Computers and Mathematics with Applications 62 (2011) 4021–4027

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Simply constructed family of a Ostrowski's method with optimal order of convergence

V. Kanwar [a,*], Ramandeep Behl [b], Kapil K. Sharma [b]

[a] University Institute of Engineering and Technology, Panjab University, Chandigarh- 160 014, India

[b] Department of Mathematics, Panjab University, Chandigarh-160 014, India

ARTICLE INFO

Article history:
Received 5 May 2011
Received in revised form 16 September 2011
Accepted 19 September 2011

Keywords:
Nonlinear equations
Newton's method
Ostrowski's method
Jarratt's method
Optimal order of convergence

ABSTRACT

In this paper, we propose a simple modification over Chun's method for constructing iterative methods with at least cubic convergence [5]. Using iteration formulas of order two, we now obtain several new interesting families of cubically or quartically convergent iterative methods. The fourth-order family of Ostrowski's method is the main finding of the present work. Per iteration, this family of Ostrowski's method requires two evaluations of the function and one evaluation of its first-order derivative. Therefore, the efficiency index of this Ostrowski's family is $E = \sqrt[3]{4} \approx 1.587$, which is better than those of most third-order iterative methods $E = \sqrt[3]{3} \approx 1.442$ and Newton's method $E = \sqrt{2} \approx 1.414$. The performance of Ostrowski's family is compared with its closest competitors, namely Ostrowski's method, Jarratt's method and King's family in a series of numerical experiments.

1. Introduction

Various problems arising in the diverse disciplines of science, engineering and nature can be described by nonlinear equations of the form

$$f(x) = 0. \tag{1.1}$$

The best known and most widely used algorithm for solving such problems is the classical Newton's method [1–4], which is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots. \tag{1.2}$$

Many researchers have developed modifications of Newton's method or Newton-like methods [5–12] in a number of ways to improve the local order of convergence of Newton's method at the expense of additional evaluations of the functions and/or derivatives, mostly at the point iterated by the method. All these modifications are targeted at increasing the local order of convergence with the view of increasing their efficiency index [2].

Kung and Traub [6] have conjectured that multipoint iteration methods without memory based on n function evaluations have optimal order of convergence 2^{n-1}. The famous Ostrowski's method [1–8] and Jarratt's method [7] are examples of fourth-order multipoint methods without memory. These methods are the most efficient fourth-order multipoint iterative

* Corresponding author.
E-mail address: vimithil@yahoo.co.in (V. Kanwar).

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.09.039

methods known to date. Another well-known example of a fourth-order multipoint method with the same number of function evaluations is King's family [8]. This family is defined as:

$$x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)} \frac{f(x_n) + \gamma f(w_n)}{f(x_n) + (\gamma - 2)f(w_n)}, \tag{1.3}$$

where $w_n = x_n - \frac{f(x_n)}{f'(x_n)}$ and $\gamma \varepsilon \mathbb{R}$.

The research of finding iterative methods with optimal fourth-order convergence, not requiring the computation of a second-order derivative, is important and interesting from the practical point of view. In this work, we contribute further to the development of the theory of iteration processes and derive many families of new third- and fourth-order multipoint iterative methods. Ostrowski's family of methods requires two evaluations of the function $f(x)$ and one of its derivatives $f'(x)$ per iteration and the efficiency index is the same as that of Ostrowski's method. These methods are obtained by introducing quadratically convergent methods, a secant line and a parabola while moving along the curve to solve nonlinear equations numerically, and the approach for deriving the formula is a different one.

2. Basic definitions

Definition 1. Let $f(x)$ be a real valued function with a simple root r and let $\{x_n\}_{n\epsilon N}$ be a sequence of real numbers that converges to r. Then, we say that the order of convergence of the sequence is p, if there exists a $p\epsilon\mathbb{R}^+$ such that

$$\lim_{n\to\infty} \frac{r - x_{n+1}}{(r - x_n)^p} = C \neq 0, \tag{2.1}$$

where C is known as the asymptotic error constant. If $p = 1, 2$ or 3, the sequence $\{x_n\}$ is said to have linear, quadratic or cubic convergence, respectively.

Definition 2. Let $e_n = x_n - r$ be the error in the nth iteration. We call the relation

$$e_{n+1} = Ce_n^p + O(e_n^{p+1}), \tag{2.2}$$

the error equation. If we can obtain the error equation for any iterative method, then the value of p is its order of convergence.

Definition 3. Let d be the number of new pieces of information (function or its derivatives) required by an iterative method per step. Then the efficiency of the method may be measured by the efficiency index introduced by Ostrowski [2] and defined as:

$$E = p^{\frac{1}{d}}. \tag{2.3}$$

3. Development of the methods

Consider a nonlinear equation

$$f(x) = 0, \tag{3.1}$$

whose one or more roots are to be found. Let $x = r$ be a simple root of the nonlinear equation (3.1) and $x = x_0$ be the initial guess to the required root.

Here we also intend to construct new families of iteration functions from available iteration functions of order two based on a geometric observation. In the sequel, whenever we mention that an iteration function ϕ is of order p, it means that the corresponding iterative method defined by $x_{n+1} = \phi(x_n)$ is of convergence order p, that is, the error $|r - x_{n+1}|$ is proportional to $|r - x_n|^p$ as $n \to \infty$. We will indicate here that ϕ is an iteration function whose order is p by writing $\phi \varepsilon I_p$.

Our proposed scheme (similar to Chun) to develop new families of iteration functions is constructed geometrically as follows:

Let β be a fixed parameter with $0 \le \beta \le 1$ and let $\phi(x_0)\varepsilon I_2$ be an iteration function of order two.

Let

$$y = f(x), \tag{3.2}$$

represents the graph of the function $f(x)$. Assume that two points, namely $\left(\frac{x_0+\phi(x_0)}{2}, \frac{f(x_0)}{2}\right)$ and $(\phi(x_0), \beta f(\phi(x_0)))$, lie on the same graph of the function $y = f(x)$. Then the approximated line of the function $f(x)$ passing through the above mentioned points is given by

$$y - \beta f(\phi(x_0)) = \frac{2\beta f(\phi(x_0)) - f(x_0)}{\phi(x_0) - x_0}(x - \phi(x_0)). \tag{3.3}$$

Draw a parabola with vertex at $(\phi(x_0), 0)$ and axis parallel to the *y-axis* on the graph of the same function (3.2). The equation of this parabola is given by

$$y = \alpha(x - \phi(x_0))^2, \tag{3.4}$$

where α is the scaling parameter. The parabola (3.4) widens as α approaches zero and narrows as $|\alpha|$ becomes large. The intersection of a line (3.3) with the parabola (3.4) is obtained by setting them equal to each other since each equals y. Therefore, we end up with a quadratic equation given by

$$\alpha(x-\phi(x_0))^2 - \frac{2\beta f(\phi(x_0)) - f(x_0)}{\phi(x_0)-x_0}(x-\phi(x_0)) - \beta f(\phi(x_0)) = 0. \tag{3.5}$$

Solving this quadratic equation for $(x-\phi(x_0))$, and after some simplification, we get the first approximation to the required root as

$$x = \phi(x_0) + \frac{\left(\frac{2\beta f(\phi(x_0))-f(x_0)}{\phi(x_0)-x_0}\right) \pm \sqrt{\left(\frac{2\beta f(\phi(x_0))-f(x_0)}{\phi(x_0)-x_0}\right)^2 + 4\alpha\beta f(\phi(x_0))}}{2\alpha}. \tag{3.6}$$

This can further be rewritten in the equivalent form (by rationalizing the numerator) as

$$x = \phi(x_0) - \frac{2\beta f(\phi(x_0))}{\left(\frac{2\beta f(\phi(x_0))-f(x_0)}{\phi(x_0)-x_0}\right) \mp \sqrt{\left(\frac{2\beta f(\phi(x_0))-f(x_0)}{\phi(x_0)-x_0}\right)^2 + 4\alpha\beta f(\phi(x_0))}}, \tag{3.7}$$

in which the sign should be chosen so as to make the denominator largest in magnitude.

Now consider the factor

$$\frac{4\alpha\beta f(\phi(x_0))}{\left(\frac{2\beta f(\phi(x_0))-f(x_0)}{\phi(x_0)-x_0}\right)^2}, \quad 0 \le \beta \le 1. \tag{3.8}$$

Since the scaling parameter α appears in the numerator of (3.8), it is clear that there exists some real values of α such that

$$\left| \frac{4\alpha\beta f(\phi(x_0))}{\left(\frac{2\beta f(\phi(x_0))-f(x_0)}{\phi(x_0)-x_0}\right)^2} \right| < 1, \tag{3.9}$$

holds.

With this assumption, the binomial theorem is applicable in Eq. (3.7) and one can get the following formula free from the square root term as

$$x = \phi(x_0) - \frac{\beta f(\phi(x_0))(x_0-\phi(x_0))(f(x_0)-2\beta f(\phi(x_0)))}{(f(x_0)-2\beta f(\phi(x_0)))^2 + \alpha\beta f(\phi(x_0))(x_0-\phi(x_0))^2}. \tag{3.10}$$

Now repeating this process until the parabola becomes *x-axis*, the general formula for successive approximation is given by

$$\varphi(x_{n+1}) \equiv x_{n+1} = \phi(x_n) - \frac{\beta f(\phi(x_n))(x_n-\phi(x_n))(f(x_n)-2\beta f(\phi(x_n)))}{(f(x_n)-2\beta f(\phi(x_n)))^2 + \alpha\beta f(\phi(x_n))(x_n-\phi(x_n))^2}. \tag{3.11}$$

The family of iteration functions $\varphi(x_{n+1})$ constructed in this manner again has order of convergence equal to at least three when $\beta = 1$. This is proved by the next theorem. A similar approach for deriving the family of Secant-like methods has been used previously by Kanwar et al. [13].

4. Order of convergence

Theorem 4.1. *Let r be a simple zero of $f(x)$ and $\phi(x)$ be an iteration function with $\phi \varepsilon I_2$, such that $\phi'''(r)$ is continuous in a neighborhood of r. Let*

$$\varphi(x_{n+1}) = \phi(x_n) - \frac{\beta f(\phi(x_n))(x_n-\phi(x_n))(f(x_n)-2\beta f(\phi(x_n)))}{(f(x_n)-2\beta f(\phi(x_n)))^2 + \alpha\beta f(\phi(x_n))(x_n-\phi(x_n))^2}. \tag{4.1}$$

Then $\varphi \varepsilon I_p$ for some p with $p \ge 3$ when $\beta = 1$. Furthermore, if $\phi''(r) = 0$ or $\phi''(r) = \frac{f''(r)}{f'(r)}$, then $\varphi \varepsilon I_p$ for some p with $p \ge 4$.

Proof. Let r be a simple zero of $f(x)$. Since $\phi(x)$ is an iteration function of order two, then we have $\phi(r) = r$ and $\phi'(r) = 0$. Expanding $f(x_n)$ and $\phi(x_n)$ about r by a Taylor series expansion, we have

$$f(x_n) = f'(r)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)\right), \tag{4.2}$$

and

$$\phi(x_n) = r + \frac{1}{2}\phi''(r)e_n^2 + \frac{1}{6}\phi'''(r)e_n^3 + O(e_n^4), \tag{4.3}$$

respectively, where $e_n = x_n - r$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(r)}{f'(r)},\ k = 2, 3, \ldots.$

Furthermore

$$x_n - \phi(x_n) = e_n - \frac{1}{2}\phi''(r)e_n^2 - \frac{1}{6}\phi'''(r)e_n^3 + O(e_n^4), \tag{4.4}$$

and in combination with the Taylor series expansion of $f(\phi(x_n))$ about r, we have

$$f(\phi(x_n)) = f'(r)\left\{\frac{1}{2}\phi''(r)e_n^2 + \frac{1}{6}\phi'''(r)e_n^3\right\} + O(e_n^4), \tag{4.5}$$

and

$$f(x_n) - 2\beta f(\phi(x_n)) = f'(r)\left\{e_n + (c_2 - \beta\phi''(r))e_n^2 + \left(c_3 - \frac{\beta}{3}\phi'''(r)\right)e_n^3\right\} + O(e_n^4). \tag{4.6}$$

Using (4.3)–(4.6), we get

$$\begin{aligned}
&\frac{\beta f(\phi(x_n))(x_n - \phi(x_n))(f(x_n) - 2\beta f(\phi(x_n)))}{(f(x_n) - 2\beta f(\phi(x_n)))^2 + \beta\alpha f(\phi(x_n))(x_n - \phi(x_n))^2} \\
&= \frac{\beta\phi''(r)}{2}e_n^2 + \beta\left\{\frac{\{6\beta - 3\}\phi''^2(r) + 2\phi'''(r) - 6\phi''(r)c_2}{12}\right\}e_n^3 \\
&\quad + \frac{1}{12f'(r)}\{\beta\phi''(r)(3\alpha\beta\phi''(r) - (2\beta - 1)f'(r)(3\beta\phi''^2(r) + 2\phi'''(r))) \\
&\quad + f'(r)c_2(3(4\beta - 1)\phi''^2(r) + 2\phi'''(r) - 6\phi''(r)c_2) + 6f'(r)\phi''(r)c_3\}e_n^4 + O(e_n^5).
\end{aligned} \tag{4.7}$$

Thus, using (4.3) and (4.7) in (4.1), we have

$$\begin{aligned}
\varphi(x_{n+1}) \equiv x_{n+1} &= r + \frac{(1 - \beta)\phi''(r)}{2}e_n^2 + \left[\frac{2\phi'''(r) - \beta\{(6\beta - 3)\phi''^2(r) + 2\phi'''(r) - 6\phi''(r)c_2\}}{12}\right]e_n^3 \\
&\quad + \frac{1}{12f'(r)}\{\beta\phi''(r)(3\alpha\beta\phi''(r) - (2\beta - 1)f'(r)(3\beta\phi''^2(r) + 2\phi'''(r))) \\
&\quad + f'(r)c_2(3(4\beta - 1)\phi''^2(r) + 2\phi'''(r) - 6\phi''(r)c_2) + 6f'(r)\phi''(r)c_3\}e_n^4 + O(e_n^5),
\end{aligned} \tag{4.8}$$

which further implies that

$$\begin{aligned}
e_{n+1} &= \frac{1 - \beta}{2}\phi''(r)e_n^2 + \frac{1}{12}[2\phi'''(r) - \beta\{(6\beta - 3)\phi''^2(r) + 2\phi'''(r) - 6\phi''(r)c_2\}]e_n^3 \\
&\quad + \frac{1}{12f'(r)}\{\beta\phi''(r)(3\alpha\beta\phi''(r) - (2\beta - 1)f'(r)(3\beta\phi''^2(r) + 2\phi'''(r))) \\
&\quad + f'(r)c_2(3(4\beta - 1)\phi''^2(r) + 2\phi'''(r) - 6\phi''(r)c_2) + 6f'(r)\phi''(r)c_3\}e_n^4 + O(e_n^5).
\end{aligned} \tag{4.9}$$

Therefore, when $\beta = 1$, we have the following error equation:

$$\begin{aligned}
e_{n+1} &= \frac{1}{4}\phi''(r)(2c_2 - \phi''(r))e_n^3 + \frac{1}{12f'(r)}\{\beta\phi''(r)(3\alpha\beta\phi''(r) - (2\beta - 1)f'(r)(3\beta\phi''^2(r) + 2\phi'''(r))) \\
&\quad + f'(r)c_2(3(4\beta - 1)\phi''^2(r) + 2\phi'''(r) - 6\phi''(r)c_2) + 6f'(r)\phi''(r)c_3\}e_n^4 + O(e_n^5).
\end{aligned} \tag{4.10}$$

This means that the iteration function φ defined by (4.1) is of order at least three. Furthermore, when $\phi''(r) = 0$ or $\phi''(r) = \frac{f''(r)}{f'(r)}$, we can observe from (4.10) that φ is of order at least four. This completes the proof. □

It should be noted that the result of Theorem 4.1 is independent of the structure of the iteration function of order two involved. Therefore, this is a further modification of Chun's basic tool for deriving the families of higher order iterative methods. This idea can further be extended for the case of multiple roots if the quadratically convergent methods for the multiple roots are taken into consideration.

5. Examples

Example 5.1. Consider the Newton's scheme defined by $\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$. In this case, $\varphi(x_{n+1})$ defined by (4.1) becomes

$$\varphi(x_{n+1}) = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{\beta f'^2(x_n) f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\left\{f(x_n) - 2\beta f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right\}}{f'^2(x_n)\left\{f(x_n) - 2\beta f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right\}^2 + \alpha\beta f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right) f^2(x_n)}\right]. \tag{5.1}$$

This is the new family of Ostrowski's method and also has order of convergence four [1–5]. This exactly agrees with the result predicted by Theorem 4.1, since $\phi''(r) = \frac{f''(r)}{f'(r)}$ and the error equation of the above family when $\beta = 1$ is

$$e_{n+1} = \left\{c_2^2\left(c_2 + \frac{\alpha}{f'(r)}\right) - c_2 c_3\right\} e_n^4 + O(e_n^5). \tag{5.2}$$

Example 5.2. Consider the Stirling's scheme of order two defined by $\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n - f(x_n))}$. In this case, $\varphi(x_{n+1})$ defined by (4.1) becomes

$$\varphi(x_{n+1}) = x_n - \frac{f(x_n)}{f'(x_n - f(x_n))} \times \left[1 + \frac{\beta f\left(x_n - \frac{f(x_n)}{f'(x_n - f(x_n))}\right) f'^2(x_n - f(x_n))\left\{f(x_n) - 2\beta f\left(x_n - \frac{f(x_n)}{f'(x_n - f(x_n))}\right)\right\}}{f'^2(x_n - f(x_n))\left\{f(x_n) - 2\beta f\left(x_n - \frac{f(x_n)}{f'(x_n - f(x_n))}\right)\right\}^2 + \alpha\beta f^2(x_n) f\left(x_n - \frac{f(x_n)}{f'(x_n - f(x_n))}\right)}\right]. \tag{5.3}$$

By Theorem 4.1, the order of convergence of the iteration function $\varphi(x_{n+1})$ defined by (5.3) is at least three. From an elementary computation for the coefficient of e_n^3 in the error equation (4.10), we have the following error equation:

$$e_{n+1} = c_2 f''(r)(1 - 2f'(r))e_n^3 + O(e_n^4). \tag{5.4}$$

Example 5.3. Consider the second-order scheme defined in [9,10] by $\phi(x_n) = x_n - \frac{f(x_n)}{f(x_n) + f'(x_n)}$. In this case the iteration scheme $\varphi(x_{n+1})$ defined by (4.1) becomes

$$\varphi(x_{n+1}) = x_n - \frac{f(x_n)}{f(x_n) + f'(x_n)} \times \left[1 + \frac{\beta f\left(x_n - \frac{f(x_n)}{f(x_n)+f'(x_n)}\right)\{f(x_n) + f'(x_n)\}^2\left\{f(x_n) - 2\beta f\left(x_n - \frac{f(x_n)}{f(x_n)+f'(x_n)}\right)\right\}}{\{f(x_n) + f'(x_n)\}^2\left\{f(x_n) - 2\beta f\left(x_n - \frac{f(x_n)}{f(x_n)+f'(x_n)}\right)\right\}^2 + \alpha\beta f\left(x_n - \frac{f(x_n)}{f(x_n)+f'(x_n)}\right) f^2(x_n)}\right]. \tag{5.5}$$

The order of convergence of this iteration function $\varphi(x_{n+1})$ is three and has the following error equation:

$$e_{n+1} = -(1 + c_2)e_n^3 + O(e_n^4). \tag{5.6}$$

Example 5.4. Consider the Steffensen's scheme defined by $\phi(x_n) = x_n - \frac{f(x_n)}{g(x_n)}$, where $g(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$, then the iteration scheme defined by (4.1) becomes

$$\varphi(x_{n+1}) = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} \times \left[1 + \frac{\beta f(y_{n+1})\{f(x_n) - 2\beta f(y_{n+1})\}\{f(x_n - f(x_n)) - f(x)\}^2}{\{f(x_n - f(x_n)) - f(x)\}^2\{f(x_n) - 2\beta f(y_{n+1})\} + \alpha\beta f(y_{n+1}) f^4(x_n)}\right], \tag{5.7}$$

where $y_{n+1} = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)}$.

This scheme is again cubically convergent and has the following error equation:

$$e_{n+1} = -\frac{1}{4} f''(r)(2c_2 + f''(r))e_n^3 + O(e_n^4). \tag{5.8}$$

Example 5.5. Consider the Mamta et al. scheme [10] defined by $\phi(x_n) = x_n - \frac{f(x_n)f'(x_n)}{f^2(x_n) + f'^2(x_n)}$. In this case we have $\phi''(r) = \frac{f''(r)}{f'(r)}$. Hence, by Theorem 4.1, the iteration function $\varphi(x_{n+1})$ defined by (4.1) becomes

$$\varphi(x_{n+1}) = x_n - \frac{f(x_n)f'(x_n)}{f^2(x_n)+f'^2(x_n)} \times \left[1 + \frac{\beta f(y_{n+1})\{f(x_n) - 2\beta f(y_{n+1})\}\{f^2(x_n)+f'^2(x_n)\}^2}{\{f^2(x_n)+f'^2(x_n)\}^2\{f(x_n)-2\beta f(y_{n+1})\}^2 + \alpha\beta f(y_{n+1})f^2(x_n)f'^2(x_n)}\right], \tag{5.9}$$

where $y_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f^2(x_n)+f'^2(x_n)}$.

Taking a particular value of $\beta = 1$, we get the following error equation of the above method as:

$$e_{n+1} = c_2\left(c_2^2 - c_3 + \frac{\alpha c_2}{f'(r)} - 1\right)e_n^4 + O(e_n^5). \tag{5.10}$$

It should be pointed out that the obtained family of methods (5.1), (5.3), (5.5), (5.7) and (5.9) converge cubically or quartically even though per iteration they require two evaluations of $f(x)$ and one of $f'(x)$ or three evaluations of $f(x)$ and none of $f'(x)$. In a similar fashion, we can continuously construct other families of iterative methods with at least cubic convergence by making use of quadratically convergent iterative methods as long as they are available.

Similarly, if we take two points $(\phi(x_0), f(\phi(x_0)))$ and $\left(\frac{\phi(x_0)+x_0}{2}, \frac{f(\phi(x_0))+f(x_0)}{2}\right)$ on the graph of the function $y = f(x)$, and using the same concept as mentioned above, we get

$$\varphi(x_{n+1}) = \phi(x_n) - \frac{f(\phi(x_n))(x_n - \phi(x_n))(f(x_n) - f(\phi(x_n)))}{(f(x_n) - f(\phi(x_n)))^2 + \alpha f(\phi(x_n))(x_n - \phi(x_n))^2}. \tag{5.11}$$

This family has the following error equation:

$$e_{n+1} = \frac{1}{2}\phi''(r)c_2e_n^3 + O(e_n^4). \tag{5.12}$$

If we take $\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$ as the Netwon's iterate, then we get the cubically convergent family of the Newton-Secant method given by

$$\varphi(x_{n+1}) = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{f(\phi(x_n))(f(x_n) - f(\phi(x_n)))f'^2(x_n)}{f'^2(x_n)(f(x_n) - f(\phi(x_n)))^2 + \alpha f(\phi(x_n))f^2(x_n)}\right]. \tag{5.13}$$

The error equation for this family is given by

$$e_{n+1} = c_2^2e_n^3 + O(e_n^4). \tag{5.14}$$

It is straightforward to see that per step these methods require three functional evaluations, viz. two evaluations of $f(x)$ and one of $f'(x)$ or three evaluations of $f(x)$ and none of $f'(x)$. In order to obtain an assessment of the efficiency of our methods, we shall make use of the efficiency index defined by Eq. (2.3). For our proposed iteration schemes, namely (5.3), (5.5) and (5.7), we find $p = 3$ and $d = 3$, yielding $E = \sqrt[3]{3} \approx 1.442$, which is better than $E = \sqrt{2} \approx 1.414$, the efficiency index of the Newton's method. For the family of methods, namely (5.1) and (5.8), we find $p = 4$ and $d = 3$, yielding $E = \sqrt[3]{4} \approx 1.587$, which is better than those of most third order methods $E \approx 1.442$ and Newton's method $E \approx 1.414$.

6. Numerical experiments

In this section, we shall present the numerical results obtained by employing various methods, namely Newton's method (NM), Traub–Ostrowski's method (also known as Ostrowski's method) (TOM), Jarratt's method (JM), modified Traub–Ostrowski's method (MTOM) (5.2) for $\alpha = -1.0, \alpha = 0.5, \alpha = 1.0, \alpha = 10$, and King's method (KM) (1.3) for $\gamma = 0.5, \gamma = 1$ respectively to solve the nonlinear equations given in Table 1. We also show the comparison of all methods mentioned above in Table 2; computations were performed using C++ in double precision arithmetic. We use $\varepsilon = 10^{-15}$ as the tolerable error. The following stopping criteria are used for the computer programs:

(i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_{n+1})| < \epsilon$.

7. Conclusions

In this paper, we have modified Chun's scheme for constructing iterative methods of order three or higher to solve nonlinear equations numerically. Now we have obtained a wide class of general methods which are without memory and include three functional evaluations per iteration. A fourth-order family of Ostrowski's method is the main finding of the present contribution in terms of speed and efficiency index. According to the Kung–Traub conjecture, the family of Ostrowski's method and the family (5.9) have the maximal efficiency index because only three function values are needed per step. The numerical results presented in Table 2 overwhelmingly support that the new family of Ostrowski's method

Table 1
Test problems.

No.	Problem	$[a, b]$	Initial guess	Root(r)
1.	$e^x - 4x^2 = 0$	[0.5, 2]	0.5 2.0	0.714805901050568
2.	$x^3 + 4x^2 - 10 = 0$	[1, 2]	1.0 2.0	1.365229964256287
3.	$\cos x - x = 0$	[0, 2]	0.0 2.0	0.739085137844086
4.	$x^2 - e^x - 3x + 2 = 0$	[0, 1]	0.0 1.0	0.257530301809311
5.	$xe^{x^2} - \sin x^2 + 3\cos x + 5 = 0$	[−1.5, −0.5]	−1.5 −0.5	−1.201576113700867
6.	$\sin^2 x - x^2 + 1 = 0$	[1, 3]	1 3	1.404491662979126
7.	$e^{x^2+7x-30} - 1 = 0$	[2.9, 3.5]	2.9 3.5	3.000000000000000

Table 2
Total number of iterations to approximate the zero of a function, total number of function evaluations for various multipoint iterative methods.

Problem	Initial guess	NM	TOM	JM	MTOM $\alpha = -1.0$	MTOM $(\alpha = 0.5)$	MTOM $(\alpha = 1)$	MTOM $(\alpha = 10)$	KM $(\gamma = 0.5)$	KM $(\gamma = 1)$
1.	0.5	(4, 8)	(2, 6)	(2, 6)	(3, 9)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(3, 9)
	2.0	(5, 10)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)
2.	1.0	(4, 8)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(4, 12)	(2, 6)
	2.0	(4, 8)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(4, 12)	(3, 9)
3.	0.0	(4, 8)	(2, 6)	(3, 9)	(3, 9)	(2, 6)	(3, 9)	(3, 9)	(3, 9)	(3, 9)
	2.0	(3, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)
4.	0.0	(3, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(3, 9)
	1.0	(3, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(4, 12)	(2, 6)
5.	−1.5	(5, 10)	(2, 6)	(3, 9)	(2, 6)	(2, 6)	(2, 6)	(2, 6)	(3, 9)	(3, 9)
	−0.5	(9, 18)	(3, 9)	(3, 9)	(3, 9)	(2, 6)	(3, 9)	(3, 9)	D	D
6.	1.0	(5, 10)	(2, 6)	(3, 9)	(3, 9)	(2, 6)	(2, 6)	(5, 15)	(3, 9)	(3, 9)
	3.0	(5, 10)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)
7.	2.9	(6, 18)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(3, 9)	(5, 15)	(8, 24)
	3.5	(11, 22)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(5, 15)	(6, 18)	(6, 18)

(D above stands for divergent).

is equally competent to Ostrowski's method, Jarratt's method and the King's family. Further, we have also determined that the family of Ostrowski's method gives a very good approximation to the required root when $|\alpha|$ (the scaling parameter) is small. This is because, for small values of α, the parabola widens along the horizontal direction. This means that our next approximation will move faster towards the desired root. For large values of α (provided that the inequality (3.9) holds), the formula still works but takes a greater number of iterations as compared to smaller values of α. This idea can be further extended to the case of multiple roots.

Acknowledgments

We would like to record our sincerest thanks to the anonymous reviewers for their constructive suggestion and remarks which have considerably contributed to the readability of this paper. Ramandeep Behl further acknowledges the financial support of CSIR, New Delhi, India.

References

[1] A.M. Ostrowski, Solution of Equations in Euclidean and Banach Space, Academic Press, New York, 1973.
[2] A.M. Ostrowski, Solutions of Equations and System of Equations, Academic Press, New York, 1960.
[3] J.F. Traub, Iterative Methods for the Solution of Equations, Prentice-Hall, Englewood Cliffs, NJ, 1964.
[4] L.W. Johnson, R.D. Roiess, Numerical Analysis, Addison-Wesley, Reading, MA, 1977.
[5] C. Chun, On the construction of iterative methods with at least cubic convergence, Appl. Math. Comput. 189 (2007) 1384–1392.
[6] H.T. Kung, J.F. Traub, Optimal order of one-point and multipoint iteration, J. ACM 21 (4) (1974) 643–651.
[7] P. Jarratt, Some efficient fourth-order multipoint methods for solving equations, BIT 9 (1969) 119–124.
[8] R.F. King, A family of fourth order methods for nonlinear equations, SIAM J. Numer. Anal. 10 (1973) 876–879.
[9] X.Y. Wu, A new continuation Newton-like method and its deformation, Appl. Math. Comput. 166 (3) (2005) 633–637.
[10] Mamta, V. Kanwar, V.K. Kukreja, S. Singh, On a class of quadratically convergent iteration formulae, Appl. Math. Comput. 166 (3) (2005) 633–637.
[11] P. Sargolzaei, F. Soleymani, Accurate fourteenth-order methods for solving nonlinear equations, Numer. Algorithms (2011) doi:10.1007/s11075-011-94674.
[12] Y.H. Geum, Y.I. Kim, A family of optimal sixteenth-order multipoint methods with a linear fraction plus a trivariate polynomial as the fourth-step weighting function, Comput. Math. Appl. 61 (11) (2011) 3278–3287.
[13] V. Kanwar, J.R. Sharma, Mamta, A new family of Secant-like methods with super-linear convergence, Appl. Math. Comput. 171 (2005) 104–107.