) < \lambda \leqslant 1$. Nevertheless, to ensure the convergence of the above method, more restrictive assumptions on the function f are required.

Then, recalling (9), we set

$$\Phi_{m+2}(f) = F(f)' - \mathscr{L}_{m+2}(F_{m+1}(f))'. \tag{12}$$

Theorem 4.2. *Let $v^{\alpha,\beta}$, $\alpha, \beta \leqslant 0$ and assume $f^{(k+1)} \in \operatorname{Lip} \lambda$, with $k \geqslant 0$, and $\max\{-2\alpha, -2\beta\} < \lambda \leqslant 1$, then*

$$\|\Phi_{m+2}(f)\| \leqslant \mathscr{C} \frac{\log^2 m}{m^{\lambda+k}},$$

where $\mathscr{C}$ is a positive constant independent of f and m.

Remark. If f is a smooth function, the condition $f^{(k+1)} \in \operatorname{Lip} \lambda$ of the Theorem 3, can be relaxed for $k \geqslant 3$. In this case, if $f \in C^{(k+1)}([-1,1])$, the following estimate holds:

$$\|\Phi_{m+2}(f)\| \leqslant \mathscr{C} \frac{\log^2 m}{m^k} E_{m-k-1}(f^{(k+1)}), \quad k \geqslant 3$$

with $\mathscr{C}$ a positive constant independent of f and m.

5. The proofs

Proof of Lemma 2.1. Let $Q_{2m+1}(x) = p_{m+1}(v^{\alpha,\beta}; x)p_m(v^{\alpha,\beta}; x)$. Since the zeros $x^{\alpha,\beta}_{m,k}$, $k = 1, \dots, m$ interlace with the zeros $x^{\alpha,\beta}_{m+1,k}$, $k = 1, \dots, m+1$, then $Q'_{2m+1}(x^{\alpha,\beta}_{m+1,k}) > 0$, and $Q'_{2m+1}(x^{\alpha,\beta}_{m,k}) < 0$, and

$$0 < Q'_{2m+1}(x^{\alpha,\beta}_{m+1,k+1}) - Q'_{2m+1}(x^{\alpha,\beta}_{m,k}) = (x^{\alpha,\beta}_{m+1,k+1} - x^{\alpha,\beta}_{m,k})Q''_{2m+1}(\xi_k), \quad x^{\alpha,\beta}_{m,k} < \xi_k < x^{\alpha,\beta}_{m+1,k+1},$$

and consequently

$$\frac{1}{x^{\alpha,\beta}_{m+1,k+1} - x^{\alpha,\beta}_{m,k}} < \frac{|Q''_{2m+1}(\xi_k)|}{Q'_{2m+1}(x^{\alpha,\beta}_{m+1,k+1})}.$$

Since (see [17])

$$|p_{m+1}(v^{\alpha,\beta}, x)| \leqslant \mathscr{C} v^{-(\alpha/2)-(1/4),-(\beta/2)-(1/4)}(x), \quad |x| \leqslant 1 - cm^{-2},$$

by the Bernstein inequality, we get

$$|Q''_{2m+1}(x)| \leqslant \mathscr{C} m^2 v^{-\alpha-(3/2),-\beta-(3/2)}(x).$$

Furthermore, taking into account

$$p_{m+1}(v^{\alpha,\beta}; x^{\alpha,\beta}_{m+1,k+1})' = \frac{\gamma_{m+1}(v^{\alpha,\beta})}{\gamma_m(v^{\alpha,\beta})} \frac{1}{\lambda^{\alpha,\beta}_{m+1,k+1} p_m(v^{\alpha,\beta}; x^{\alpha,\beta}_{m+1,k+1})},$$

where $\gamma_m(v^{\alpha,\beta})$ is the leading coefficient of $p_m(v^{\alpha,\beta})$, and being

$$\lambda^{\alpha,\beta}_{m+1,k+1} \sim \frac{1}{m} v^{\alpha+(1/2),\beta+(1/2)}(x^{\alpha,\beta}_{m+1,k+1}),$$

we get

$$\frac{1}{x^{\alpha,\beta}_{m+1,k+1} - x^{\alpha,\beta}_{m,k}} < \mathscr{C} m \frac{v^{-\alpha-(3/2),-\beta-(3/2)}(\xi_k)}{v^{-\alpha-(1/2),-\beta-(1/2)}(x^{\alpha,\beta}_{m+1,k+1})}.$$

Since

$$1 \pm x^{\alpha,\beta}_{m+1,k+1} \sim 1 \pm \xi_k,$$

It follows

$$x^{\alpha,\beta}_{m+1,k+1} - x^{\alpha,\beta}_{m,k} > \frac{1 - x^2_{m+1,k+1}}{m}.$$

Since $x_{m+1,k+1} \in [a,b] \subset (-1,1)$, we have $1 - x^2_{m+1,k+1} \geqslant \min_{x\in[a,b]}(1-a^2, 1-b^2) = c$, and the lemma easily follows. □

Proof of Theorem 3.1. We denote by $\delta_{k,r} = [t_0, t_r]$, where t_0 and t_r are the points defined in (6). Then

$$|\delta_{k,r}| \leqslant (x_{k+[(r+1)/2]+1} - x_{k-[(r/2)]}) \sim (r+1)\frac{\sqrt{1-x^2}}{N}, \quad x \in \delta_{k,r}.$$

We get

$$\begin{aligned}|\Phi_r^{\mathrm{loc}}(f;t)| &= |F(f;t)' - \mathscr{L}_{r+1}(F_N(f);t)'| \\ &\leqslant |F(f;t)' - \mathscr{L}_{r+1}(F(f);t)'| + |\mathscr{L}_{r+1}(F(f) - F_N(f);t)'| \\ &=: \mathscr{I}_1 + \mathscr{I}_2.\end{aligned}$$

For any polynomial $p \in \mathbb{P}_r$, we have:

$$\begin{aligned}\mathscr{I}_1 &= |F(f;t)' - p'(t) - \mathscr{L}_{r+1}(F(f) - p;t)'| \\ &\leqslant |F(f;t)' - p'(t)| + |\mathscr{L}_{r+1}(F(f) - p;t)'|.\end{aligned}$$

Using the Markov–Bernstein inequality, we obtain

$$\begin{aligned}\mathscr{I}_1 &\leqslant \mathscr{C}\|(F(f) - p)'\|_{\delta_{k,r}} + r^2\|\mathscr{L}_{r+1}(F(f) - p)\|_{\delta_{k,r}} \\ &\leqslant \mathscr{C}\|(F(f) - p)'\|_{\delta_{k,r}} + r^2\|\mathscr{L}_{r+1}\|_{\delta_{k,r}}\|(F(f) - p)\|_{\delta_{k,r}},\end{aligned}$$

where $\|g\|_{\delta_{k,r}} = \sup_{x\in\delta_{k,r}}|g(x)|$ and $\|\mathscr{L}_{r+1}\|_{\delta_{k,r}}$ is the Lebesgue constant. Since we use a local interpolant on $r+1$ knots, r fixed, then $r^2\|\mathscr{L}_{r+1}\| = \mathscr{O}(1)$. Then

$$\mathscr{I}_1 \leqslant \|(F(f) - p)'\|_{\delta_{k,r}} + \mathscr{C}\|(F(f) - p\|_{\delta_{k,r}},$$

where $\mathscr{C}$ depends only on r. In particular, using the polynomial $p \in \mathbb{P}_r$ of [20, Th. 3. 19] we have

$$\|(F(f)p')\|_{\delta_{k,r}} \leqslant \frac{\tilde{\mathscr{C}}_r}{N^{r-1}}\,\omega((F(f))^{(r)}, \Delta_N(t)),$$

$$\|F(f) - p\|_{\delta_{k,r}} \leqslant \frac{\mathscr{C}_r}{N^r}\,\omega((F(f))^{(r)}, \Delta_N(t)).$$

Then

$$\mathscr{I}_1 \leqslant \frac{\mathscr{C}_r}{N^{r-1}}\,\omega((F(f))^{(r)}, \Delta_N(t)).$$

Furthermore, making use of the Markov–Bernstein inequality, we obtain

$$\mathscr{I}_2 \leqslant \|F(f) - F_N(f)\|_{\delta_{k,r}} = e_N(f). \qquad \square$$

We need the following

Lemma 5.1 (Gopengauz [13]). *If $f \in C^{(r)}([-1,1])$, $r \geqslant 0$, then for each $n \in \mathbb{N}$ there exists a polynomial q_n of degree at most $n \geqslant 4(r+1)$ such that*

$$|f^{(k)}(x) - q_n^{(k)}(x)| \leqslant \mathscr{C}[n^{-1}\sqrt{1-x^2}]^{r-k}\omega(f^{(r)}; n^{-1}\sqrt{1-x^2}),$$

uniformly for $0 \leqslant k \leqslant r$, $-1 \leqslant x \leqslant 1$.

Proof of Theorem 4.1. Let q_{m+3} be the polynomial of Lemma 5.1 related to the function f. Setting $r_{m+3} = f - q_{m+3}$, since $\Phi_{m+3}(f) = 0$, $\forall f \in \Pi_{m+3}$, we get by (11)

$$\|\Phi_{m+3}(f)\| = \|\Phi_{m+3}(f - q_{m+3})\| \leqslant \mathscr{C}\|F_m(f)' - \mathscr{L}_{m+3}(F_m(f))'\| + \|F(r_{m+3})'\| + \|F_m(r_{m+3})'\|.$$

By [4, Lemmas 8 and 9, p. 146] we have

$$\|F(r_{m+3})'\|, \|F_m(r_{m+3})'\| \leqslant \mathscr{C}\omega\left(f'; \frac{1}{m}\right)\log m. \tag{13}$$

Moreover, making use of [14, Theorem 3.1, p. 40] for $h = q = i = 1$, $r = s = 1$ we get

$$\|\mathscr{L}_{m+3}(F_m(r_{m+3}))' - F(r_{m+3})'\| \leqslant \mathscr{C}\|F_m(r_{m+3})'\|\log m,$$

and taking into account (13) we have

$$\|\Phi_{m+3}(f)\| \leqslant \mathscr{C}\omega\left(f'; \frac{1}{m}\right)\log^2 m.$$

Now let p^*_{m+2} denote the best uniform approximation polynomial of f' and P_{m+3} denote a polynomial such that $P'_{m+3} = p^*_{m+2}$. Then we have

$$\|\Phi_{m+3}(f)\| = \|\Phi_{m+3}(f - P_{m+3})\| \leqslant \mathscr{C}\omega\left(f' - p^*_{m+2}; \frac{1}{m}\right)\log^2 m$$
$$\leqslant \mathscr{C}\|f' - p^*_{m+2}\|\log^2 m = \mathscr{C}E_{m+2}(f')\log^2 m,$$

and the theorem follows. □

Proof of Theorem 4.2. To prove the theorem, we start from (12). Let q_{m+2} be the polynomial of Lemma 5.1 related to the function f. Setting $r_{m+2} = f - q_{m+2}$, since $\Phi(P_{m+2}) = 0$, $\forall P_{m+2} \in \Pi_{m+2}$, by (12) we get

$$\|\Phi_{m+2}(f)\| = \|\Phi_{m+2}(f - q_{m+2})\| \leqslant \mathscr{C}\|F_{m+1}(f)' - \mathscr{L}_{m+2}(F_{m+1}(f))'\| + \|F(r_{m+2})'\| + \|F_{m+1}(r_{m+2})'\|. \tag{14}$$

At first we estimate $\|F'(r_{m+2})\|$. In the case $k = 0$ the estimate can be found in [4, Lemma 9, p. 146]. Assume $k \geqslant 1$.

We have for $|t| \leqslant 1$

$$|F'(r_{m+2}; t)| \leqslant \mathscr{C}\left\{\int_{-1}^{t-(1+t)/m} + \int_{t-(1+t)/m}^{t+(1-t)/m} + \int_{t+(1-t)/m}^{1}\right\} \frac{|r_{m+2}(x) - r_{m+2}(t) - r'_{m+2}(t)(x-t)|}{(x-t)^2} v^{\alpha,\beta}(x)\,\mathrm{d}x$$
$$:= B_1(t) + B_2(t) + B_3(t), \tag{15}$$

and

$$B_1(t) \leqslant \mathscr{C}\left\{\int_{-1}^{t-(1+t)/m} \frac{|r_{m+2}(x)|}{(x-t)^2} v^{\alpha,\beta}(x)\,\mathrm{d}x + |r_{m+2}(t)| \int_{-1}^{t-(1+t)/m} \frac{v^{\alpha,\beta}(x)}{(x-t)^2}\mathrm{d}x \right.$$
$$\left. + |r'_{m+2}(t)| \int_{-1}^{t-(1+t)/m} \frac{v^{\alpha,\beta}(x)}{|x-t|}\mathrm{d}x\right\}$$
$$:= I_1 + I_2 + I_3.$$

Applying Lemma 5.1,

$$\begin{aligned} I_1(t) &\leqslant \frac{\mathscr{C}}{m^{k+\lambda+1}} \int_{-1}^{t-(1+t)/m} \frac{v^{(\lambda+k+1)/2+\alpha,(\lambda+k+1)/2+\beta}(x)}{(x-t)^2}\mathrm{d}x \\ &\leqslant \mathscr{C}\frac{(1-t)^{(k+\lambda+1)/2+\alpha}}{m^{k+\lambda+1}} \int_{-1}^{t-(1+t)/m} \frac{(1+x)^{(\lambda+k+1)/2+\beta}}{(x-t)^2}\mathrm{d}x \\ &= \mathscr{C}\frac{(1-t)^{(k+\lambda+1)/2+\alpha}(1+t)^{(\lambda+k+1)/2+\beta-1}}{m^{k+\lambda+1}} \int_0^{1-(1/m)} \frac{u^{(k+1+\lambda)/2+\beta}}{(1-u)^2}\mathrm{d}u \\ &\leqslant \mathscr{C}\frac{v^{(k+\lambda+1)/2+\alpha,(k+\lambda-1)/2+\beta}(t)}{m^{k+\lambda}} \end{aligned}$$

since by the assumptions on k and λ, $k+\lambda-1+2\beta>0$, $k+\lambda+1+2\alpha>0$, we have

$$I_1(t) \leqslant \frac{\mathscr{C}}{m^{k+\lambda}}. \tag{16}$$

To estimate I_2 we use Lemma 5.1 again and by similar developments used for I_1 we get

$$I_2(t) \leqslant \frac{\mathscr{C}}{m^{k+\lambda}}. \tag{17}$$

Applying Lemma 5.1 and taking into account [5, Lemma 3.3, p. 453], it follows that

$$I_3(t) \leqslant \mathscr{C}\frac{\log m}{m^{k+\lambda}}. \tag{18}$$

Combining (16)–(18), we have

$$B_1(t) \leqslant \mathscr{C}\frac{\log m}{m^{k+\lambda}}. \tag{19}$$

By analogous developments one proves

$$B_3(t) \leqslant \mathscr{C}\frac{\log m}{m^{k+\lambda}}. \tag{20}$$

Now we estimate $B_2(t)$:

$$B_2(t) \leqslant \mathscr{C}\int_{t-(1+t)/m}^{t+(1-t)/m} \left|\frac{r'_{m+2}(\xi)-r'_{m+2}(t)}{x-t}\right| v^{\alpha,\beta}(x)\,\mathrm{d}x \leqslant \mathscr{C}\frac{\|r''_{m+2}\|}{m}.$$

Then by Lemma 5.1 we have

$$B_2(t) \leqslant \frac{\mathscr{C}}{m^{k+\lambda}}. \tag{21}$$

Combining (19)–(21), we have

$$\| F(r_{m+2})' \| \leqslant \mathscr{C} \frac{\log m}{m^{k+\lambda}}. \tag{22}$$

By similar developments we can prove

$$\| F_{m+1}(r_{m+2})' \| \leqslant \mathscr{C} \frac{\log m}{m^{k+\lambda}}. \tag{23}$$

Moreover, making use of [14, Theorem 3.1, p. 40] for $h = i = 1$, $r = s = 1$, $q = k + 1$ we get

$$\| \mathscr{L}_{m+2}(F_{m+1}(r_{m+2}))' - F_{m+1}(r_{m+2})' \| \leqslant \mathscr{C} \| F_{m+1}(r_{m+2})' \| \log m,$$

Combining the last inequality with (23) we get

$$\| \mathscr{L}_{m+2}(F_{m+1}(r_{m+2}))' - F_{m+1}(r_{m+2})' \| \leqslant \mathscr{C} \frac{\log^2 m}{m^{k+\lambda}}. \tag{24}$$

Taking into account (22), (23), (14), and (24), we have

$$\| \Phi_{m+2}(f) \| \leqslant \mathscr{C} \frac{\log^2 m}{m^{k+\lambda}}. \qquad \square$$

6. Numerical results

In this section we state some numerical results obtained using the described numerical methods for evaluating some Hadamard integrals. We show that the computed errors agree with the theoretical error estimates.

In the following we denote by N the number of knots of the Gaussian rule $F_N(f; t)$ used in the second and third algorithm. For the first algorithm we have considered $F_N(f)'$ or $F_{N+1}(f)'$ according to the position of the singularity t. Furthermore, we have considered the "local" Lagrange polynomial with degree equal to 3.

Among the proposed algorithms, the first appears the most efficient, although, in many cases, the last two algorithms are also very fast.

In Tables 1–14 the columns denoted by Φ_1, Φ_2, Φ_3, contain the errors due to the first, the second and the third algorithm, respectively. Moreover, for some examples, to compare the behaviour of the proposed numerical methods with other algorithms, we give in the columns denoted by Φ_4, Φ_5, the errors obtained by making use of the product type rules described in [1] and [9], respectively.

For the third algorithm, in the examples, we choose, as interpolation knots,

$$\{x_{N-1,k}^{1,1}\} \cup \{\pm 1\} \quad \text{if } v^{\alpha,\beta}(x) = 1,$$

$$\{x_{N-1,k}^{1/2,1/2}\} \cup \{\pm 1\} \quad \text{if } v^{\alpha,\beta}(x) = \frac{1}{\sqrt{1-x^2}}.$$

Example 1.

$$\oint_{-1}^{1} \frac{e^x}{(x-t)^2}\,dx,$$

$$f(x) = e^x, \qquad v^{\alpha,\beta} = 1.$$

In this case, the theoretical error for the first and third method geometrically goes to zero. For the second method the theoretical error is $\mathcal{O}(N^{-3})$.

In Table 1 we give the absolute errors for various values of t obtained by the three algorithms.

As we can see, the global Lagrange interpolation works very well in this case, since the function f is an analytical function.

The following two examples can be found, for instance, in [1].

Example 2.

$$\oint_{-1}^{1} \frac{x^{4/3}}{(x-t)^2}\,dx, \tag{25}$$

$$f(x) = x^{4/3}, \qquad v^{\alpha,\beta} = 1.$$

The order of theoretical error for the three algorithms is $\mathcal{O}(1/N^{1/3})$.

These results (Table 2–4) are compared with those obtained using the product rule described in [1].

Example 3.

$$\oint_{-1}^{1} \frac{(1-x^2)^{5/2}}{(x-t)^2}\,dx,$$

$$f(x) = (1-x^2)^{5/2}, \qquad v^{\alpha,\beta} = 1.$$

The order of the theoretical error for the proposed method is $\mathcal{O}(1/N^3)$. The results are given in Tables 5–7.

We can see that, in the last two examples, the errors of the considered algorithms are of the same order, although the second algorithm does not evaluate f' in the points t and requires less computational cost of the product rule.

Table 1
$N = 10$

t	Φ_1	Φ_2	Φ_3
0.1	1.22×10^{-14}	2.64×10^{-5}	2.39×10^{-12}
0.2	1.77×10^{-15}	3.49×10^{-5}	2.00×10^{-12}
0.3	3.55×10^{-15}	1.83×10^{-5}	9.49×10^{-13}
0.5	5.55×10^{-15}	3.09×10^{-5}	7.31×10^{-13}
0.8	1.04×10^{-14}	2.41×10^{-5}	3.33×10^{-12}
0.99	2.17×10^{-14}	8.89×10^{-8}	5.88×10^{-12}

Table 2
$N = 10$

t	Φ_1	Φ_2	Φ_3	Φ_5
0.01	4.96×10^{-1}	4.32×10^{-2}	7.41×10^{-1}	1.69×10^{-1}
0.1	3.50×10^{-2}	1.15×10^{-2}	5.11×10^{-1}	4.08×10^{-1}
0.25	2.52×10^{-2}	4.17×10^{-2}	2.14×10^{-2}	1.15×10^{-1}
0.5	9.54×10^{-3}	1.35×10^{-2}	5.65×10^{-2}	2.70×10^{-2}
0.8	4.18×10^{-3}	5.80×10^{-3}	2.46×10^{-1}	1.23×10^{-2}
0.9	3.36×10^{-3}	4.64×10^{-3}	9.06×10^{-2}	8.97×10^{-3}
0.99	2.81×10^{-3}	3.85×10^{-3}	3.05×10^{-1}	8.94×10^{-3}

Table 3
$N = 100$

t	Φ_1	Φ_2	Φ_3	Φ_5
0	9.05×10^{-1}	7.21×10^{-1}	1.13×10^{-0}	9.05×10^{-1}
0.01	1.59×10^{-2}	1.40×10^{-1}	2.41×10^{-1}	1.95×10^{-1}
0.1	1.39×10^{-3}	2.48×10^{-3}	7.07×10^{-2}	1.78×10^{-3}
0.25	2.35×10^{-4}	3.94×10^{-4}	2.02×10^{-2}	4.84×10^{-4}
0.5	5.93×10^{-5}	9.55×10^{-5}	4.96×10^{-3}	9.32×10^{-5}
0.8	2.32×10^{-5}	3.76×10^{-5}	1.89×10^{-3}	1.32×10^{-5}
0.9	1.83×10^{-5}	2.97×10^{-5}	8.78×10^{-3}	1.01×10^{-5}
0.99	1.51×10^{-5}	2.45×10^{-5}	1.40×10^{-2}	4.50×10^{-6}

Table 4
$N = 400$

t	Φ_1	Φ_2	Φ_5
0	5.71×10^{-1}	4.56×10^{-1}	5.71×10^{-1}
0.01	4.31×10^{-3}	9.41×10^{-3}	1.26×10^{-2}
0.1	5.88×10^{-5}	9.48×10^{-5}	9.79×10^{-5}
0.25	9.44×10^{-6}	1.57×10^{-5}	1.20×10^{-5}
0.8	9.23×10^{-7}	1.52×10^{-6}	1.02×10^{-6}
0.5	2.36×10^{-6}	3.88×10^{-6}	2.70×10^{-6}
0.9	1.20×10^{-6}	1.20×10^{-6}	8.11×10^{-7}
0.99	9.93×10^{-7}	9.93×10^{-7}	7.12×10^{-7}

Example 4. The following example can be found in [18]:

$$\neq\!\!\!\!\!\!\int_{-1}^{1} \frac{1}{(x^2+\gamma^2)(x-t)^2} \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x, \tag{26}$$

$$f(x) = \frac{1}{(x^2+\gamma^2)}, \qquad v^{\alpha,\beta}(x) = \frac{1}{\sqrt{1-x^2}}.$$

Table 5
$N = 10$

t	Φ_1	Φ_2	Φ_3	Φ_5
0.0	1.62×10^{-6}	8.20×10^{-5}	9.10×10^{-5}	1.46×10^{-6}
0.10	1.63×10^{-6}	1.36×10^{-4}	1.36×10^{-4}	2.94×10^{-5}
0.25	1.85×10^{-6}	1.52×10^{-3}	2.91×10^{-4}	7.65×10^{-5}
0.50	3.56×10^{-5}	2.99×10^{-3}	4.97×10^{-4}	2.03×10^{-4}
0.80	1.45×10^{-3}	5.29×10^{-3}	3.00×10^{-3}	9.74×10^{-4}
0.99	1.66×10^{-6}	1.38×10^{-3}	1.32×10^{-2}	1.07×10^{-2}

Table 6
$N = 100$

t	Φ_1	Φ_2	Φ_3	Φ_5
0.0	1.63×10^{-7}	1.73×10^{-7}	4.99×10^{-5}	1.66×10^{-13}
0.10	1.62×10^{-7}	7.10×10^{-7}	4.79×10^{-5}	1.61×10^{-12}
0.25	1.24×10^{-7}	1.84×10^{-6}	3.79×10^{-5}	3.87×10^{-12}
0.50	8.03×10^{-8}	2.12×10^{-6}	3.30×10^{-5}	2.03×10^{-11}
0.80	1.43×10^{-6}	1.20×10^{-6}	8.31×10^{-6}	1.11×10^{-10}
0.99	1.57×10^{-7}	7.46×10^{-6}	2.33×10^{-5}	5.80×10^{-8}

Table 7
$N = 400$

t	Φ_1	Φ_2	Φ_5
0.0	1.63×10^{-7}	1.63×10^{-7}	3.64×10^{-14}
0.10	1.62×10^{-7}	1.72×10^{-7}	8.30×10^{-14}
0.25	1.24×10^{-7}	9.34×10^{-8}	9.94×10^{-14}
0.50	8.03×10^{-8}	6.12×10^{-8}	2.11×10^{-13}
0.80	3.78×10^{-8}	2.20×10^{-7}	1.06×10^{-14}
0.99	1.57×10^{-7}	1.64×10^{-8}	2.73×10^{-12}

We observe that the function f has poles at $\pm \gamma i$, with residuals $\mp i/2\gamma$. If $\gamma \to 0$, the residuals tend to ∞. Consequently, the evaluation of the integral (26) presents serious problems for small values of γ [21].

For this example, we have chosen $\{x_{N-1,k}^{1/2,1/2}\} \cup \{\pm 1\}$ as interpolation knots for the global Lagrange polynomial.

In the Tables 8–10 we report the absolute errors computed by the proposed algorithms, with $\gamma = 5$ and $\gamma = 0.1$.

As we can see, all the methods work for $\gamma = 5$. Nevertheless, for $\gamma = 0.1$ we must consider N sufficiently large to obtain a good approximation of (26) in all cases.

Table 8
$N = 10, \gamma = 5$

t	Φ_1	Φ_2	Φ_3	Φ_4
0.0	2.25×10^{-17}	1.17×10^{-9}	2.39×10^{-12}	2.27×10^{-10}
0.10	1.77×10^{-16}	2.10×10^{-9}	2.00×10^{-12}	1.25×10^{-10}
0.30	7.80×10^{-17}	1.77×10^{-9}	9.49×10^{-13}	1.94×10^{-10}
0.50	1.19×10^{-16}	2.43×10^{-8}	7.31×10^{-13}	2.47×10^{-10}
0.80	1.77×10^{-16}	1.00×10^{-8}	3.33×10^{-12}	1.36×10^{-10}
0.99	2.05×10^{-13}	3.95×10^{-12}	5.88×10^{-12}	5.97×10^{-10}

Table 9
$N = 100, \gamma = 0.1$

t	Φ_1	Φ_2	Φ_4	Φ_5
0.00	1.33×10^{-5}	4.43×10^{-1}	2.90	1.33×10^{-5}
0.10	3.23×10^{-13}	1.04×10^{-2}	1.13	1.45×10^{-1}
0.30	1.06×10^{-6}	3.21×10^{-4}	2.00×10^{-1}	1.82×10^{-2}
0.50	4.72×10^{-7}	8.05×10^{-4}	7.79×10^{-2}	4.93×10^{-3}
0.80	1.62×10^{-7}	4.24×10^{-5}	6.68×10^{-3}	1.82×10^{-3}
0.99	1.07×10^{-7}	1.56×10^{-7}	1.23×10^{-1}	4.15×10^{-3}

Table 10
$N = 400, \gamma = 0.1$

t	Φ_1	Φ_2	Φ_4	Φ_5
0.00	2.27×10^{-11}	1.83×10^{-3}	1.58×10^{-10}	1.12×10^{-10}
0.10	7.09×10^{-13}	1.74×10^{-2}	2.75×10^{-12}	6.06×10^{-11}
0.30	2.84×10^{-12}	8.48×10^{-6}	4.69×10^{-10}	1.61×10^{-11}
0.50	4.40×10^{-13}	1.32×10^{-5}	2.84×10^{-11}	6.55×10^{-12}
0.80	2.84×10^{-14}	2.84×10^{-7}	1.89×10^{-9}	3.41×10^{-12}
0.99	3.90×10^{-13}	2.40×10^{-9}	9.88×10^{-9}	1.19×10^{-11}

Example 5.

$$\int\!\!\!\!\!\!=_{-1}^{1} \frac{x|x|}{(x-t)^2} \frac{1}{\sqrt{1-x^2}}\, dx,$$

$$f(x) = x|x|, \qquad v^{\alpha,\beta}(x) = \frac{1}{\sqrt{1-x^2}}.$$

Table 11
$N = 10$

t	Φ_1	Φ_2	Φ_3	Φ_4
0.0	7.31×10^{-17}	3.83×10^{-15}	1.34×10^{-13}	3.36×10^{-14}
0.1	2.98×10^{-2}	4.22×10^{-2}	2.30×10^{-1}	1.47×10^{-1}
0.3	5.43×10^{-3}	5.57×10^{-3}	5.80×10^{-2}	1.57×10^{-1}
0.5	1.68×10^{-3}	8.78×10^{-4}	1.39×10^{-2}	8.72×10^{-2}
0.8	4.90×10^{-4}	4.63×10^{-4}	2.06×10^{-2}	2.44×10^{-1}
0.9	3.54×10^{-4}	2.79×10^{-4}	1.09×10^{-4}	1.54×10^{-2}
0.99	2.71×10^{-4}	2.15×10^{-4}	1.11×10^{-2}	2.03×10^{-2}

Table 12
$N = 100$

t	Φ_1	Φ_2	Φ_3	Φ_4
0.0	2.91×10^{-15}	4.31×10^{-14}	1.05×10^{-10}	3.19×10^{-12}
0.1	2.58×10^{-5}	8.33×10^{-5}	1.58×10^{-3}	5.52×10^{-3}
0.3	1.04×10^{-6}	3.18×10^{-6}	1.41×10^{-4}	1.70×10^{-3}
0.5	2.26×10^{-7}	6.84×10^{-7}	4.32×10^{-5}	1.04×10^{-3}
0.8	6.08×10^{-8}	3.56×10^{-8}	1.98×10^{-5}	1.25×10^{-4}
0.9	3.89×10^{-8}	4.36×10^{-8}	6.56×10^{-7}	1.90×10^{-3}
0.99	3.21×10^{-8}	3.19×10^{-8}	2.26×10^{-6}	3.36×10^{-3}

Table 13
$N = 400$

t	Φ_1	Φ_2	Φ_4
0.0	1.17×10^{-15}	1.59×10^{-13}	9.13×10^{-11}
0.1	1.10×10^{-7}	6.27×10^{-7}	3.55×10^{-4}
0.3	4.10×10^{-9}	4.31×10^{-8}	1.42×10^{-4}
0.5	8.87×10^{-10}	1.42×10^{-8}	6.49×10^{-5}
0.8	2.16×10^{-10}	8.85×10^{-10}	1.64×10^{-4}
0.9	1.70×10^{-10}	2.81×10^{-11}	6.37×10^{-5}
0.99	1.31×10^{-10}	1.53×10^{-10}	2.45×10^{-3}

The interpolation knots, for the global Lagrange interpolation, are the same as of the previous example. The results are given in Tables 11–13.

We observe that the behaviour of the Φ_1 and Φ_2 are similar, since $f' \in \text{Lip}\,1$. Furthermore, the theoretical convergence of the third method is not assured, although it numerically converges.

Remarks. All the algorithms make use of zeros and Christoffel contants with respect to Jacobi polynomials and they can be computed efficiently (see [12]). The methods introduced above can be

Table 14
$N = 17$

t	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5
0.01	1.52×10^{-6}	8.35×10^{-7}	9.96×10^{-5}	6.9×10^{-6}	6.93×10^{-6}
0.05	7.64×10^{-6}	6.78×10^{-6}	2.88×10^{-4}	2.5×10^{-5}	3.47×10^{-5}
0.1	1.54×10^{-5}	1.42×10^{-5}	3.64×10^{-4}	5.4×10^{-6}	3.96×10^{-5}
0.2	3.17×10^{-5}	2.53×10^{-5}	1.44×10^{-3}	7.4×10^{-5}	8.16×10^{-5}
0.3	5.02×10^{-5}	4.28×10^{-5}	3.07×10^{-3}	1.6×10^{-4}	2.33×10^{-4}
0.4	7.25×10^{-5}	6.74×10^{-5}	6.85×10^{-4}	2.3×10^{-4}	3.36×10^{-4}
0.5	1.01×10^{-4}	7.65×10^{-5}	4.36×10^{-3}	2.7×10^{-4}	2.82×10^{-4}
0.6	1.42×10^{-4}	1.30×10^{-4}	8.46×10^{-3}	2.9×10^{-4}	7.11×10^{-4}
0.7	2.07×10^{-4}	2.19×10^{-4}	9.32×10^{-3}	6.6×10^{-4}	1.02×10^{-3}
0.8	3.33×10^{-4}	3.21×10^{-4}	1.21×10^{-2}	9.0×10^{-4}	1.07×10^{-3}
0.9	6.94×10^{-4}	9.16×10^{-4}	2.90×10^{-2}	3.1×10^{-3}	3.85×10^{-3}
0.95	1.16×10^{-3}	1.01×10^{-3}	3.78×10^{-2}	5.7×10^{-3}	5.46×10^{-3}
0.99	4.81×10^{-3}	2.86×10^{-3}	1.20×10^{-1}	3.4×10^{-2}	3.59×10^{-2}

used to approximate weakly singular integrals of the type

$$\int_{-1}^{1} \frac{f(x)}{|x-t|^{\mu}} v^{\alpha,\beta}(x)\,dx, \quad 0 < \mu < 1.$$

Furthermore, by the same techniques we can approximate Cauchy principal value integrals, and with little change, also $H_p(fv^{\alpha,\beta})$, for $p > 1$. For the case of Cauchy principal value integrals, we give an example.

Example 6.

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{(x-t)}\,dx,$$

$$f(x) = \sqrt{1-x^2}, \qquad v^{\alpha,\beta} = 1.$$

The results are given in Table 14.

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