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Total restrained domination in claw-free graphs with minimum degree at least two

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ABSTRACT

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a total restrained dominating set if every vertex is adjacent to a vertex in S and every vertex in $V - S$ is adjacent to a vertex in $V - S$. The total restrained domination number of G, denoted $\gamma_{tr}(G)$, is the smallest cardinality of a total restrained dominating set of G. We will show that if G is claw-free, connected, has minimum degree at least two and G is not one of nine exceptional graphs, then $\gamma_{tr}(G) \le \frac{4n}{7}$.

1. Introduction

For notation and graph theory terminology we, in general, follow [6]. Specifically, let $G = (V, E)$ be a graph with vertex set V and edge set E. For a set $S \subseteq V$, the subgraph induced by S is denoted $\langle S \rangle_G$ or just $\langle S \rangle$ if the context is clear. If G_1 is an induced subgraph of G, then $G - G_1$ will denote the induced subgraph $\langle V(G) - V(G_1) \rangle$. If $\mathcal{K}$ is a set of graphs and G has a component that is isomorphic to a graph in $\mathcal{K}$, then we will say that G has a component in $\mathcal{K}$. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$).

A set $S \subseteq V$ is a *dominating set* of G, denoted *DS*, if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a *DS*. The concept of domination in graphs, with its many variations, is now well studied in graph theory. A thorough study of domination appears in [6,7].

A set $S \subseteq V$ is a *total restrained dominating set*, denoted *TRDS*, if every vertex is adjacent to a vertex in S and every vertex in $V - S$ is adjacent to a vertex in $V - S$. Every graph without isolated vertices has a total restrained dominating set, since $S = V$ is such a set. The *total restrained domination number* of G, denoted $\gamma_{tr}(G)$, is the minimum cardinality of a *TRDS* of G. If $|S| = \gamma_{tr}(G)$, then S will be referred to as a γ_{tr}-set. Total restrained domination was introduced by Telle and Proskurowski [10], albeit indirectly, as a vertex partitioning problem and further studied, for example, in [1,9,2,3,5,4,8,11]. A specific application of the total restrained domination number of a graph is discussed in [1].

The number of vertices (edges respectively) of a graph is denoted by n (m respectively). A graph G is said to be *claw-free* if for any vertex u of degree at least three, we have that if $v, w, x \in N(u)$, then $\langle \{v, w, x, u\} \rangle$ is not isomorphic to $K_{1,3}$. In the other case, $\langle \{v, w, x, u\} \rangle$ is a claw and u is called the center of the claw.

In [8], total restrained domination was studied in graphs with minimum degree at least two. The following bound was derived.

Theorem 1. *If G is a connected graph with $n \ge 4$, $\delta \ge 2$ and $\Delta \le n - 2$ then* $\gamma_{tr}(G) \le n - \frac{\Delta}{2} - 1$.

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In [9] the bound in Theorem 1 was further improved for claw-free graphs with $\Delta \geq 6$, with the following result:

Theorem 2. *If G is a connected claw-free graph with $n \geq 4$, $\delta \geq 2$ and $\Delta \leq n-2$ then $\gamma_{\rm tr}(G) \leq n - \Delta + 1$.*

Let $\mathcal{B}$ be the graph obtained by joining a degree two vertex of a C_3 to the two vertices of a K_2. Let $\mathcal{K} = \{\mathcal{B}, C_3, C_5, C_6, C_7, C_{10}, C_{11}, C_{15}, C_{19}\}$. In [1] the following result is derived:

Proposition 3. *If $n = 4q + r$ for some positive integer q and $r \in \{0, 1, 2, 3\}$, then $\gamma_{\rm tr}(C_n) = 2q + r$.*

Proposition 4. *Let G be a claw-free graph with $\delta(G) \geq 2$. The following conditions hold:*

1. *If $G = \mathcal{B}$ then $\gamma_{\rm tr}(G) > \frac{4n}{7}$.*
2. *If $G = C_n$ then $\gamma_{\rm tr}(C_n) \leq \frac{4n}{7}$ if and only if $n \notin \{3, 5, 6, 7, 10, 11, 15, 19\}$.*

Consider a path $P : x = v_0, v_1, v_2, \ldots, v_{j+1} = y$, where $\deg(v_i) = 2$, for $i = 1, 2, \ldots, j$, and $\deg(x) \geq 3$ and $\deg(y) \geq 3$. The path P will be called a 2-path. If we set $x = y$ then P will be called a 2-cycle. For a 2-path P define $\mathcal{H}(P) = \{v \in V(G) - V(P) \mid v \in N(x) \cap N(y)\}$.

Throughout the paper, if G has an induced subgraph D in $\mathcal{K}$ then we will let the vertices $u_1, u_2, \ldots, u_i$ denote the Hamiltonian path of D (by default), with $i \in \{3, 5, 6, 7, 10, 11, 15, 19\}$, provided that the vertex labels have not already been used. Clearly, if $D = \mathcal{B}$ then $\deg(u_3) = 4$.

Consider a path $P' : v_1, v_2, \ldots, v_j$, with $2 \leq j \leq 4$. Let $\deg(v_k) = 1$ for $k \in \{1, j\}$ and $\deg(v_k) = 2$ for $k \in \{2, 3, \ldots, j-1\}$. Let $D \in \mathcal{K}$. We describe the following construction.

Construction:

1. If $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$ then join v_j to at least u_1 and u_2 in D.
2. If $D = C_3$ and $j \geq 3$ then join v_j to at least u_1 in D.
3. If $D = \mathcal{B}$ then join v_j to at least u_1 in D.

Additional edges may be added between the vertices of $P' - \{v_j\}$ and D and between the non-adjacent vertices of P'.

If a graph is obtained by using the construction mentioned above, we will refer to it as a *necklace*. The path P' will be referred to as the *attachment*. We will refer to v_1 (v_j respectively) as the *initial vertex* (*end vertex* respectively) of the necklace.

We are now ready to proceed to the main result of this paper. We shall show the following:

Theorem 5. *Let G be a connected claw-free graph with minimum degree at least two. If $G \notin \mathcal{K}$, then $\gamma_{\rm tr}(G) \leq \frac{4n}{7}$.*

2. Proof of Theorem 5

We will prove our main result by contradiction. Suppose, to the contrary, that there is at least one connected claw-free graph J of order n and minimum degree at least two, such that $J \notin \mathcal{K}$ and $\gamma_{\rm tr}(J) > \frac{4n}{7}$. Among all the counter examples, choose G to have minimum size.

The proof of Theorem 5 will follow from a series of key lemmas. We will start by making a few observations that will play an essential part in the proofs of these lemmas.

Suppose that G has a necklace G_1 as an induced subgraph. For the purpose of the next observation, we will label the vertices of G_1 in exactly the same way as described in our construction. Let $G_2 = G - G_1$.

Observation 6. *Suppose that $V(G_2) = \emptyset$, or $\delta(G_2) \geq 2$ and G_2 has no components in $\mathcal{K}$. Then:*

1. *The attachment of the necklace G_1 has length at most two.*
2. *The initial vertex of the necklace G_1 has no neighbor in $V(G_2)$.*

Proof. Let S be any $\gamma_{\rm tr}$-set of G_2. Note that G_2 is claw-free. Suppose that $V(G_2) = \emptyset$, or $\delta(G_2) \geq 2$ and G_2 has no components in $\mathcal{K}$. Suppose, to the contrary, that the attachment either has length three or that it has length at most two, with v_1 having a neighbor v in $V(G_2)$. Note that in the case where the attachment has length three, v_1 has a neighbor v in $V(G_2) \cup (V(G_1) - \{v_1, v_2\})$. Since G_2 has smaller size than G, it follows that $|S| \leq \frac{4n(G_2)}{7} = \frac{4(n-i-j)}{7}$. We will produce a contradiction by showing that $\gamma_{\rm tr}(G) \leq \frac{4n}{7}$. Let $i = 4q + r$, where q is a positive integer and $r \in \{1, 2, 3\}$.

Case 1: $i = 4q + 3$.

We will first look at the case where $j = 4$. If $q \geq 1$ then the set $S \cup \{v_1, v_2, u_1, u_i\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+3}, u_{4k+4}\}$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \leq |S| + 4 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q+7) + \frac{14q}{7} + \frac{28}{7} \leq \frac{4n}{7}$. If $q = 0$ then if $v \in V(G_2) - S$ or $v \in \{u_1, v_4\}$ then $S \cup \{v_2, v_3, u_2, u_3\}$ is a *TRDS* of G. If $v \in S$ or $v = v_3$ then $S \cup \{v_3, v_4, u_1\}$ is a *TRDS* of G. If $v \in \{u_2, u_3\}$ then $S \cup \{v_2, v_3, v_4, u_1\}$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} = \frac{4n}{7}$.

Suppose that $j = 3$. If $q \geq 1$ and $v \notin S$ then the set $S \cup \{v_2, v_3, u_{i-1}\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$ is a *TRDS* of G. If $v \in S$ then the set $S \cup \{v_3, u_1\} \cup \bigcup_{k=0}^{q-1} \{u_{4k+4}, u_{4k+5}\}$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \leq |S| + 3 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q+6) + \frac{14q}{7} + \frac{21}{7} \leq \frac{4n}{7}$. If $q = 0$ then if $v \in S$ then $S \cup \{v_3, u_1\}$ is a *TRDS* of G. If $v \notin S$ then $S \cup \{v_2, v_3, u_1\}$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \leq |S| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} \leq \frac{4n}{7}$.

Suppose $j = 2$. If $q = 0$ then $i = 3$. By the second point of the construction mentioned above, we must have that $j \geq 3$ which is a contradiction. It follows that $q \geq 1$. The set $S \cup \{v_1, v_2, u_1\} \cup \bigcup_{k=0}^{q-1}\{u_{4k+4}, u_{4k+5}\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 3 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q+5) + \frac{14q}{7} + \frac{21}{7} \leq \frac{4n}{7}$.

Case 2: $i = 4q + 2$.

We will first look at the case where $j = 4$. If $v \in V(G_2) - S$ or $v \in V(G_1) - \{v_1, v_2, v_3, v_4\} - \bigcup_{k=0}^{q-1}\{u_{4k+4}, u_{4k+5}\}$ then the set $S \cup \{v_2, v_3, v_4\} \cup \bigcup_{k=0}^{q-1}\{u_{4k+4}, u_{4k+5}\}$ is a *TRDS* of G. If $v \in S$ or $v \in \{v_3, v_4\} \cup \bigcup_{k=0}^{q-1}\{u_{4k+4}, u_{4k+5}\}$ then the set $S \cup \{v_3, v_4\} \cup \bigcup_{k=0}^{q-1}\{u_{4k+4}, u_{4k+5}\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 3 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q+6) + \frac{14q}{7} + \frac{21}{7} \leq \frac{4n}{7}$.

Suppose that $j = 3$. If $v \notin S$ then the set $S \cup \{v_2, v_3\} \cup \bigcup_{k=0}^{q-1}\{u_{4k+4}, u_{4k+5}\}$ is a *TRDS* of G. If $v \in S$ then the set $S \cup \{v_1, u_1, u_i\} \cup \bigcup_{k=0}^{q-1}\{u_{4k+4}, u_{4k+5}\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 3 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q+5) + \frac{14q}{7} + \frac{21}{7} \leq \frac{4n}{7}$.

Suppose $j = 2$. The set $S \cup \{v_1, v_2\} \cup \bigcup_{k=0}^{q-1}\{u_{4k+4}, u_{4k+5}\}$ is a *TRDS* of G when $v \in S$ or $v \notin S$. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 2 + 2q \leq \frac{4n}{7} - \frac{4}{7}(4q+4) + \frac{14q}{7} + \frac{14}{7} \leq \frac{4n}{7}$.

Case 3: $i = 5$.

We consider first the case where $D = C_5$. Suppose first that $j = 4$. The set $S \cup \{v_1, v_2, u_1, u_4, u_5\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 5 \leq \frac{4n}{7} - \frac{36}{7} + \frac{35}{7} \leq \frac{4n}{7}$.

Suppose that $j = 3$. If $v \notin S$ then the set $S \cup \{v_2, v_3, u_4, u_5\}$ is a *TRDS* of G. If $v \in S$ then the set $S \cup \{v_3, u_1, u_4, u_5\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{32}{7} + \frac{28}{7} \leq \frac{4n}{7}$.

Suppose $j = 2$. If $v \notin S$ then the set $S \cup \{u_1, u_5, u_4, v_2\}$ is a *TRDS* of G. If $v \in S$ then the set $S \cup \{u_1, u_5, u_4\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} = \frac{4n}{7}$.

We now consider the case where $D = \mathcal{B}$. If $j = 4$ then the set $S \cup \{v_1, v_2, u_1, u_2, u_3\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 5 \leq \frac{4n}{7} - \frac{36}{7} + \frac{35}{7} \leq \frac{4n}{7}$.

If $j = 3$ then the set $S \cup \{v_1, v_2, u_2, u_3\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{32}{7} + \frac{28}{7} \leq \frac{4n}{7}$.

Suppose $j = 2$. If $v \notin S$ then the set $S \cup \{u_1, u_5, u_4, v_2\}$ is a *TRDS* of G. If $v \in S$ then the set $S \cup \{v_1, u_3, u_4, u_5\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} = \frac{4n}{7}$. □

Consider a path $P_A : v_1, v_2, v_3, v_4$ of G, where either v_1 and v_4 are adjacent or v_1 is adjacent to a vertex $v_0 \in V(G) - V(P_A)$ and v_4 is adjacent to a vertex $v_5 \in V(G) - V(P_A)$. Let $G_2 = G - \langle\{v_1, v_2, v_3, v_4\}\rangle$. We form the graph G_A from G_2 as follows. If v_1 is adjacent to v_4 or $v_0 = v_5$, then let $G_A = G_2$. If $v_0 \neq v_5$ then either let $G_A = G_2$ if v_0 is adjacent to v_5 or form G_A from G_2 by joining v_0 and v_5. For a path $P_B : v_1, v_2, v_3, v_4$ where v_4 is adjacent to v_2, we let $G_B = G - \langle\{v_1, v_2, v_3, v_4\}\rangle$.

Observation 7. *If G_A (G_B respectively) is claw-free, has no components in $\mathcal{K}$ and $\delta(G_A) \geq 2$ ($\delta(G_B) \geq 2$ respectively) then $\gamma_{\mathrm{tr}}(G) \leq \frac{4n}{7}$, a contradiction.*

Proof. Suppose that G_A (G_B respectively) is claw-free, has no components in $\mathcal{K}$ and $\delta(G_A) \geq 2$ ($\delta(G_B) \geq 2$ respectively). Since G_A (G_B respectively) is of smaller size than G it follows that $\gamma_{\mathrm{tr}}(G_A) \leq \frac{4(n-4)}{7}$ $\left(\gamma_{\mathrm{tr}}(G_B) \leq \frac{4(n-4)}{7} \text{ respectively}\right)$.

Now for a γ_{tr}-set S of G_B we have that $S \cup \{v_1, v_2\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{16}{7} + \frac{14}{7} \leq \frac{4n}{7}$.

For a γ_{tr}-set S of G_A we have that if $v_0, v_5 \notin S$ or v_1 is adjacent to v_4, then $S \cup \{v_2, v_3\}$ is a *TRDS* of G. If, without loss of generality, $v_0 \notin S$ and $v_5 \in S$, then $S \cup \{v_1, v_2\}$ is a *TRDS* of G. If $v_0, v_5 \in S$ then $S \cup \{v_1, v_4\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{16}{7} + \frac{14}{7} \leq \frac{4n}{7}$. □

Observation 8. *Suppose that $D \in \mathcal{K}$ is an induced subgraph of G. If $v \in V(G) - V(D)$ and v is adjacent to a vertex of D then we have the following:*

1. *If $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$ then v is adjacent to two consecutive vertices on the cycle D.*
2. *If $D = \mathcal{B}$ then $N(v) \cap (V(D) - \{u_3\}) \neq \emptyset$.*

Proof. Suppose that D is an induced subgraph of G, where $D \in \mathcal{K}$. Furthermore, let $v \in V(G) - V(D)$ where v is adjacent to a vertex of D. If $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$ then suppose that v is adjacent to say u_1. Since $\langle\{v, u_i, u_2, u_1\}\rangle$ is not a claw and u_i is not adjacent to u_2 we have that v is adjacent to say u_2. Hence, v is adjacent to two consecutive vertices on the cycle D. If $D = \mathcal{B}$ and v is adjacent to u_3 then since $\langle\{v, u_4, u_1, u_3\}\rangle$ is not a claw and u_4 is not adjacent to u_1 we have that v is adjacent to a vertex in $V(D) - \{u_3\}$. □

Let $P : v_1, v_2, v_3, v_4$ be a path of G. Define $G_1 = \langle V(P)\rangle$ and $G_2 = G - G_1$.

Observation 9. *Suppose that $\delta(G_2) \geq 2$ and that G_2 is disconnected. If G_2 consists of exactly two components $\mathcal{U}$ and $\mathcal{U}'$ and v_1 (v_4 respectively) is adjacent to a vertex of $\mathcal{U}$ ($\mathcal{U}'$ respectively) then $\mathcal{U} \notin \{C_3, \mathcal{B}\}$ or $\mathcal{U}' \notin \{C_3, \mathcal{B}\}$.*

Proof. Let G_2 be disconnected and let $\mathcal{U}$ and $\mathcal{U}'$ be the two components of G_2. Suppose that v_1 (v_4 respectively) is adjacent to a vertex of $\mathcal{U}$ ($\mathcal{U}'$ respectively). Assume, to the contrary, that $\mathcal{U} \in \{C_3, \mathcal{B}\}$ and $\mathcal{U}' \in \{C_3, \mathcal{B}\}$. Let $P' : u_1, \ldots, u_i$ ($P'' : w_1, \ldots, w_j$ respectively) be the Hamiltonian path of $\mathcal{U}$ ($\mathcal{U}'$ respectively). Using Observation 8, we have that v_1 (v_4 respectively) is adjacent to say u_1 (w_1 respectively).

By symmetry, we need only to consider the case where $\mathcal{U} = C_3$ and $\mathcal{U}' \in \{C_3, \mathcal{B}\}$ and the case where $\mathcal{U} = \mathcal{U}' = \mathcal{B}$. If $\mathcal{U} = \mathcal{U}' = C_3$ then $n(G) = 10$ and the set $\{v_1, u_1, v_4, w_1\}$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq 4 < \frac{4.10}{7} = \frac{4n}{7}$ which is a contradiction. If $\mathcal{U} = C_3$ and $\mathcal{U}' = \mathcal{B}$ then $n(G) = 12$ and the set $\{v_1, u_1, v_4, w_1, w_2, w_3\}$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq 6 < \frac{4.12}{7} = \frac{4n}{7}$ which is a contradiction. If $\mathcal{U} = \mathcal{U}' = \mathcal{B}$ then $n(G) = 14$ and the set $\{v_1, u_1, u_2, u_3, v_4, w_1, w_2, w_3\}$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq 8 = \frac{4.14}{7} = \frac{4n}{7}$ which is a contradiction. □

Suppose that $D \in \mathcal{K}$ is an induced subgraph of G and let $v \in V(G) - V(D)$ be adjacent to a vertex of D. Define $G_1 = \langle V(D) \cup \{v\}\rangle$ and $G_2 = G - G_1$, and suppose that $V(G_2) \neq \emptyset$.

Observation 10. *We have the following:*

1. *If $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$ then G_2 has at least one component in $\mathcal{K}$ or $\delta(G_2) \leq 1$.*
2. *If $D = \mathcal{B}$ and v is adjacent to a vertex $w \in V(G_2)$ then G_2 has at least one component in $\mathcal{K}$ or $\delta(G_2) \leq 1$.*

Proof. If $D \in \mathcal{K} - \{\mathcal{B}, C_3\}$ then suppose, to the contrary, that G_2 has no component in $\mathcal{K}$ and $\delta(G_2) \geq 2$. By Observation 8, $G_1 \notin \mathcal{K}$. The graph G_2 (G_1 respectively) has a *TRDS* S_2 (S_1 respectively) of cardinality at most $\frac{4n(G_2)}{7}$ $\left(\frac{4n(G_1)}{7}\text{ respectively}\right)$. The set $S_1 \cup S_2$ is a *TRDS* of G and so $\gamma_{tr}(G) \leq |S_1| + |S_2| \leq \frac{4n(G_1)}{7} + \frac{4n(G_2)}{7} = \frac{4n}{7}$, a contradiction. Hence, G_2 has at least one component in $\mathcal{K}$ or $\delta(G_2) \leq 1$.

Suppose that $D = \mathcal{B}$ and that v is adjacent to a vertex $w \in V(G_2)$. Assume, to the contrary, that G_2 has no component in $\mathcal{K}$ and $\delta(G_2) \geq 2$. The graph G_2 has a *TRDS* S of cardinality at most $\frac{4n(G_2)}{7}$. By Observation 8, v is adjacent to say u_1. If $w \in S$ ($w \notin S$ respectively) the set $S \cup \{u_3, u_2\}$ ($S \cup \{u_1, u_3, u_2\}$ respectively) is a *TRDS* of G and so $\gamma_{tr}(G) \leq |S| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} < \frac{4n}{7}$, a contradiction. Hence, G_2 has at least one component in $\mathcal{K}$ or $\delta(G_2) \leq 1$. □

Observation 11. *Let e be an edge of G. If the graph $G - e$ is claw-free and has no component isomorphic to C_3, then $\delta(G - e) \leq 1$.*

Proof. Let $e = xy$ be an edge of G. Suppose that $G - e$ is claw-free and that $G - e$ has no component isomorphic to C_3. Suppose, to the contrary, that $\delta(G - e) \geq 2$.

Case 1: $G - e$ is connected. If $G - e \notin \mathcal{K}$ then since $G - e$ has smaller size than G, we have that $G - e$ has a *TRDS* S of cardinality at most $\frac{4n}{7}$. Since S is also a *TRDS* of G we have that $\gamma_{tr}(G) = |S| \leq \frac{4n}{7}$, a contradiction. Suppose first that $G - e \in \mathcal{K} - \{\mathcal{B}, C_3\}$. If x and y are at distance more than two apart on $G - e$, then x, y and the two vertices adjacent to y on $G - e$ form a claw in G, which is a contradiction. Thus x and y are at distance two apart on $G - e$. Hence $\gamma_{tr}(G) \leq \frac{4n}{7}$, which is a contradiction. If $G - e = \mathcal{B}$ then $n(G) = 5$ and $\gamma_{tr}(G) = 2 < \frac{4n}{7}$, a contradiction.

Case 2: $G - e$ is disconnected. Let D and D' be the two components of $G - e$. Neither D nor D' is in $\mathcal{K} - \{\mathcal{B}, C_3\}$, since otherwise G will not be claw-free. Let $x \in V(D)$ and $y \in V(D')$. Certainly, D is not isomorphic to C_3. If $D = \mathcal{B}$ then, by Observation 8, x is say u_1. Note that $G' = G - xu_3$ is claw-free, connected, $G' \notin \mathcal{K}$ and $\delta(G') \geq 2$. Furthermore, G' has smaller size than G and so G' has a *TRDS* S of cardinality at most $\frac{4n(G')}{7}$. Since S is also a *TRDS* of G we have that $\gamma_{tr}(G) \leq \frac{4n}{7}$, which is a contradiction. Hence, neither D nor D' is in $\mathcal{K}$. Since D (D' respectively) has smaller size than G it follows that D (D' respectively) has a *TRDS* S_1 (S_2 respectively) of cardinality at most $\frac{4n(D)}{7}$ $\left(\frac{4n(D')}{7}\text{ respectively}\right)$. Then $S_1 \cup S_2$ is a *TRDS* of G, and so $\gamma_{tr}(G) \leq |S_1| + |S_2| \leq \frac{4n(D)}{7} + \frac{4n(D')}{7} = \frac{4n}{7}$ which is a contradiction. □

To complete the proof of Theorem 5 we will first state our key lemmas, then briefly explain the importance of each lemma and then use them to prove our main result. The proof of each of the following lemmas will be provided in the next section and each proof will rely on the observations mentioned above.

Lemma 12. *The graph G has no 2-paths of length greater than three.*

Lemma 13. *Suppose that G has a path $P : v_1, v_2, v_3, v_4$ of which* $\deg(v_1) = \deg(v_4) = 2$. *If $G_1 = \langle V(P)\rangle$ and $G_2 = G - G_1$ then $\delta(G_2) \leq 1$.*

Lemma 14. *Suppose that G has a path $P : v_1, v_2, v_3, v_4$ of which the vertices v_1 and v_4 have degree at least two and* $\deg(v_3) = 2$. *If $G_1 = \langle\{v_1, v_2, v_3, v_4\}\rangle$ and $G_2 = G - G_1$ then $\delta(G_2) \leq 1$.*

Lemma 15. *The graph G has no 2-paths of length three.*

Lemma 16. *The graph G either has no 2-cycles or it has a 2-path of length two.*

Lemma 17. *If P is a 2-path of G that has length two then $\mathcal{H}(P) = \emptyset$.*

Lemma 18. *The graph G has no 2-paths of length two.*

Lemma 19. *The graph G does not have minimum degree at least three.*

The proof of Theorem 5 will follow from Lemmas 16, 18 and 19. Lemmas 12–15 will be used to prove Lemmas 16–18. Lemma 17 will be essential in the proof of Lemma 18. We are now ready to complete the proof of Theorem 5.

Suppose first that $\delta(G) = 2$. If $G = C_n$ then Proposition 4 provides a contradiction. Hence, G has either a 2-path or a 2-cycle. If G has a 2-cycle then, by Lemma 16, we have that G must have a 2-path of length two. But this fact contradicts Lemma 18. Hence, G has no 2-cycles. It follows, by Lemmas 12 and 15, that G must have a 2-path of length two and so we contradict Lemma 18. Hence, $\delta(G) \geq 3$ and so we contradict Lemma 19. It immediately follows that G does not exist and so we obtain a contradiction which proves Theorem 5. □

3. Proofs of key lemmas

Proof of Lemma 12. Suppose, to the contrary, that G has a 2-path $P : x = v_0, v_1, v_2, \ldots, v_{j+1} = y$, with $j \geq 3$.

Suppose that $j \geq 4$.

If $j \geq 5$ or $j = 4$ and x is not adjacent to y then we let $G_2 = G - \langle\{v_1, v_2, \ldots, v_4\}\rangle$ and we form G_A from G_2 by joining the vertices x and v_5. The graph G_A is claw-free and $\delta(G_A) \geq 2$. Furthermore, G_A is connected and $G_A \notin \mathcal{K}$. By Observation 7 we get a contradiction.

Hence, $j = 4$ and x is adjacent to y. Let $G_2 = G - \langle\{v_1, v_2, \ldots, v_4\}\rangle$. The graph G_2 is claw-free, connected and $\delta(G_2) \geq 2$. If $G_2 \notin \mathcal{K}$ then, by Observation 7, we get a contradiction. Hence, $G_2 \in \mathcal{K}$.

If $G_2 \in \{C_3, \mathcal{B}\}$ then G is a necklace with an attachment that has length three. By the first part of Observation 6 we get a contradiction. Hence, $G_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$. For $u \in N(x) - \{v_1, y\}$ the graph $\langle\{u, v_1, y, x\}\rangle$ is a claw, which is a contradiction.

Hence, we may assume that $j = 3$.

Claim 1. The only degree two vertex adjacent to x (y respectively) is v_1 (v_3 respectively).

Proof. Let $u \in N(y) \cap (V(G) - V(P))$ and suppose, to the contrary, that $\deg(u) = 2$.

Case 1. The vertex u is adjacent to x.

If x (y respectively) has a neighbor $v \in V(G) - V(P) - \{u\}$ ($v' \in V(G) - V(P) - \{u\}$ respectively), then the graph $\langle\{u, v, v_1, x\}\rangle$ ($\langle\{u, v', v_3, y\}\rangle$ respectively) induces a claw, a contradiction. Hence, $N(x) = \{u, v_1, y\}$ and $N(y) = \{u, v_3, x\}$. Furthermore, $n(G) = 6$ and the set $\{x, v_1, v_2\}$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \leq 3 < \frac{4.6}{7} = \frac{4n}{7}$ which is a contradiction.

Case 2. The vertex u is not adjacent to x.

If x is adjacent to y then $\langle\{u, x, v_3, y\}\rangle$ is a claw which is a contradiction. Thus, x and y are not adjacent. The fact that $\langle N(y) - \{v_3\}\rangle$ is complete, implies that $\deg(y) = 3$. Let $u' \in N(y) - \{v_3, u\}$. Clearly, u' is adjacent to u. The graph $G' = G - yu'$ is claw-free and connected. If $\deg(u') \geq 3$ then $\delta(G') \geq 2$ and this contradicts Observation 11. Hence, $\deg(u') = 2$. Note that the graph $G_1 = \langle V(P) \cup \{u, u'\} - \{x\}\rangle$ is a necklace, where the path $P - x - y$ is the attachment with initial vertex v_1. Let $G_2 = G - G_1$. Now G_2 is connected and $\delta(G_2) \geq 2$. Note that v_1 has a neighbor in $V(G_2)$. If $G_2 \notin \mathcal{K}$ then we contradict Observation 6. Hence, $G_2 \in \mathcal{K}$. Since $\deg(v_1) = 2$ we have, by the first part of Observation 8, that $G_2 \in \{C_3, \mathcal{B}\}$.

If $G_2 = C_3$ then $n(G) = 9$ and the set $\{x, v_1, v_2, v_3, y\}$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$, a contradiction.

If $G_2 = \mathcal{B}$ we have, by Observation 8, that x is say u_1. The graph $G' = G - xu_3$ is claw-free and connected. In addition, $\delta(G') \geq 2$ and this contradicts Observation 11. Our claim follows by symmetry. □

Let $G_1 = \langle V(P) - \{x\}\rangle$ and let $G_2 = G - G_1$. Note that $\delta(G_2) \geq 1$.

Case 1. $\delta(G_2) = 1$.

Let v be a degree one vertex of G_2. If $v \in V(G) - V(P)$ then v is adjacent to y and $\deg(v) = 2$, contradicting Claim 1. It follows that $v = x$. Hence, for some $u \in N(x) \cap (V(G) - V(P))$ we have that $N(x) = \{y, v_1, u\}$. Now if $\mathcal{H}(P) = \emptyset$ then $G' = G - xy$ is claw-free, connected and has minimum degree at least two. This contradicts Observation 11. Hence, $u \in \mathcal{H}(P)$ and $\deg(u) \geq 3$. If y has a neighbor $u' \in N(y) - \{v_3, x, u\}$ then $\langle\{x, v_3, u', y\}\rangle$ induces a claw, which is a contradiction. Hence, $N(y) = \{x, v_3, u\}$.

Case 1.1. $\deg(u) = 3$.

Let $w_1 \in N(u) - \{y, x\}$. In this case $N(u) = \{w_1, x, y\}$. Assume first that $\deg(w_1) \geq 3$. Let $G'_1 = \langle V(P) \cup \{u\}\rangle$ and $G'_2 = G - G'_1$. Note that G'_2 is claw-free, connected and $\delta(G'_2) \geq 2$. Suppose first that $G'_2 \in \mathcal{K}$. By Observation 8 and the fact that $\deg(u) = 3$, we have that $G'_2 \in \{C_3, \mathcal{B}\}$. Observation 8 implies that say $w_1 = u_1$.

If $G'_2 = C_3$ then $n(G) = 9$ and $\{u_1, u, x, v_1, v_2\}$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$, a contradiction.

If $G'_2 = \mathcal{B}$ then the graph $G' = G - u_1u_3$ is claw-free, connected and has minimum degree at least two. This contradicts Observation 11.

Hence, $G'_2 \notin \mathcal{K}$. Note that G'_2 has smaller size than G and so G'_2 has a *TRDS* S_2 with cardinality at most $\frac{4(n-6)}{7}$. The set $S_2 \cup \{x, v_1, v_2\}$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \leq |S_2| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} < \frac{4n}{7}$ which is a contradiction.

We may assume that $\deg(w_1) = 2$. Let $w_2 \in N(w_1) - \{u\}$. Since $N(u) = \{x, y, w_1\}$ we can find a 2-path P' : $u, w_1, w_2, \ldots, w_{j+1}$, where $w_i \in V(G) - V(P) - \{u\}$ for $i = 2, \ldots, j+1$. Since every 2-path has length at most four we have that $j = 1, 2, 3$. We let $G'_1 = \langle V(P) \cup V(P') - \{w_{j+1}\}\rangle$ and $G'_2 = G - G'_1$. Note that G'_2 is claw-free and $\delta(G'_2) \geq 2$. Also, w_j has a neighbor in $V(G'_2)$. Suppose first that $G'_2 \notin \mathcal{K}$. If $j \geq 2$ then G'_1 is a necklace and so we contradict Observation 6. Hence, $j = 1$. The graph G'_2 has a *TRDS* S_2 of cardinality at most $\frac{4(n-7)}{7}$. If $w_2 \in S_2$ then $\{x, v_1, v_2\} \cup S_2$ is a *TRDS* of G. If $w_2 \notin S_2$

then $\{x, v_1, v_2, u\} \cup S_2$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \le |S_2| + 4 \le \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} = \frac{4n}{7}$ which is a contradiction. Therefore, $G_2' \in \mathcal{K}$.

Since $\deg(w_j) = 2$ we have, by Observation 8, that $G_2' \in \{C_3, \mathcal{B}\}$. Let $G_1'' = \langle V(G_2') \cup V(P') - \{u\}\rangle$ and $G_2'' = G - G_1''$. Note that if $G_2' = \mathcal{B}$ and $j \ge 2$ or if $G_2' = C_3$ and $j = 3$ then the graph G_1'' is a necklace with attachment $P' - u - w_{j+1}$ and initial vertex w_1. If $j = 1$ and $G_2' = C_3$ then by Observation 7 we get a contradiction. If $j = 1$ and $G_2' = \mathcal{B}$ then by Observation 8 we have that w_2 is say u_1. Furthermore, $G' = G - u_1u_3$ is claw-free, connected and $\delta(G') \ge 2$. This contradicts Observation 11. If $j = 3$ or $j = 2$ and $G_2' = \mathcal{B}$, then we contradict Observation 6. Hence, $j = 2$ and $G_2' = C_3$. Note that $n(G) = 11$ and so $\{w_2, w_3, y, v_3, v_2\}$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \le 5 < \frac{4.11}{7} = \frac{4n}{7}$, a contradiction.

Case 1.2. $\deg(u) \ge 4$.

Let $w, w' \in N(u) - \{x, y\}$. Since G is claw-free we have that $\langle N(u) - \{x, y\}\rangle$ is complete. If $\deg(w) = 2$ then w is adjacent to w' and $\deg(u) = 4$. If $\deg(w') \ge 3$ then $G' = G - uw'$ is claw-free, connected and $\delta(G') \ge 2$. This contradicts Observation 11. Hence, $\deg(w') = 2$ and so $n(G) = 8$. The set $\{u, x, v_1, v_2\}$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \le 4 < \frac{4.8}{7} = \frac{4n}{7}$, a contradiction. Hence, u is not adjacent to any degree two vertex and so if we define $G_1' = \langle V(P) \cup \{u\}\rangle$ and $G_2' = G - G_1'$ we see that $\delta(G_2') \ge 2$. Now since $\langle N(u) - \{x, y\}\rangle$ is complete we have that G_2' is connected. If $G_2' \notin \mathcal{K}$ then note that G_2' has smaller size than G and so G_2' has a *TRDS* S_2 of cardinality at most $\frac{4(n-6)}{7}$. The set $S_2 \cup \{x, v_1, v_2\}$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \le |S_2| + 3 \le \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} < \frac{4n}{7}$ which is a contradiction. It follows that $G_2' \in \mathcal{K}$. By Observation 8, u is adjacent to say u_1.

If $G_2' \in \mathcal{K} - \{C_3, \mathcal{B}\}$ then, by Observation 8, u is adjacent to say u_2. If there are edges between the vertex u and the set $V(G_2') - \{u_1, u_2\}$, then we may form G', from G, by removing all these edges. The graph G' is claw-free, connected, $G' \notin \mathcal{K}$ and G' has smaller size than G. Hence, $\gamma_{\rm tr}(G) \le \frac{4n}{7}$ which is a contradiction. It can be concluded that $P : u_2, u_3, \ldots, u_i, u_1$ is a 2-path. By all previous arguments we have that $i = 5$. Hence, $n(G) = 11$ and so $\{y, v_3, v_2, u_1, u_2, u_3\}$ is a *TRDS* of G. Thus, $\gamma_{\rm tr}(G) \le 6 < \frac{4.11}{7} = \frac{4n}{7}$ which is a contradiction.

If $G_2' = C_3$ then $n(G) = 9$ and so $\{y, v_3, v_2, u_1, u\}$ is a *TRDS* of G, and so $\gamma_{\rm tr}(G) \le 5 < \frac{4.9}{7} = \frac{4n}{7}$ which is a contradiction.

If $G_2' = \mathcal{B}$ then clearly $n(G) = 11$. Constructing a $\gamma_{\rm tr}$-set of G of cardinality at most six will suffice since then $\gamma_{\rm tr}(G) \le 6 < \frac{4.11}{7} = \frac{4n}{7}$, a contradiction. The set $\{y, v_3, v_2, u_1, u_2, u_3\}$ is a *TRDS* of G.

Case 2. $\delta(G_2) \ge 2$.

If G_2 has no components in $\mathcal{K}$ then, by Observation 7, we get a contradiction. We may assume that G_2 has a component $\mathcal{U} \in \mathcal{K}$.

Case 2.1. The vertex x lies on the component $\mathcal{U}$.

Note that since $\delta(G_2) \ge 2$ we have that x is adjacent to a vertex $u \in V(\mathcal{U})$. If x is adjacent to a vertex $w \in V(G) - V(\mathcal{U}) - \{y, v_1, v_2, v_3\}$ then $\langle\{w, u, v_1, x\}\rangle$ is a claw which is a contradiction. Hence, $N[x] \subseteq V(\mathcal{U}) \cup \{v_1, y\}$. Since $\deg(v_1) = 2$ we have, by Observation 8, that $\mathcal{U} \in \{C_3, \mathcal{B}\}$.

By Observation 8, x is say u_1. Assume first that y is adjacent to a vertex $z \in V(\mathcal{U})$. Note that $N[z] \subseteq V(\mathcal{U}) \cup \{v_1, y\}$. If y has a neighbor $w \in V(G) - V(P) - V(\mathcal{U})$ then $\langle\{v_3, z, w, y\}\rangle$ would induce a claw, a contradiction. Hence, $N(y) \subseteq V(\mathcal{U}) \cup \{v_3\}$. It follows that G is a necklace with attachment $P - x$ and initial vertex y. Since the attachment has length three, we contradict Observation 6. Thus, y is adjacent to no vertex of $V(\mathcal{U})$ and so u_2 will have degree two in G. This contradicts Claim 1.

Case 2.2. The vertex x does not lie on the component $\mathcal{U}$.

By the previous case we may assume that x lies on a component that is not in $\mathcal{K}$. Furthermore, y is adjacent to some vertex of $\mathcal{U}$ and $V(\mathcal{U}) \subseteq V(G) - V(P)$. Using Observation 8, we can assume that y is adjacent to say u_1. If $\mathcal{U} \in \mathcal{K} - \{C_3, \mathcal{B}\}$ then y must be adjacent to u_2. Also note that if y has a neighbor $u \in V(G_2) - V(\mathcal{U})$ then $\langle\{u, u_1, v_3, y\}\rangle$ induces a claw which is a contradiction. Hence, $N(y) \subseteq \{v_3\} \cup V(\mathcal{U})$. The graph $G_1' = \langle V(\mathcal{U}) \cup V(P) - \{x\}\rangle$ is a necklace with attachment $P - x$ and initial vertex v_1. The graph $G_2' = G - G_1'$ is claw-free, has no components in $\mathcal{K}$, v_1 has a neighbor in $V(G_2')$ and $\delta(G_2') \ge 2$. By Observation 6 we get a contradiction. □

Proof of Lemma 13. Let $P : v_1, v_2, v_3, v_4$ be a path of which $\deg(v_1) = \deg(v_4) = 2$. Define $G_1 = \langle V(P)\rangle$ and $G_2 = G - G_1$. If $V(G_2) = \emptyset$ then $\gamma_{\rm tr}(G) \le \frac{4n}{7}$, a contradiction. We may assume, to the contrary, that $\delta(G_2) \ge 2$. If G_2 has no component in $\mathcal{K}$ then, by Observation 7, we have that G has a *TRDS* of cardinality at most $\frac{4n}{7}$, a contradiction. Hence, G_2 has a component $\mathcal{U}$ in $\mathcal{K}$. The following claim will be useful.

Claim 1. Let D be a component of G_2 that is in $\mathcal{K}$. If D has a vertex adjacent to v_2, neither v_1 nor v_4 is adjacent to a vertex of D and v_1 is not adjacent to v_3, then $D \in \{C_3, \mathcal{B}\}$.

Proof. Assume that v_2 is adjacent to a vertex of D, neither v_1 nor v_4 is adjacent to a vertex of D and that v_1 and v_3 are not adjacent. Suppose, to the contrary, that $D \in \mathcal{K} - \{C_3, \mathcal{B}\}$. By Observation 8, v_2 must be adjacent to say u_1 and u_2. In addition, the fact that $\langle\{v_3, u_1, v_1, v_2\}\rangle$ ($\langle\{v_3, u_2, v_1, v_2\}\rangle$ respectively) is not a claw, implies that v_3 is adjacent to both u_1 and u_2. Furthermore, if v_2 is adjacent to u_j where $j = 3, \ldots, i$, then either $\langle\{v_1, u_j, u_1, v_2\}\rangle$ or $\langle\{v_1, u_j, u_2, v_2\}\rangle$ will induce a claw which is a contradiction. By symmetry we have that v_3 is not adjacent to u_j, with $j = 3, \ldots, i$. Hence, the path $P' : u_2, u_3, \ldots, u_i, u_1$ is a 2-path of length greater than three and so we contradict Lemma 12. Hence, $D \in \{C_3, \mathcal{B}\}$. □

Case 1. The vertex v_1 is adjacent to a vertex of $\mathcal{U}$.

Since $\deg(v_1) = 2$, we have, by Observation 8, that $\mathcal{U} \in \{C_3, \mathcal{B}\}$ and v_1 is adjacent to say u_1.

Case 1.1. $V(G_2) - V(\mathcal{U}) = \emptyset$.

Note that $G_2 = \mathcal{U}$. Hence, G is a necklace with an attachment that has length three. By the first part of Observation 6 we get a contradiction.

Case 1.2. $V(G_2) - V(\mathcal{U}) \neq \emptyset$.

Let $z \in V(G_2) - V(\mathcal{U})$.

Case 1.2.1. v_2 is adjacent to either z or v_4.

Let $G'_1 = \langle\{v_1\} \cup V(\mathcal{U})\rangle$ and $G'_2 = G - G'_1$. Clearly, G'_2 is connected and $\delta(G'_2) \geq 1$. Note that if v_2 is adjacent to z then since $\langle\{z, v_1, v_3, v_2\}\rangle$ does not induce a claw, we have that v_3 is adjacent to z and so G'_2 has a triangle. If v_2 is adjacent to v_4 then G'_2 contains a triangle. We may therefore assume that G'_2 has an induced triangle.

Suppose first that $\delta(G'_2) \geq 2$. If $G'_2 \in \mathcal{K}$ then since G'_2 has a triangle and at least four vertices, we must have that $G'_2 = \mathcal{B}$. Now G'_2 has a *TRDS* S of cardinality three. If $\mathcal{U} = C_3$, then $n(G) = 9$. The set $\{v_1, u_1\} \cup S$ is a *TRDS* of G and so $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$, which is a contradiction. If $\mathcal{U} = \mathcal{B}$ then $n(G) = 11$. If $v_2 \in S$ ($v_2 \notin S$ respectively) we have that $\{u_3, u_2\} \cup S$ ($\{u_3, u_2, u_1\} \cup S$ respectively) is a *TRDS* of G. Hence, $\gamma_{\text{tr}}(G) \leq 6 < \frac{4.11}{7} = \frac{4n}{7}$ which is a contradiction.

We may assume that $G'_2 \notin \mathcal{K}$. If $\mathcal{U} = \mathcal{B}$ then since v_1 is adjacent to the vertex $v_2 \in V(G'_2)$, we have that the second part of Observation 10 implies that $\delta(G'_2) \leq 1$ or $G'_2 \in \mathcal{K}$ which is a contradiction. It follows that $\mathcal{U} = C_3$. Since G'_1 has a Hamiltonian path and $n(G'_1) = 4$ we obtain, by Observation 7, a contradiction.

We may assume that $\delta(G'_2) = 1$.

Since $\deg_{G'_2}(v_2) \geq 2$ and $\deg_{G'_2}(v_3) \geq 2$, we have that $\deg_{G'_2}(v_4) = 1$. Hence, v_4 must be adjacent to a vertex of $\mathcal{U}$ and so v_2 is adjacent to z. Recall that v_3 is also adjacent to z. If v_4 is adjacent to u_1 then $\langle\{v_4, v_1, u_3, u_1\}\rangle$ induces a claw. Hence, if $\mathcal{U} = C_3$ then v_4 must be adjacent to say u_2. If $\mathcal{U} = \mathcal{B}$ then since $\deg(v_4) = 2$, we have, by Observation 8, that v_4 is adjacent to u_2 or say u_4.

Let $G''_1 = \langle V(P) \cup V(\mathcal{U})\rangle$ and $G''_2 = G - G''_1$.

Suppose first that G''_2 is disconnected. Let $\mathcal{U}'$ ($\mathcal{U}''$ respectively) be a component of G''_2 that contains (does not contain respectively) the vertex z. Without loss of generality, v_2 is adjacent to some $z' \in V(\mathcal{U}'')$ and so $\langle\{z', z, v_1, v_2\}\rangle$ induces a claw which is a contradiction. Hence, G''_2 is connected. Since $\delta(G_2) \geq 2$, we must have that $\delta(G''_2) \geq 2$. Note that $\delta(G''_1) \geq 2$. Furthermore, the fact that $G''_1 \notin \mathcal{K}$ implies that G''_1 has a *TRDS* S_1 of cardinality at most $\frac{4n(G''_1)}{7}$.

If $G''_2 \notin \mathcal{K}$ then G''_2 has a *TRDS* S_2 of cardinality at most $\frac{4n(G''_2)}{7}$. Hence, $\gamma_{\text{tr}}(G) \leq |S_1| + |S_2| \leq \frac{4n(G''_1)}{7} + \frac{4n(G''_2)}{7} = \frac{4n}{7}$ which is a contradiction.

Hence, $G''_2 \in \mathcal{K}$. Let $P' : w_1, \ldots, w_j$ be the Hamiltonian path of G''_2. Now G''_2 is clearly a component of G_2. Furthermore, every vertex of G''_2 is adjacent to neither v_1 nor v_4, v_1 is not adjacent to v_3 and v_2 is adjacent to the vertex z in $V(G''_2)$. By applying Claim 1 to the component G''_2, we have that $G''_2 \in \{C_3, \mathcal{B}\}$.

Suppose that $G''_2 = \mathcal{B}$. Note that G''_2 has a *TRDS* S of cardinality three. If $\mathcal{U} = \mathcal{B}$ and v_4 is adjacent to u_2 then $\{u_3, u_4, u_5, v_2, v_3\} \cup S$ is a *TRDS* of G. If $\mathcal{U} = \mathcal{B}$ and v_4 is adjacent to u_4 then $\{u_3, u_5, v_2, v_3\} \cup S$ is a *TRDS* of G. Clearly, $n(G) = 14$ and $\gamma_{\text{tr}}(G) \leq 5 + |S| = 8 = \frac{4.14}{7} = \frac{4n}{7}$ which is a contradiction. If $\mathcal{U} = C_3$ then $n(G) = 12$. By Observation 8, we may assume that v_2 is adjacent to say w_1. The fact that $\langle\{w_1, v_1, v_3, v_2\}\rangle$ is not a claw implies that v_3 is adjacent to w_1 also. The set $\{w_1, w_2, w_3, u_1, u_2, u_3\}$ is a *TRDS* of G and so $\gamma_{\text{tr}}(G) \leq 6 < \frac{4.12}{7} = \frac{4n}{7}$, which is a contradiction.

Suppose that $G''_2 = C_3$. If $\mathcal{U} = \mathcal{B}$ then $n(G) = 12$. Without loss of generality, let $z = w_1$. If v_4 is adjacent to u_2 then $\{z, v_2, v_3, u_3, u_4, u_5\}$ is a *TRDS* of G. If v_4 is adjacent to u_4 then $\{u_5, u_3, v_2, v_3, z\}$ is a *TRDS* of G. Hence, $\gamma_{\text{tr}}(G) \leq 6 < \frac{4.12}{7} = \frac{4n}{7}$ which is a contradiction. If $\mathcal{U} = C_3$ then $n(G) = 10$. The set $\{u_1, v_1, v_2, v_3, z\}$ is a *TRDS* of G and so $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.10}{7} = \frac{4n}{7}$, which is a contradiction.

Case 1.2.2. $N(v_2) \subseteq \{v_1, v_3\} \cup V(\mathcal{U})$.

Let $G'_1 = \langle V(P) \cup V(\mathcal{U})\rangle$ and $G'_2 = G - G'_1$. Note that G'_1 is a necklace. Since G is connected we have that G'_2 has a vertex that is adjacent to either v_3 or v_4. If a vertex $z \in V(G'_2)$ is adjacent to v_3 then since $\langle\{z, v_2, v_4, v_3\}\rangle$ is not a claw we have that v_4 is adjacent to z. This fact implies that v_4 has a neighbor in $V(G'_2)$ and since $\deg(v_4) = 2$, we have that G'_2 is connected. Since $\delta(G_2) \geq 2$ we have that $\delta(G'_2) \geq 2$. If $G'_2 \notin \mathcal{K}$ then, by Observation 6, we are done.

If $G'_2 \in \mathcal{K}$ then since $\deg(v_4) = 2$, we have, by Observation 8, that $G'_2 \in \{C_3, \mathcal{B}\}$. Clearly, G_2 consists out of two components which are both in the set $\{C_3, \mathcal{B}\}$. By Observation 9 we get a contradiction.

Case 2. The vertex v_1 is not adjacent to a vertex in $V(\mathcal{U})$.

By symmetry, we have that v_4 is not adjacent to any vertex in $V(\mathcal{U})$. Since G is connected, we have, without loss of generality, that v_2 is adjacent to some vertex in $V(\mathcal{U})$. By Observation 8, v_2 is adjacent to say u_1. Define $G'_1 = \langle\{v_1, v_2, v_3, v_4\} \cup V(\mathcal{U})\rangle$ and $G'_2 = G - G'_1$. Note that if $V(G'_2) \neq \emptyset$ then since $\delta(G_2) \geq 2$ we have that $\delta(G'_2) \geq 2$.

If G'_2 has a component $\mathcal{U}' \in \mathcal{K}$ then $\mathcal{U}'$ is also a component of G_2, and, by Case 1, every vertex of $\mathcal{U}'$ is adjacent to neither v_1 nor v_4. Since G is connected, we must have that some z in $V(\mathcal{U}')$ is adjacent to either v_2 or v_3. If z is adjacent to v_2 then $\langle\{z, v_1, u_1, v_2\}\rangle$ is a claw, a contradiction. Hence, z is adjacent to v_3 and no vertex of $\mathcal{U}'$ is adjacent to v_2. If v_1 is adjacent to v_3 then $\langle\{z, v_4, v_1, v_3\}\rangle$ is a claw, a contradiction. Hence, v_1 is not adjacent to v_3. The fact that $\langle\{u_1, v_3, v_1, v_2\}\rangle$ is a not a claw implies that v_3 is adjacent to u_1 and so $\langle\{u_1, v_4, z, v_3\}\rangle$ is a claw, a contradiction.

We may assume that either $V(G_2') = \emptyset$, or $V(G_2') \neq \emptyset$ with G_2' not having any component in $\mathcal{K}$. In both cases G_2' has a *TRDS* S of cardinality at most $\frac{4n(G_2')}{7}$.

Case 2.1. v_1 is adjacent to v_3.

Clearly, G_1' is a necklace with initial vertex v_4 and end vertex v_2. By Observation 6 we get a contradiction.

Case 2.2. v_1 and v_3 are not adjacent.

Since $\langle\{u_1, v_3, v_1, v_2\}\rangle$ is not a claw we have that u_1 and v_3 are adjacent. By symmetry, v_4 is not adjacent to v_2. By Claim 1 we have that $\mathcal{U} \in \{C_3, \mathcal{B}\}$. Suppose that $\mathcal{U} = \mathcal{B}$ ($\mathcal{U} = C_3$ respectively). If v_1 or v_4, say v_1, is adjacent to a vertex in S then $S \cup \{v_3, v_4, u_2, u_3\}$ ($S \cup \{v_3, v_4, u_1\}$ respectively) is a *TRDS* of G. If v_1 is adjacent to v_4 or v_1 and v_4 both have neighbors in $V(G_2') - S$, then $S \cup \{v_2, v_3, u_1, u_2, u_3\}$ ($S \cup \{v_2, v_3, u_1\}$ respectively) is a *TRDS* of G. Hence, if $\mathcal{U} = \mathcal{B}$ then $\gamma_{\mathrm{tr}}(G) \leq 5 + |S| < \frac{4n}{7} - \frac{36}{7} + \frac{35}{7} < \frac{4n}{7}$ and if $\mathcal{U} = C_3$ then $\gamma_{\mathrm{tr}}(G) \leq 3 + |S| < \frac{4n}{7} - \frac{28}{7} + \frac{21}{7} < \frac{4n}{7}$. In both cases, we get a contradiction. □

Proof of Lemma 14. Let $P : v_1, v_2, v_3, v_4$ be a path of which the vertices v_1 and v_4 have degree at least two and $\deg(v_3) = 2$ and define $G_1 = \langle\{v_1, v_2, v_3, v_4\}\rangle$ and $G_2 = G - G_1$. If $V(G_2) = \emptyset$ then $\gamma_{\mathrm{tr}}(G) \leq \frac{4n}{7}$, a contradiction. We may assume, to the contrary, that $\delta(G_2) \geq 2$. If G_2 has no component in $\mathcal{K}$ then, by Observation 7, we have that G has a *TRDS* of cardinality at most $\frac{4n}{7}$, a contradiction. Hence, G_2 has a component $\mathcal{U}$ in $\mathcal{K}$.

Claim 1. Any component in G_2 has a vertex that is adjacent to either v_1 or v_4.

Proof. Let D be a component of G_2. Since G is connected there is some vertex $v \in V(D)$ adjacent to either v_1, v_4 or v_2. If v is adjacent to say v_2 then since $\langle\{v, v_3, v_1, v_2\}\rangle$ is not a claw we must have that v is adjacent to v_1 and so we are done. □

Case 1. $V(G_2) - V(\mathcal{U}) = \emptyset$.

In this case $G_2 = \mathcal{U}$. By Claim 1 and Observation 8, the graph G is a necklace with attachment that has length three. This contradicts Observation 6.

Case 2. $V(G_2) - V(\mathcal{U}) \neq \emptyset$.

Case 2.1. v_4 is adjacent to say u_1.

If v_4 is adjacent to a vertex $v \in V(G_2) - V(\mathcal{U})$, then $\langle\{v, u_1, v_3, v_4\}\rangle$ is a claw which is a contradiction. Hence, $N(v_4) \subset V(\mathcal{U}) \cup V(P)$. Note that if $\mathcal{U} \notin \{C_3, \mathcal{B}\}$ then v_4 is adjacent to say u_2. Clearly, the path P together with $\mathcal{U}$ forms a necklace and so we define $G_1' = \langle V(P) \cup V(\mathcal{U})\rangle$ and $G_2' = G - G_1'$. If $V(G_2') = \emptyset$ then we obtain a contradiction by Observation 6. Now clearly $\delta(G_2') \geq 2$ and if G_2' has no components in $\mathcal{K}$ then we get, by Observation 6, a contradiction. Hence, G_2' has a component $\mathcal{U}'$ in $\mathcal{K}$. Since $N(v_4) \subset V(\mathcal{U}) \cup V(P)$, we have, by Claim 1, that v_1 must be adjacent to a vertex of $\mathcal{U}'$.

Suppose that either $V(G_2') - V(\mathcal{U}') \neq \emptyset$ or v_1 is adjacent to two vertices of $\mathcal{U}'$. The path $P - v_1$ together with $\mathcal{U}$ forms a necklace and so we define $G_1'' = \langle V(P) \cup V(\mathcal{U}) - \{v_1\}\rangle$ and $G_2'' = G - G_1''$. If v_1 is adjacent to two vertices of $\mathcal{U}'$ then obviously $\deg(v_1) \geq 3$. If $V(G_2') - V(\mathcal{U}') \neq \emptyset$ then since $N(v_4) \subset V(\mathcal{U}) \cup V(P)$, we have, by Claim 1, that there is a vertex $v \in V(G_2') - V(\mathcal{U}')$ adjacent to v_1. Hence, $\deg(v_1) \geq 3$. It follows that $\delta(G_2'') \geq 2$. Since $N(v_4) \subset V(\mathcal{U}) \cup V(P)$ we have, by Claim 1, that G_2'' is connected and $G_2'' \notin \mathcal{K}$. By Observation 6 we are done.

Hence, $V(G_2') - V(\mathcal{U}') = \emptyset$ and v_1 is adjacent to exactly one vertex of $\mathcal{U}'$. The first part of Observation 8 must imply that $\mathcal{U}' \in \{C_3, \mathcal{B}\}$. The path $P - v_4$ together with $\mathcal{U}'$ forms a necklace and so we define $G_1'' = \langle V(P) \cup V(\mathcal{U}') - \{v_4\}\rangle$ and $G_2'' = G - G_1''$. If v_4 is adjacent to two vertices of $\mathcal{U}$ then $\delta(G_2'') \geq 2$. Furthermore, $G_2'' \notin \mathcal{K}$. By Observation 6 we get a contradiction. Therefore, v_4 is adjacent to exactly one vertex of $\mathcal{U}$. The first part of Observation 8 must imply that $\mathcal{U} \in \{C_3, \mathcal{B}\}$.

It follows that $V(G_2) = V(\mathcal{U}') \cup V(\mathcal{U})$ and that v_1 (v_4 respectively) is adjacent to exactly one vertex of $\mathcal{U}'$ ($\mathcal{U}$ respectively). By Observation 9 we get a contradiction.

Case 2.2. v_1 is adjacent to u_1.

Suppose that $\mathcal{U}'$ is a component of G_2, with $\mathcal{U}' \in \mathcal{K}$. If $\mathcal{U}' \neq \mathcal{U}$ and v_4 is adjacent to a vertex of $\mathcal{U}'$, then we may re-label $\mathcal{U}'$ as $\mathcal{U}$ and so by the previous case we are done. It follows that if G_2 has a component in $\mathcal{K}$, then no vertex of this component can be adjacent to v_4. Note that the path P together with $\mathcal{U}$ is a necklace. Let $G_1' = \langle V(P) \cup V(\mathcal{U})\rangle$ and $G_2' = G - G_1'$. If $V(G_2') = \emptyset$ then we obtain a contradiction by Observation 6. Since $\delta(G_2) \geq 2$, we may assume that $\delta(G_2') \geq 2$. If G_2' has no components in $\mathcal{K}$, then, by Observation 6, we get a contradiction. Hence, G_2' has a component $\mathcal{U}'$ in $\mathcal{K}$.

Note that $\mathcal{U}'$ is also a component of G_2. By Claim 1 and the fact that v_4 is not adjacent to any vertex of $\mathcal{U}'$, we have that v_1 is adjacent to a vertex v of $\mathcal{U}'$. In addition, $N(v_4) \subseteq V(G) - \{v_4\} - V(\mathcal{U}) - V(\mathcal{U}')$. Since $\langle\{v, u_1, v_2, v_1\}\rangle$ is not a claw we have that v_2 is adjacent to v or v_2 is adjacent to u_1. We also claim that v_1 is not adjacent to a vertex $w \in V(G_2) - V(\mathcal{U}) - V(\mathcal{U}')$. Suppose, to the contrary, that v_1 is adjacent to w. Then $\langle\{w, v, u_1, v_1\}\rangle$ induces a claw which is a contradiction. Also, if v_4 is adjacent to v_1 then $\langle\{u_1, v, v_4, v_1\}\rangle$ induces a claw. Hence, $N(v_1) \subseteq \{v_2\} \cup V(\mathcal{U}) \cup V(\mathcal{U}')$ and $N(v_4) \subseteq \{v_3, v_2\} \cup V(G_2) - V(\mathcal{U}) - V(\mathcal{U}')$.

If v_2 is adjacent to v, then let $G_1'' = \langle\{v_1\} \cup V(\mathcal{U})\rangle$. If v_2 is adjacent to u_1, then re-label $\mathcal{U}'$ as $\mathcal{U}$ and $\mathcal{U}$ as $\mathcal{U}'$ and set $G_1'' = \langle\{v_1\} \cup V(\mathcal{U})\rangle$. Define $G_2'' = G - G_1''$. Note that $\delta(G_2'') \geq 2$. The fact that $N(v_1) \subseteq \{v_2\} \cup V(\mathcal{U}) \cup V(\mathcal{U}')$ implies that G_2'' is connected. Clearly, $G_2'' \notin \mathcal{K}$. If $\mathcal{U} = C_3$ ($\mathcal{U} = \mathcal{B}$ respectively) then by Observation 7 (second part of Observation 10 respectively) we are done. Hence, $\mathcal{U} \notin \{C_3, \mathcal{B}\}$. By the first part of Observation 10 we get a contradiction. □

Proof of Lemma 15. Suppose, to the contrary, that G has a 2-path $P : x, v_1, v_2, y$. Let $G_1 = \langle V(P)\rangle$ and $G_2 = G - G_1$. Since x and y both have degree greater than two and $\deg(v_2) = 2$, we may apply Lemma 14 to deduce that $\delta(G_2) \le 1$. Let v be a vertex of G_2, with $\deg_{G_2}(v) \le 1$.

Case 1. $\deg_{G_2}(v) = 0$.

Obviously, $N(v) = \{x, y\}$. If x is adjacent to a vertex w in $V(G) - V(P) - \{v\}$, then $\langle\{v, w, v_1, x\}\rangle$ is a claw which is a contradiction. Hence, $N(x) = \{v_1, v, y\}$. We can show, in similar fashion, that $N(y) = \{v_2, v, x\}$. Hence, $n(G) = 5$ and $\{x, v_1\}$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \le 2 \le \frac{4.5}{7} = \frac{4n}{7}$ which is a contradiction.

Case 2. $\deg_{G_2}(v) = 1$.

Case 2.1. The vertex v is adjacent to both x and y.

Since $\deg_{G_2}(v) = 1$ we must have that $\deg(v) = 3$.

Case 2.1.1. x is adjacent to a vertex $w \in V(G) - V(P) - \{v\}$.

Since $\langle N[x] - \{v_1\}\rangle$ is complete, we must have that v is adjacent to w and so $N(v) = \{x, y, w\}$. Furthermore, $\{v, w, v_1\} \subseteq N(x) \subseteq \{v, w, v_1, y\}$. Since $\langle N[y]-\{v_2\}\rangle$ is complete and $\deg(v) = 3$, we can conclude that $N(y)\cap(V(G)-V(P)-\{v, w\}) = \emptyset$. Hence, $\{v, v_2\} \subset N(y) \subseteq \{v, v_2, w, x\}$. We may form G' by deleting from G the edge yw (if it exists) and xy (if it exists). Note that $\delta(G') \ge 2$ and $G' \notin \mathcal{K}$. If G' is a claw-free graph then G' has smaller size than G and so it follows that G' has a *TRDS* S' of cardinality at most $\frac{4n(G')}{7}$. Clearly, S' is a *TRDS* of G and so $\gamma_{\rm tr}(G) \le |S'| \le \frac{4n}{7}$ which is a contradiction. Hence, G' has a claw $\langle\{w_4, w_3, w_2, w_1\}\rangle$. Note that, without loss of generality, $w_2 = y$ and $w_3 \in \{x, w\}$.

If $w_3 = x$ then the fact that $\{v, v_2\} \subset N(y) \subseteq \{v, v_2, w, x\}$ implies that $w_1 = v$. But since $N(v) = \{x, y, w\}$, we have that $w_4 = w$. Hence, w_4 is adjacent to w_3 which is a contradiction. Hence, $w_3 = w$. It follows once more that $w_1 = v$ and so $w_4 = x$. But then w_4 and w_3 are adjacent which is a contradiction.

Case 2.1.2. $N(x) = \{v_1, y, v\}$ and $N(y) = \{v_2, x, v\}$.

Let $w_1 \in N(v) - \{x, y\}$. Suppose first that $\deg(w_1) \ge 3$. Let $G'_1 = \langle V(P) \cup \{v\}\rangle$ and we consider the connected claw-free graph $G'_2 = G - G'_1$. Now G'_2 has minimum degree at least two. If $G'_2 \notin \mathcal{K}$ then G'_2 has a *TRDS* S_2 with cardinality at most $\frac{4(n-5)}{7}$. The set $\{x, v_1\} \cup S_2$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \le |S_2| + 2 \le \frac{4n}{7} - \frac{20}{7} + \frac{14}{7} \le \frac{4n}{7}$, a contradiction. Hence, $G'_2 \in \mathcal{K}$. By the first part of Observation 8, and the fact that $\deg(v) = 3$, we can conclude that $G'_2 \in \{C_3, \mathcal{B}\}$. By Observation 8 we may also assume that v is adjacent to say u_1.

If $G'_2 = C_3$ then $n(G) = 8$ and $\{v, u_1, x, v_1\}$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \le 4 < \frac{4.8}{7} = \frac{4n}{7}$, which is a contradiction.

If $G'_2 = \mathcal{B}$ then we form $G' = G - u_1u_3$. The graph G' is connected and claw-free. By Observation 11, we are done.

Hence, $\deg(w_1) = 2$. Let $w_2 \in N(w_1) - \{v\}$. We have that there exists a 2-path $P' : v, w_1, w_2, \ldots, w_j, w_{j+1}$, where $w_i \in V(G) - V(P) - \{v\}$ for $i = 1, 2, \ldots, j+1$. By Lemma 12 we have that $j = 1$ or $j = 2$.

Case 2.1.2.1. $j = 2$.

We let $G''_1 = \langle V(P') \cup V(P) - \{w_3\}\rangle$ and $G''_2 = G - G''_1$. If $G''_2 \notin \mathcal{K}$ then G''_2 has a *TRDS* S_2 with cardinality at most $\frac{4(n-7)}{7}$. If $w_3 \in S_2$ then $\{v, x, v_1\} \cup S_2$ is a *TRDS* of G. If $w_3 \notin S_2$ then $\{v, w_1, x, v_1\} \cup S_2$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \le |S_2| + 4 \le \frac{4n}{7} - \frac{28}{7} + \frac{28}{7} \le \frac{4n}{7}$ which is a contradiction. We may therefore conclude that $G''_2 \in \mathcal{K}$.

By the first part of Observation 8, and the fact that $\deg(w_2) = 2$, we can deduce that $G''_2 \in \{C_3, \mathcal{B}\}$. By Observation 8 we may also assume that say $u_1 = w_3$.

If $G''_2 = C_3$ then $n(G) = 10$ and $\{u_1, w_2, x, v_1\}$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \le 4 \le \frac{4.10}{7} = \frac{4n}{7}$, a contradiction.

If $G''_2 = \mathcal{B}$ then we form $G' = G - u_1u_3$. The graph G' is connected and claw-free. By Observation 11, we obtain a contradiction.

Case 2.1.2.2. $j = 1$.

We let $G''_1 = \langle V(P) \cup \{v, w_1\}\rangle$ and $G''_2 = G - G''_1$. Note that G''_2 is claw-free, connected and has minimum degree at least two. If $G''_2 \notin \mathcal{K}$ then G''_2 has a *TRDS* S_2 with cardinality at most $\frac{4(n-6)}{7}$. If $w_2 \notin S_2$ then $\{v, x, v_1\} \cup S_2$ is a *TRDS* of G. If $w_2 \in S_2$ then $\{x, v_1\} \cup S_2$ is a *TRDS* of G. Hence, $\gamma_{\rm tr}(G) \le |S_2| + 3 \le \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} < \frac{4n}{7}$ which is a contradiction. Hence, $G''_2 \in \mathcal{K}$.

By the first part of Observation 8, and the fact that $\deg(w_1) = 2$, we can deduce that $G''_2 \in \{C_3, \mathcal{B}\}$. By Observation 8 we may also assume that say $u_1 = w_2$.

If $G''_2 = C_3$ then $n(G) = 9$ and $\{u_1, w_1, x, v_1\}$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \le 4 < \frac{4.9}{7} = \frac{4n}{7}$.

If $G''_2 = \mathcal{B}$ then we form $G' = G - u_1u_3$. The graph G' is connected and claw-free. By Observation 11, we obtain a contradiction.

Case 2.2. The vertex v is adjacent to only x.

Clearly, $\deg(v) = 2$. If x is adjacent to y then, since $\langle N(x) - \{v_1\}\rangle$ is complete, we must have that v is adjacent to y, which is a contradiction. Hence, x and y are not adjacent. It follows, since $\deg(x) \ge 3$, that there is a vertex w in $N(x) \cap (V(G) - V(P) - \{v\})$. The completeness of $\langle N(x) - \{v_1\}\rangle$ also implies that $N(x) = \{v_1, v, w\}$. If $\deg(w) \ge 3$ then the graph $G' = G - xw$ is claw-free and connected. By Observation 11, we get a contradiction. So $\deg(w) = 2$.

If y has a degree two neighbor $v' \in V(G) - V(P) - \{v, w\}$ then, by the same argument, there is a vertex $w' \in V(G) - V(P) - \{v, w, v'\}$, such that $N(y) = \{v_2, v', w'\}$ and $\deg(w') = 2$. Hence, $n(G) = 8$. The set $\{x, v_1, v_2, y\}$ is a *TRDS* of G and so $\gamma_{\rm tr}(G) \le 4 < \frac{4.8}{7} = \frac{4n}{7}$, a contradiction.

We may assume that the only degree two vertex adjacent to y is v_2. Note that $G_1' = \langle V(P) \cup N(x)\rangle$ induces a necklace with attachment $P - x$. Let $G_2' = G - G_1'$ and note that since G is claw-free we have that G_2' is connected. Also, G_2' has minimum degree at least two. If $G_2' \notin \mathcal{K}$ then, by Observation 6, we have a contradiction. Hence, $G_2' \in \mathcal{K}$.

Suppose that $G_2' \in \mathcal{K} - \{\mathcal{B}\}$. We may assume that y is adjacent to say u_1. If $i \geq 5$ then we have, since $N(y) - \{v_2\}$ induces a clique, that $\deg(y) = 3$. Furthermore, by the first part of Observation 8, y is adjacent to say u_1 and u_2. Also, $u_2, \ldots, u_i, u_1$ is a 2-path of length greater than or equal to four. This will contradict Lemma 12. Hence, $i = 3$ and so $n(G) = 9$. The set $\{x, v_1, v_2, y, u_1\}$ is a *TRDS* of G and so $\gamma_{tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$.

If $G_2' = \mathcal{B}$ then $n(G) = 11$. By the second part of Observation 8, y is adjacent to u_1. The set $\{u_1, u_2, u_3, x, v_1\}$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq 5 < \frac{4.11}{7} = \frac{4n}{7}$. □

Proof of Lemma 16. If G has a 2-path of length greater than or equal to two, then, by Lemmas 12 and 15, we are done. Hence, we may assume that G has no 2-path. Suppose, to the contrary, that G has a 2-cycle $P : x, v_1, v_2, \ldots, v_{j+1} = x$. Let $w \in N(x) - \{v_1, v_j\}$.

Claim 1. There are no 2-cycles on more than three vertices.

Proof. For the 2-cycle $P : x, v_1, v_2, \ldots, v_{j+1} = x$, suppose, to the contrary, that $j \geq 3$. Then $\langle\{w, v_1, v_j, x\}\rangle$ is a claw, a contradiction. Hence, $j = 2$. □

Claim 2. No vertex in $N(x) - \{v_1, v_2\}$ has degree two.

Proof. Suppose, to the contrary, that a vertex $v \in N(x) - \{v_1, v_2\}$ has degree two. Let $N(v) = \{x, v'\}$. By Claim 1 and the fact that G has no 2-path, $\deg(v') = 2$ and $N(v') = \{x, v\}$. Since G is claw-free we have that $\deg(x) = 4$ and so $G = \mathcal{B}$, a contradiction. □

Case 1. $\deg(x) = 3$.

By Claim 2, $\deg(w) \geq 3$. Let $w', w'' \in N(w) - \{x\}$. Since G is claw-free, w' is adjacent to w''. Suppose first that $\deg(w') = 2$. If $\deg(w'') \geq 3$ then $ww'w''$ is a 2-path, a contradiction. Hence, $\deg(w'') = 2$. It follows that $n(G) = 6$ and the set $\{x, w\}$ is a *TRDS* of G and so $\gamma_{tr}(G) \leq 2 < \frac{4.6}{7} = \frac{4n}{7}$. Hence, $\deg(w') \geq 3$ and by symmetry we have that $\deg(w'') \geq 3$. We now define $G_1 = \langle\{x, v_1, v_2, w\}\rangle$ and $G_2 = G - G_1$. Clearly, $\delta(G_2) \geq 2$ and G_2 is claw-free and connected. If $G_2 \notin \mathcal{K}$ then, by Observation 7, we get a contradiction. Hence, $G_2 \in \mathcal{K}$.

If $G_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$ then G_2 has a 2-path of length greater than three which contradicts Lemma 12.

If $G_2 = C_3$ then $n(G) = 7$ and $\{x, w, w'\}$ is a *TRDS* of G and so $\gamma_{tr}(G) \leq 3 < \frac{4.7}{7} = \frac{4n}{7}$, which is a contradiction.

If $G_2 = \mathcal{B}$ then $n(G) = 9$. Clearly, G_2 has a *TRDS* S of cardinality at most three. The set $S \cup \{x, w\}$ is a *TRDS* of G and so $\gamma_{tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$, which is a contradiction.

Case 2. $\deg(x) \geq 4$.

Case 2.1. The vertex w is adjacent to a degree two vertex $v \in V(G) - N[x]$.

Let $v' \in N(v) - \{w\}$. If $\deg(v') \geq 3$ then G has a 2-path of length two, which is a contradiction. Hence, v' has degree two. To avoid a 2-path of length greater than or equal to two, or a 2-cycle on more than three vertices, we must have that $N(v') = \{v, w\}$. If w is adjacent to a vertex $w' \in V(G) - N[x] - \{v', v\}$, then $\langle\{w', v, x, w\}\rangle$ will induce a claw which is a contradiction. Hence, $N[w] = \{v, v'\} \cup (N[x] - \{v_1, v_2\})$ and so $N[w] - \{v, v'\} = N[x] - \{v_1, v_2\}$. Let $G_1 = \langle\{x, v_1, v_2, w, v, v'\}\rangle$ and $G_2 = G - G_1$. Since $\langle N(x) - \{v_1, v_2\}\rangle$ is complete, we must have that G_2 is connected.

Case 2.1.1. $\delta(G_2) \leq 1$.

Let $w' \in N(x) - \{v_1, v_2, w\}$. We may assume, without loss of generality, that $\deg_{G_2}(w') \leq 1$. Since $\langle N(x) - \{v_1, v_2\}\rangle$ is complete, we have that $N[x] - \{v_1, v_2, w'\} \subseteq N(w')$. In addition, by Claim 2, we have that $\deg_{G_2}(w') = 1$ and so $\deg(w') = 3$.

Case 2.1.1.1. w' has a neighbor $z \in V(G) - N[x] - \{v, v'\}$.

Note that, since $\langle N(x) - \{v_1, v_2\}\rangle$ is complete and $N[w] - \{v, v'\} = N[x] - \{v_1, v_2\}$, we have that $N(w') = \{w, x, z\}$, $N(x) = \{v_1, v_2, w, w'\}$ and $N(w) = \{v, v', w', x\}$. We let $G_1' = \langle N[x] \cup \{v, v'\}\rangle$ and $G_2' = G - G_1'$. Clearly, G_2' is connected.

Case 2.1.1.1.1. $\delta(G_2') \geq 2$.

If $G_2' \notin \mathcal{K}$, then note that G_2' has smaller size than G and so G_2' has a *TRDS* S_2 of cardinality at most $\frac{4(n-7)}{7}$. The set $S_2 \cup \{x, w, w'\}$ is a *TRDS* of G and so $\gamma_{tr}(G) \leq |S_2| + 3 \leq \frac{4n}{7} - \frac{28}{7} + \frac{21}{7} < \frac{4n}{7}$, which is a contradiction. Hence, $G_2' \in \mathcal{K}$.

If $G_2' \in \mathcal{K}$ then, without loss of generality, $z = u_1$. If $G_2' \in \mathcal{K} - \{C_3, \mathcal{B}\}$ then, since $N(w') = \{w, x, z\}$, the first part of Observation 8 produces a contradiction.

If $G_2' = C_3$ then $n(G) = 10$. The set $\{x, w, w', u_1\}$ is a *TRDS* of G and so $\gamma_{tr}(G) \leq 4 < \frac{4.10}{7} = \frac{4n}{7}$, which is a contradiction.

If $G_2' = \mathcal{B}$ then $n(G) = 12$. The set $\{x, w, w', u_1, u_2, u_3\}$ is a *TRDS* of G. Hence $\gamma_{tr}(G) \leq 6 < \frac{4.12}{7} = \frac{4n}{7}$, which is a contradiction.

Case 2.1.1.1.2. $\delta(G_2') \leq 1$.

In this case we may assume that the vertex z has degree one in G_2'. Since $\deg(w') = 3$, we have that G has a 2-path of length at least two, a contradiction.

Case 2.1.1.2. w' has no neighbor in $V(G) - N[x] - \{v, v'\}$.

Since $\langle N(x) - \{v_1, v_2\}\rangle$ is a clique and $\deg(w') = 3$, we have that $\deg(x) = 5$. Let $w'' \in N(x) - \{v_1, v_2, w, w'\}$. If $\deg(w'') = 3$ then $N(w'') = \{x, w, w'\}$ and $n(G) = 8$. The set $\{v_1, v_2, x, w\}$ is a *TRDS* of G and so $\gamma_{tr}(G) \le 4 < \frac{4.8}{7} = \frac{4n}{7}$, which is a contradiction. Therefore, $\deg(w'') \ge 4$ and so w'' is adjacent to at least one vertex in $V(G) - N[x] - \{v, v'\}$. Note that the path $P' : w', x, v_1, v_2$ is such that w' and v_2 both have degree at least two and $\deg(v_1) = 2$. We form $G_1'' = \langle\{x, w', v_1, v_2\}\rangle$ and $G_2'' = G - G_1''$. Note that $\delta(G_2'') \ge 2$ and so we obtain a contradiction by Lemma 14.

Case 2.1.2. $\delta(G_2) \ge 2$.

If $G_2 \notin \mathcal{K}$ then it follows that G_2 has a *TRDS* S_2 of cardinality at most $\frac{4(n-6)}{7}$. The set $S_2 \cup \{x, w\}$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \le |S_2| + 2 \le \frac{4n}{7} - \frac{24}{7} + \frac{14}{7} < \frac{4n}{7}$. Thus, $G_2 \in \mathcal{K}$.

Suppose that $G_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$. By the first part of Observation 8, and the fact that $N[w] - \{v, v'\} = N[x] - \{v_1, v_2\}$, we have that w and x are both adjacent to say u_1 and u_2. Also note that if u_3 has degree two, then G will have a 2-path of length at least two, a contradiction. Hence, both x and w are adjacent to u_3. Since $\langle N(x) - \{v_1, v_2\}\rangle$ is a clique, we have that u_1 and u_3 are adjacent. This is a contradiction.

If $G_2 = C_3$ then w (and x) are both adjacent to say u_1. The set $\{x, w, u_1\}$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \le 3 < \frac{4.9}{7} = \frac{4n}{7}$ which is a contradiction.

If $G_2 = \mathcal{B}$ then $n(G) = 11$ and the set $\{x, w, u_1, u_2, u_3\}$ is a *TRDS* of G. Hence $\gamma_{tr}(G) \le 5 < \frac{4.11}{7} = \frac{4n}{7}$, which is a contradiction.

Case 2.2. No vertex in $N(x) - \{v_1, v_2\}$ has a degree two neighbor in $V(G) - N[x]$.

Let $G_1 = \langle V(P) \cup \{w\}\rangle$ and $G_2 = G - G_1$. Note that the path $P' : w, x, v_1, v_2$ is such that w and v_2 both have degree at least two and $\deg(v_1) = 2$. By Lemma 14 it follows that $\delta(G_2) \le 1$. Since no vertex in $N(x) - \{v_1, v_2\}$ has a degree two neighbor in $V(G) - N[x]$, we have that any vertex of degree at most one in G_2 must be adjacent to x. Furthermore, Claim 2 implies that $\delta(G_2) = 1$. Let w' be a degree one vertex of G_2. Note that $\deg(w') = 3$.

Case 2.2.1. $N(x) - \{w, w', v_1, v_2\} \ne \emptyset$.

Let $w'' \in N(x) - \{v_1, v_2, w, w'\}$. Hence, $N(w') = \{w, x, w''\}$ and $\deg(x) = 5$. Furthermore, $N(x) = \{v_1, v_2, w', w, w''\}$.

Case 2.2.1.1. $\deg(w) = 3$.

Clearly, $N(w) = \{w', x, w''\}$. If $\deg(w'') = 3$ then $n(G) = 6$ and the set $\{x, v_1, v_2\}$ is a *TRDS* of G. Therefore, $\gamma_{tr}(G) \le 3 < \frac{4.6}{7} = \frac{4n}{7}$ which is a contradiction. Hence, $\deg(w'') \ge 4$. We form the graph G' by removing, from G, the edges xw'' and ww''. The graph G' is claw-free, connected and has minimum degree at least two. Also, $G' \notin \mathcal{K}$ and so G' has a *TRDS* S of cardinality at most $\frac{4n}{7}$. Since S is also a *TRDS* of G we have that $\gamma_{tr}(G) \le |S| \le \frac{4n}{7}$, which is a contradiction.

Case 2.2.1.2. $\deg(w) \ge 4$.

By symmetry, $\deg(w'') \ge 4$. We re-label w' as w and so $\delta(G_2) \ge 2$, which is impossible by Lemma 14.

Case 2.2.2. $N(x) - \{w, w', v_1, v_2\} = \emptyset$.

Hence, $\deg(x) = 4$. If $\deg(w) \ge 4$ then we may re-label w' as w and so $\delta(G_2) \ge 2$, which is impossible by Lemma 14. Hence, $\deg(w) = 3$. We let $G_1' = \langle N[x]\rangle$ and $G_2' = G - G_1'$. The graph G_2' is claw-free.

Case 2.2.2.1. $\delta(G_2') \le 1$.

The fact that no vertex in $N(x) - \{v_1, v_2\}$ has a degree two neighbor in $V(G) - N[x]$, implies that there is a vertex $v \in V(G) - N[x]$, adjacent to both w and w', that has degree three. Hence, $N(w) = \{v, w', x\}$ and $N(w') = \{v, w, x\}$. The graph $G' = G - wv$ is claw-free, connected and has minimum degree at least two. By Observation 11, we get a contradiction.

Case 2.2.2.2. $\delta(G_2') \ge 2$.

If $G_2' \notin \mathcal{K}$ then G_2' has a *TRDS* S_2 of cardinality at most $\frac{4(n-5)}{7}$. If w (w' respectively) has a neighbor in $V(G_2') - S_2$, then $\{x, w'\} \cup S_2$ ($\{x, w\} \cup S_2$ respectively) is a *TRDS* of G. If w (and w') has a neighbor in S_2 then $\{v_1, v_2\} \cup S_2$ is a *TRDS* of G. Hence $\gamma_{tr}(G) \le 2 + |S_2| < \frac{4n}{7} - \frac{20}{7} + \frac{14}{7} < \frac{4n}{7}$, which is a contradiction. We may assume that G_2' has a component $\mathcal{U}$ that is isomorphic to a graph in $\mathcal{K}$. Without loss of generality, u_1 is adjacent to say w. Furthermore, the fact that $\deg(w) = 3$ implies that $N(w) = \{x, w', u_1\}$.

If $\mathcal{U} \in \mathcal{K} - \{C_3\}$, then the first part of Observation 8 implies that w is adjacent to u_2. Hence, $\deg(w) \ge 4$ which is a contradiction.

Suppose that $\mathcal{U} = C_3$. If $V(G_2') - V(\mathcal{U}) = \emptyset$, then $n(G) = 8$. The set $\{x, w, w', u_1\}$ is a *TRDS* of G and so $\gamma_{tr}(G) \le 4 < \frac{4.8}{7} = \frac{4n}{7}$, which is a contradiction. If $V(G_2') - V(\mathcal{U}) \ne \emptyset$ then w' is adjacent to a vertex $v \in V(G_2') - V(\mathcal{U})$. We let $G_1'' = \langle V(\mathcal{U}) \cup \{w\}\rangle$ and $G_2'' = G - G_1''$. The graph G_2'' is claw-free, has degree at least two and $G_2'' \notin \mathcal{K}$. By Observation 7, we get a contradiction.

We may assume that $\mathcal{U} = \mathcal{B}$. The second part of Observation 8 implies that w is adjacent to say u_1. If $V(G_2') - V(\mathcal{U}) = \emptyset$ then $n(G) = 10$. The set $\{x, w', u_3, u_2\}$ is a *TRDS* of G and so $\gamma_{tr}(G) \le 4 < \frac{4.10}{7} = \frac{4n}{7}$, which is a contradiction. If $V(G_2') - V(\mathcal{U}) \ne \emptyset$ then w' is adjacent to a vertex $v \in V(G_2') - V(\mathcal{U})$. We let $G_1'' = \langle V(\mathcal{U}) \cup \{w\}\rangle$ and $G_2'' = G - G_1''$. The graph G_2'' is claw-free, has degree at least two and $G_2'' \notin \mathcal{K}$. This contradicts the second part of Observation 10. □

Proof of Lemma 17. Consider the 2-path $P : x, v_1, y$ and suppose, to the contrary, that $\mathcal{H}(P) \ne \emptyset$. Let $w \in \mathcal{H}(P)$. Define $G_1 = \langle V(P) \cup \{w\}\rangle$ and $G_2 = G - G_1$. Since the path $P' : w, y, v_1, x$ is such that x and w have degree at least two and $\deg(v_1) = 2$, we may apply Lemma 14 and deduce that $\delta(G_2) \le 1$. Let v be a vertex such that $\deg_{G_2}(v) \le 1$.

Case 1: $\deg_{G_2}(v) = 0$.

Since $\deg(v) \geq 2$, we must have that v is adjacent to at least two vertices in the set $\{x, y, w\}$. It follows, without loss of generality, that v is adjacent to x. Since $\langle N[x] - \{v_1, y\}\rangle$ is complete, we must have that $\{w, x\} \subseteq N(v) \subseteq \{x, y, w\}$ and $\{v, w, v_1\} \subseteq N(x) \subseteq \{v, w, v_1, y\}$. We form the graph G' by removing, from G, the edge xw. Clearly, G' is connected and $\delta(G') \geq 2$. If G' is claw-free then, by Observation 11, we are done. So we may assume that G' has a claw.

Let $\langle\{w_4, w_3, w_2, w_1\}\rangle$ be a claw of G'. We may also assume that $w_2 = x$ and $w_3 = w$. Clearly, $w_1 \in \{v, y\}$. If $w_1 = v$ then $w_4 = y$. But then w_4 is adjacent to w_3, which is a contradiction. Hence, $w_1 = y$. If $w_4 \in \{v_1, v\}$ then w_4 is adjacent to w_2 which is impossible. Hence, $w_4 \in V(G) - V(P) - \{v, w\}$. Since $\langle N[y] - \{v_1, x\}\rangle$ is complete, we must have that w_4 is adjacent to w_3 which is a contradiction.

We may assume that no vertex of G_2 has degree 0.

Case 2: $\deg_{G_2}(v) = 1$.

Case 2.1. The vertex v is adjacent to x.

Since $\langle N[x] - \{v_1, y\}\rangle$ is a clique, we have that v is adjacent to w. Furthermore, v has exactly one neighbor z in $V(G) - V(P) - \{w, v\}$. Hence, $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$ and $\deg(w) \geq 3$.

If $\deg(w) = 3$ then $N(w) = \{x, y, v\}$ and, since $\langle N[x] - \{v_1, y\}\rangle$ and $\langle N[y] - \{v_1, x\}\rangle$ are complete, we have that $\{v, w, v_1\} \subseteq N(x) \subseteq \{v, w, y, v_1\}$ and $\{v_1, w\} \subset N(y) \subseteq \{v, w, x, v_1\}$. The fact that $\deg(y) \geq 3$, implies that either vy or xy exists. We form the graph G' by removing, from G, the edges vy (if it exists) and xy (if it exists). If G' has a claw then, since $\langle V(P) \cup \{v, w\}\rangle_{G'}$ and $\langle V(G) - V(P) - \{w\}\rangle_{G'}$ are claw-free in G', the center of this claw must be v. But then G has a claw which is a contradiction. Hence, G' is claw-free. Furthermore, $\delta(G') \geq 2$ and $G' \notin \mathcal{K}$. The graph G' has a *TRDS* S of cardinality at most $\frac{4n}{7}$. Hence, $\gamma_{\mathrm{tr}}(G) \leq |S| \leq \frac{4n}{7}$ which is a contradiction. We may therefore assume that $\deg(w) \geq 4$.

Let $G'_1 = \langle\{x, y, v_1, v\}\rangle$ and $G'_2 = G - G'_1$. From Lemma 14 it follows that $\delta(G'_2) \leq 1$. Let v' be a vertex of G'_2 that has degree at most one. If $\deg_{G'_2}(v') = 0$ then, since $\deg(w) \geq 4$, we have that $v' \neq w$ and v' is adjacent to two vertices in the set $\{x, y, v\}$. Since both $\langle N[x] - \{v_1, y\}\rangle$ and $\langle N[y] - \{v_1, x\}\rangle$ are complete, we have that v' is adjacent to w. But then $\deg_{G'_2}(v') \geq 1$, a contradiction. It follows that $\deg_{G'_2}(v') = 1$.

We claim that $v' \in \{z, w\}$. Suppose, to the contrary, that $v' \notin \{z, w\}$. Note that v' is adjacent to either x or y. If v' is adjacent to x then, since $\langle N[x] - \{v_1, y\}\rangle$ is complete and $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$, we have that $v' \in \{w, z\}$ which is a contradiction. Hence, v' is adjacent to y and not to x. The fact that $\langle N[y] - \{v_1, x\}\rangle$ is complete implies that $N(v') = \{y, w\}$ and so $\deg_{G_2}(v') = 0$, which is a contradiction. Hence, $v' \in \{z, w\}$.

Case 2.1.1. The vertex z is adjacent to x.

If $N(x) \cap (V(G) - V(P) - \{w, v, z\})$ has a vertex u, then, since $\langle N[x] - \{v_1, y\}\rangle$ is complete, we will have that $u \in N(v)$. Hence, $\deg_{G_2}(v) \geq 2$ which is a contradiction. It immediately follows that $\{v_1, w, v, z\} \subseteq N(x) \subseteq \{v_1, w, v, z, y\}$. Since $\langle N[x] - \{v_1, y\}\rangle$ is complete, we have that w is adjacent to z.

Case 2.1.1.1. $v' = z$.

Note that $\{w, v, x\} \subseteq N(z) \subseteq \{w, v, x, y\}$. If $\deg_{G'_2}(w) = 1$ then $N(w) = \{x, y, z, v\}$ and if $N(y) \cap (V(G) - V(P) - \{w, v, z\})$ has a vertex y' then, since $\langle N[y] - \{v_1, x\}\rangle$ is complete, we will have that $y' \in N(w)$ which is a contradiction. Hence, $N(y) \subseteq \{x, w, v, z, v_1\}$ and so $n(G) = 6$. The set $\{x, v_1, y\}$ is a *TRDS* of G and so $\gamma_{\mathrm{tr}}(G) \leq 3 < \frac{4.6}{7} = \frac{4n}{7}$, a contradiction. It follows that $\deg_{G'_2}(w) \geq 2$ and so w has a neighbor in $V(G) - V(P) - \{v, z, w\}$. We let $G''_1 = \langle\{x, v_1, z, v\}\rangle$ and $G''_2 = G - G''_1$. Note that G''_2 is connected.

If $\delta(G''_2) = 1$ then the fact that w has a neighbor in $V(G) - V(P) - \{v, z, w\}$, implies that $\deg_{G''_2}(y) = 1$ and so $\{w, v_1\} \subset N(y) \subseteq \{w, v, z, v_1, x\}$. We form the graph G' by first removing, from G, the edges between w and the vertices v, z and x and then removing the possible edges between y and the vertices v, z and x. If G' has a claw then, since $\langle V(P) \cup \{z, v, w\}\rangle_{G'}$ and $\langle V(G) - V(P) - \{z, v\}\rangle_{G'}$ are claw-free in G', the center of this claw must be w. But then G has a claw, which is a contradiction. The graph G' is therefore claw-free and $\delta(G') \geq 2$. Furthermore, $G' \notin \mathcal{K}$ and so G' has a *TRDS* S of cardinality at most $\frac{4n}{7}$. It follows that $\gamma_{\mathrm{tr}}(G) \leq |S| \leq \frac{4n}{7}$, which is a contradiction.

Hence, $\delta(G''_2) \geq 2$. If $G''_2 \notin \mathcal{K}$ then, by Observation 7, we are done. Hence, $G''_2 \in \mathcal{K}$.

If $G''_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$ then suppose, without loss of generality, that $u_1 = y$ and $u_2 = w$. The fact that $\langle N[y] - \{v_1, x\}\rangle$ is complete will imply that u_2 is adjacent to u_i, a contradiction.

If $G''_2 = C_3$ then $n(G) = 7$. The set $\{x, v_1, y\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq 3 < \frac{4.7}{7} = \frac{4n}{7}$ which is a contradiction.

If $G''_2 = \mathcal{B}$ then $n(G) = 9$. The set $\{x, v_1, u_1, u_2, u_3\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$ which is a contradiction.

Case 2.1.1.2. $v' = w$.

Clearly, $N(w) = \{x, v, z, y\}$. The fact that $\langle N[y] - \{v_1, x\}\rangle$ is complete must imply that $\{w, v_1\} \subset N(y) \subseteq \{x, v_1, z, v, w\}$. We form the graph G' by first removing, from G, the possible edges between y and the vertices v, z and x and then removing the edges xw and wz. If G' has a claw then, since $\langle V(P) \cup \{z, v, w\}\rangle_{G'}$ and $\langle V(G) - V(P) - \{v, w\}\rangle_{G'}$ are claw-free in G', the center of this claw must be z. But then G will contain a claw, a contradiction. Hence, the graph G' is claw-free and $\delta(G') \geq 2$. Furthermore, $G' \notin \mathcal{K}$ and so G' has a *TRDS* S of cardinality at most $\frac{4n}{7}$. It follows that $\gamma_{\mathrm{tr}}(G) \leq |S| \leq \frac{4n}{7}$, which is a contradiction.

Case 2.1.2. The vertex z is not adjacent to x.

Since $\langle N[x] - \{v_1, y\}\rangle$ is complete and $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$, we have that $\{v_1, v, w\} \subseteq N(x) \subseteq \{v_1, v, y, w\}$.

Claim 1. z is adjacent to w.

Proof. We form the graph G' by removing, from G, the edges vw, xw and vy (if it exists). Since $\deg(w) \geq 4$, we have that $\delta(G') \geq 2$. Suppose first that G' is claw-free. If $G' \notin \mathcal{K}$ then we have that G' has a *TRDS* S of cardinality at most $\frac{4n}{7}$. It follows that $\gamma_{\mathrm{tr}}(G) \leq |S| \leq \frac{4n}{7}$, a contradiction. We may therefore assume that $G' \in \mathcal{K} - \{C_3\}$.

If $G' = \mathcal{B}$ then it is easy to verify that joining at least one pair of non-adjacent vertices of $\mathcal{B}$ will result in a graph that has a *TRDS* of cardinality at most $\frac{4n}{7}$. Hence, G has a *TRDS* of cardinality at most $\frac{4n}{7}$ which is a contradiction.

If $G' \in \mathcal{K} - \{C_3, \mathcal{B}\}$ then G has a Hamiltonian cycle. Recall that G contains a triangle. If at least one edge is added to a graph in $\mathcal{K} - \{C_3, \mathcal{B}\}$ such that the resulting graph contains a triangle, then it is easy to verify that this resulting graph has a *TRDS* of cardinality at most $\frac{4n}{7}$. Hence, G has a *TRDS* of cardinality at most $\frac{4n}{7}$ which is a contradiction.

We may assume that G' is not claw-free. Let $\langle\{w_4, w_3, w_2, w_1\}\rangle$ be a claw in G'. Suppose first, without loss of generality, that $w_2 = x$ and $w_3 = w$. Since $\{v_1, v, w\} \subseteq N(x) \subseteq \{v_1, v, y, w\}$ and $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$, we have that $w_1 = y$ and so y and x are adjacent. If $w_4 = v_1$ then w_4 and w_2 are adjacent, which is a contradiction. Hence, $w_4 \in V(G) - V(P) - \{w, v\}$. Since $\langle N[y] - \{v_1, x\}\rangle$ is complete, we must have that w_4 is adjacent to w_3 which is a contradiction. We may assume that $w_2 = v$ and that $w_3 \in \{y, w\}$.

If $w_3 = w$ then the fact that $\{x, w, z\} \subseteq N(v) \subseteq \{x, y, w, z\}$, implies that $w_1 = z$ and so z is adjacent to w. Hence, $w_2 = v$ and $w_3 = y$. Clearly, $w_1 \in \{x, z\}$. Suppose first that $w_1 = x$. Then x is adjacent to y. The fact that $\{v_1, v, w\} \subseteq N(x) \subseteq \{v_1, v, y, w\}$, implies that $w_4 = v_1$. But then w_4 is adjacent to w_3, a contradiction. Hence, $w_1 = z$ and so z must be adjacent to y. Since $\langle N[y] - \{v_1, x\}\rangle$ is complete, we have that z is adjacent to w. □

Case 2.1.2.1. $v' = z$.

Clearly, $\deg(z) = 2$ or $\deg(z) = 3$ and $N(z) = \{v, w, y\}$. If w has a neighbor w' in $V(G) - V(P) - \{w, v, z\}$, then $\langle\{z, x, w', w\}\rangle$ will induce a claw which is a contradiction. Hence $N(w) = \{v, z, x, y\}$ and since $\langle N[y] - \{v_1, x\}\rangle$ is complete, we have that $\{v_1, w\} \subset N(y) \subseteq \{v, z, v_1, x, w\}$. Hence, $n(G) = 6$ and the set $\{x, w\}$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \leq 2 < \frac{4.6}{7} = \frac{4n}{7}$ which is a contradiction.

Case 2.1.2.2. $v' = w$.

Clearly, $N(w) = \{x, y, v, z\}$. Let $G' = G - xw$. It follows that $\delta(G') \geq 2$ and if G' is claw-free then, by Observation 11, we are done. Hence, G' is not claw-free. Let $\langle\{w_4, w_3, w_2, w_1\}\rangle$ be a claw in G'. We may assume, without loss of generality, that $x = w_2$ and $w = w_3$. Since z is not adjacent to x and $\{v_1, v, w\} \subseteq N(x) \subseteq \{v_1, v, y, w\}$, we have that $w_1 \in \{y, v\}$.

Suppose that $w_1 = v$. If $w_4 = y$ then w_4 is adjacent to w_3, a contradiction. Hence, $w_4 = z$. But then w_4 is adjacent to w_3, a contradiction. Hence, $w_1 = y$ and y is adjacent to x. If $w_4 = v_1$ then w_2 is adjacent to w_4, a contradiction. Hence, $w_4 \in V(G) - V(P) - \{w\}$. Since $\langle N[y] - \{v_1, x\}\rangle$ is complete, we must have that w_4 is adjacent to w_3 which is a contradiction.

Case 2.2 The vertex v is not adjacent to x.

We may assume, by symmetry, that any vertex of G_2 that has degree at most one, is adjacent to neither x nor y. Hence, v is not adjacent to y. Clearly, $\deg(v) = 2$. Let $z \in N(v) - \{w\}$. Since G has no 2-paths of length greater than two we have that $\deg(z) \geq 3$, or $\deg(z) = 2$ and $N(z) = \{w, v\}$. If $N(x) \cap (V(G) - V(P)) - \{w\} = N(y) \cap (V(G) - V(P)) - \{w\} = \emptyset$ then it follows that $N(x) = \{v_1, y, w\}$ and $N(y) = \{v_1, x, w\}$. The graph $G' = G - wy$ is claw-free and $\delta(G') \geq 2$. By Observation 11 we are done. Hence, $N(x) \cap (V(G) - V(P)) - \{w\} \neq \emptyset$ or $N(y) \cap (V(G) - V(P)) - \{w\} \neq \emptyset$.

Claim 2. If u is adjacent to either x or y and $u \notin \{z, w\}$, then $u \in \mathcal{H}(P)$.

Proof. Suppose, without loss of generality, that u is adjacent to x and $u \notin \{z, w\}$. Since $\langle N[x] - \{v_1, y\}\rangle$ is complete, we have that u is adjacent to w. If $z = y$ ($z = x$ respectively) then v is adjacent to y (x respectively), a contradiction. The fact that $\langle\{v, y, u, w\}\rangle$ does not induce a claw, implies that u is also adjacent to y. Hence, $u \in \mathcal{H}(P)$ and our claim is verified. □

Case 2.2.1. z is adjacent to either x or y.

Without loss of generality, suppose that z is adjacent to x. Since $\langle N[x] - \{v_1, y\}\rangle$ is complete, we have that z is adjacent to w. Note that if z (w respectively) has a neighbor u in $V(G) - N[x] - N[y] - \{v\}$, then $\langle\{x, v, u, z\}\rangle$ ($\langle\{x, v, u, w\}\rangle$ respectively) induces a claw which is a contradiction. Hence, $N(z) \cap (V(G) - N[x] - N[y]) = N(w) \cap (V(G) - N[x] - N[y]) = \{v\}$. Let $G'_1 = \langle\{v, z, x, v_1\}\rangle$ and $G'_2 = G - G'_1$. Since $\deg(v_1) = \deg(v) = 2$ we have, by Lemma 13, that $\delta(G'_2) \leq 1$.

Let v' be a vertex of G'_2 that has degree at most one. We claim that $v' \in \{y, w\}$. Suppose, to the contrary, that $v' \notin \{y, w\}$. If v' is adjacent to x then, since $\langle N(x) - \{v_1, y\}\rangle$ is complete, we have that v' is adjacent to w and z. Since $v' \notin \{z, w\}$ we have, by Claim 2, that v' is adjacent to y. Hence, $\deg_{G'_2}(v') \geq 2$ which is a contradiction. We may assume that v' is not adjacent to x. Hence, v' is adjacent to z. Since $N(z) \cap (V(G) - N[x] - N[y]) = \{v\}$, we can conclude that $v' \in N(y)$. Claim 2 implies that v' must be adjacent to x which is impossible. We can conclude that $v' \in \{y, w\}$.

If $v' = y$ then $\{v_1, w\} \subset N(y) \subseteq \{v_1, x, z, w\}$. If x is adjacent to a vertex $u \in V(G) - V(P) - \{z, w, v\}$, then since $u \notin \{z, w\}$ we have, by Claim 2, that u is adjacent to y and so this contradicts the fact that $N(y) \subseteq \{v_1, x, z, w\}$. Hence, $\{v_1, z, w\} \subseteq N(x) \subseteq \{v_1, y, z, w\}$ and $\{w, v, x\} \subseteq N(z) \subseteq \{y, x, v, w\}$. Note that z is adjacent to x and $\deg_{G_2}(z) = 1$. This contradicts our earlier assumption.

If $v' = w$ then $N(w) = \{x, y, v, z\}$. Since $\langle N[x] - \{v_1, y\}\rangle$ and $\langle N[y] - \{v_1, x\}\rangle$ are complete, we must have that $\{v_1, w\} \subset N(y) \subseteq \{v_1, x, z, w\}$ and $\{v_1, w, z\} \subseteq N(x) \subseteq \{v_1, y, z, w\}$. We have, once more, that z is adjacent to x and $\deg_{G_2}(z) = 1$, contradicting our earlier assumption.

Case 2.2.2. z is adjacent to neither x nor y.

By Claim 2, y (x respectively) is adjacent to every vertex of $N(x)$ ($N(y)$ respectively). Hence, $N(x)\cap(V(G)-V(P)) = N(y)\cap(V(G)-V(P))$. Furthermore, since $\langle N(x)-\{v_1, y\}\rangle$ and $\langle N(y)-\{v_1, x\}\rangle$ are complete, we have that $\langle N(x)\cup N(y)-\{v_1, x, y\}\rangle$ is complete. Since $N(x) \cap (V(G) - V(P)) - \{w\} \neq \emptyset$ or $N(y) \cap (V(G) - V(P)) - \{w\} \neq \emptyset$ we let, without loss of generality, u be a vertex in $N(x) \cap (V(G) - V(P) - \{w\})$. The vertex u is adjacent to w and y. Note that if w is adjacent to a vertex w' in $V(G) - N[x] - N[y] - \{v, z\}$, then $\langle\{x, w', v, w\}\rangle$ induces a claw which is a contradiction. Hence, $N(w)\cap(V(G)-N[x]-N[y]) \subseteq \{z, v\}$. If x and y are not adjacent then $\langle\{x, y, v, w\}\rangle$ induces a claw which is a contradiction. Hence, $\langle N(w) - \{z, v\}\rangle$ is complete.

Case 2.2.2.1. $\deg(z) = 2$.

Since G has no 2-paths of length greater than two, we have that $N(z) = \{w, v\}$. Due to the fact that $\langle N(x) \cup N(y) - \{v_1, x, y\}\rangle$ is complete and $N(w)\cap(V(G)-N[x]-N[y]) \subseteq \{z, v\}$, we can deduce that $N(w) = N[x]\cup N[y]\cup\{z, v\}-\{v_1, w\}$. Let $G'_1 = \langle V(P)\cup\{w, z, v\}\rangle$ and note that G'_1 induces a necklace where the vertices of the attachment are the vertices of $V(P)$. Let $G'_2 = G - G'_1$. If $\delta(G'_2) \leq 1$ then there will be a vertex v', of G'_2, of degree at most one. Furthermore, since $\deg(v') \geq 2$ we will have that v' is adjacent to at least one vertex in $\{x, y\}$. Hence, $\deg_{G_2}(v') \leq 1$ and this contradicts our earlier assumption. Hence, $\delta(G'_2) \geq 2$.

We first claim that G'_2 is connected. Suppose, to the contrary, that G'_2 has two components $\mathcal{U}$ and $\mathcal{U}'$. Since G is connected and $N(w) = N[x] \cup N[y] \cup \{z, v\} - \{v_1, w\}$, we have that $\mathcal{U}$ ($\mathcal{U}'$ respectively) has a vertex u' (u'' respectively) in $\langle N(x) \cup N(y) - \{v_1, w, x, y\}\rangle$. Since $\langle N(x) \cup N(y) - \{v_1, x, y\}\rangle$ is complete we have that $\langle N(x) \cup N(y) - \{v_1, w, x, y\}\rangle$ is complete and so u' is adjacent to u'' which is a contradiction. Hence, G'_2 is connected.

If $G'_2 \notin \mathcal{K}$ then, by Observation 6, we are done. Hence, $G'_2 \in \mathcal{K}$.

If $G'_2 = C_3$, then $n(G) = 9$. Without loss of generality, u_1 is adjacent to x or w and so we have that the set $\{x, w, u_1\}$ is a *TRDS* of G and so $\gamma_{\mathrm{tr}}(G) \leq 3 < \frac{4.9}{7} = \frac{4n}{7}$, which is a contradiction.

If $G'_2 = \mathcal{B}$ then $n(G) = 11$. The set $\{x, w, u_1, u_2, u_3\}$ is a *TRDS* of G and so $\gamma_{\mathrm{tr}}(G) \leq 5 < \frac{4.11}{7} = \frac{4n}{7}$, which is a contradiction.

Hence, $G'_2 \in \mathcal{K} - \{C_3, \mathcal{B}\}$. Note that, without loss of generality, u_1 is adjacent to some vertex in $\{x, y, w\}$. Since $N(w) = N[x] \cup N[y] \cup \{z, v\} - \{w, v_1\}$ and $N(x) \cap (V(G) - V(P)) = N(y) \cap (V(G) - V(P))$, we have that x, y and w are adjacent to say u_1. Hence, we can say that any vertex of G'_2 that is adjacent to a vertex in $\{x, y, w\}$ is adjacent to x, y and w. Hence, Observation 8 implies, without loss of generality, that u_2 is adjacent to x, y and w. Furthermore, if $v'' \in \{x, y, w\}$ is adjacent to a vertex u_j, for $3 \leq j \leq i-1$, then $u_j \in N(x)$ and since $\langle N[x] - \{v_1, y\}\rangle$ is complete, we have that u_j and u_1 are adjacent which is a contradiction. Similarly, if v'' is adjacent to u_i then $u_i \in N(x)$ and so u_2 and u_i will be adjacent which is a contradiction. Hence, $P' : u_2, \ldots, u_i, u_1$ is a 2-path with length greater than three which, by Lemma 12, is a contradiction.

Case 2.2.2.2. The vertex $\deg(z) \geq 3$.

Let $G'_1 = \langle\{v, w, x, v_1\}\rangle$ and $G'_2 = G - G'_1$. If $\delta(G'_2) \geq 2$ then, by Lemma 13, we are done. We may assume that $\delta(G'_2) \leq 1$. Let v' be the vertex of G'_2 that has degree at most one. We claim that $v' \in \{y, z\}$. Suppose, to the contrary, that $v' \notin \{y, z\}$. Since $\langle N[x] - \{v_1, y\}\rangle$ is complete and $N(w) \cap (V(G) - N[x] - N[y]) \subseteq \{z, v\}$, we have that $N(v') = \{w, x, y\}$. But then $\deg_{G_2}(v') = 0$, a contradiction. Hence, $v' \in \{y, z\}$.

Suppose that $v' = y$. Clearly, $N(y) = \{v_1, u, w, x\}$. Since $N(x) \cap (V(G) - V(P)) = N(y) \cap (V(G) - V(P))$, we can deduce that $N(x) = \{v_1, u, w, y\}$. Since $N(w) \cap (V(G) - N[x] - N[y]) \subseteq \{z, v\}$, we have that $\{u, x, y, v\} \subseteq N(w) \subseteq \{u, x, y, v, z\}$. We form G' by removing, from G, the edges wy and uy. The graph G' is such that $\delta(G') \geq 2$ and $G' \notin \mathcal{K}$. If G' is claw-free then G' has a *TRDS* S of cardinality at most $\frac{4n}{7}$. It follows that $\gamma_{\mathrm{tr}}(G) \leq |S| \leq \frac{4n}{7}$, a contradiction. Hence, G' has a claw $\langle\{w_4, w_3, w_2, w_1\}\rangle$. Clearly, $w_2 = y$ and $w_3 \in \{w, u\}$.

If $w_3 = u$ then since $N(y) = \{v_1, u, w, x\}$, we have that $w_1 = x$. Since $N(x) = \{v_1, u, w, y\}$ we have that $w_4 \in \{v_1, w\}$. If $w_4 = v_1$ then w_4 is adjacent to w_2, a contradiction. If $w_4 = w$ then w_4 is adjacent to w_3, a contradiction.

Hence, $w_3 = w$. Since $N(y) = \{v_1, u, w, x\}$ and $N(x) = \{v_1, u, w, y\}$, we have that $w_1 = x$ and $w_4 \in \{v_1, u\}$. If $w_4 = v_1$ then w_4 is adjacent to w_2, a contradiction. If $w_4 = u$ then w_4 is adjacent to w_3, a contradiction.

Hence $v' = z$, $\deg(z) = 3$ and z is adjacent to v and w. The fact that $N(w) \cap (V(G) - N[x] - N[y]) \subseteq \{z, v\}$, implies that $N(w) = N[x] \cup N[y] \cup \{z, v\} - \{w, v_1\}$.

Case 2.2.2.2.1. z is adjacent to a vertex in $N(x) \cup N(y) - \{w, v_1, x, y\}$.

Without loss of generality, z is adjacent to say u. Hence, $N(z) = \{w, u, v\}$. Let $G' = G - uz$. Clearly, G' is connected and $\delta(G') \geq 2$. If G' is claw-free then, by Observation 11, we are done. Hence, G' has a claw $\langle\{w_4, w_3, w_2, w_1\}\rangle$. Furthermore, $w_2 = u$ and $w_3 = z$. Since $N(z) = \{w, u, v\}$, we must have that $w_1 = w$. Furthermore, $w_4 \in N[x] \cup N[y] \cup \{v\} - \{w, v_1\}$. If $w_4 = v$, then w_4 and w_3 will be adjacent which is a contradiction. Hence, $w_4 \in N[x] \cup N[y] - \{w, v_1\}$. The fact that $\langle N(w) - \{z, v\}\rangle$ is complete, implies that w_4 is adjacent to w_2 which is a contradiction.

Case 2.2.2.2.2. z is not adjacent to any vertex in $N(x) \cup N(y) - \{v_1, w, x, y\}$.

We may assume that z has a neighbor z' in $V(G)-N[x]-N[y]-\{v, z\}$. Hence, $N(z) = \{w, v, z'\}$. Let $G' = G - wz$. Clearly, G' is connected and $\delta(G') \geq 2$. If G' is claw-free then, by Observation 11, we are done. Hence, G' has a claw $\langle\{w_4, w_3, w_2, w_1\}\rangle$. Furthermore, $w_2 = w$ and $w_3 = z$. Clearly, $w_1 = z'$. But since $N(w) = N[x] \cup N[y] \cup \{z, v\} - \{w, v_1\}$, we have that w_1 cannot be adjacent to w_2 which is impossible. □

Proof of Lemma 18. Suppose, to the contrary, that G has a 2-path $P : x, v_1, y$. We may assume, by Lemma 17, that $\mathcal{H}(P) = \emptyset$. If x is adjacent to y then, since $\mathcal{H}(P) = \emptyset$, we have that the graph $G' = G - xy$ is claw-free. Furthermore, G' is connected and $\delta(G') \geq 2$ and so, by Observation 11, we are done. Thus, x is not adjacent to y. We may assume, henceforth, that for any 2-path of length two, the end vertices are not adjacent and the only vertex that is adjacent to both end vertices is the degree two vertex on the path. The proof of Lemma 18 will follow from a series of claims.

Claim 1. The only degree two vertex adjacent to x (y respectively) is v_1.

Proof. Suppose, without loss of generality, that u is a degree two vertex in $N(x) - \{v_1\}$. Since $\deg(x) \geq 3$, there is a vertex v in $N(x) - \{v_1, u\}$. Furthermore, the fact that $\langle N(x) - \{v_1\}\rangle$ is complete must imply that $N(u) = \{v, x\}$ and $N(x) = \{v, u, v_1\}$. If $\deg(v) \geq 3$ then the path $P' : x, u, v$ is a 2-path of length two such that x is adjacent to v. This contradicts our earlier assumption. Hence, $\deg(v) = 2$. Define $G_1 = \langle\{u, v, x, v_1\}\rangle$ and $G_2 = G - G_1$. Observe that $\delta(G_2) \geq 2$. This contradicts Lemma 14. □

Let u be an arbitrary vertex in $N(x) - \{v_1\}$. Define $G_1 = \langle\{y, u, x, v_1\}\rangle$ and $G_2 = G - G_1$. We have the following:

Claim 2. If no neighbor of y has degree one in G_2, then at least one of the following holds:

1. There is a vertex in $N(x) - \{v_1, u\}$ that has degree one in G_2.
2. There are two adjacent degree two vertices in $V(G) - N[x] - N[y]$, both being adjacent to u.

Proof. Suppose that no neighbor of y has degree one in G_2. By Lemma 14, we must have that $\delta(G_2) \leq 1$. Let v be a vertex of G_2 that has degree at most one. If $v \in N(x) - \{v_1, u\}$ then by Claim 1, and since $\mathcal{H}(P) = \emptyset$, we have that $\deg_{G_2}(v) = 1$. Hence, we are done.

It follows that $v \in V(G) - N[x] - N[y]$ and $\deg(v) = 2$. Let $z \in N(v) - \{u\}$. If $\deg(z) = 2$, then, since G has no 2-paths of length greater than two, we have that z is adjacent to u and so we are done. Hence, $\deg(z) \geq 3$. Let $G'_1 = \langle\{u, v, x, v_1\}\rangle$ and $G'_2 = G - G'_1$. By Lemma 13 we have that $\delta(G'_2) \leq 1$. Let v' be a vertex of G'_2 that has degree at most one. Since $\langle N(y) - \{v_1\}\rangle$ is complete and $\deg(y) \geq 3$, we have that $v' \in N(x) - \{v_1, u\}$ or $v' \in V(G) - N[x] - N[y] - \{v\}$.

Case 1. $v' \in N(x) - \{v_1, u\}$.

If $v' \neq z$ then the fact that v' is not adjacent to y must imply that $\deg_{G_2}(v') \leq 1$. If $\deg_{G_2}(v') = 0$ then $\deg(v') = 2$, which, by Claim 1, is a contradiction. Hence, $\deg_{G_2}(v') = 1$ and so we are done. Hence, $v' = z$. Note that the path $P' : u, v, v'$ is a 2-path of length two such that $x \in \mathcal{H}(P')$. This contradicts Lemma 17.

Case 2. $v' \in V(G) - N[x] - N[y] - \{v\}$.

If v' is adjacent to v but not to u, then $\deg(v') = 2$ and $v' = z$ and so we contradict the fact that $\deg(z) \geq 3$. Hence, v' is adjacent to u. If v' is not adjacent to v then $\langle\{x, v, v', u\}\rangle$ is a claw. Hence, v' is adjacent to v and so $v' = z$. The path $P' : u, v, z$ is a 2-path of length two such that u and z are adjacent. This contradicts our earlier assumption.

This completes the proof of our claim. □

Case 1. $\deg(x) \geq 4$.

Let u be an arbitrary vertex of $N(x) - \{v_1\}$. Let $v, w \in N(x) - \{v_1, u\}$. Consider, once again, the graph G_2. By Lemma 14 we have that $\delta(G_2) \leq 1$.

Claim 3. If G_2 has no degree one vertex in $N(y) - \{v_1\}$, then G_2 has no degree one vertex in $N(x) - \{v_1, u\}$.

Proof. Suppose that G_2 has no degree one vertex in $N(y) - \{v_1\}$. Suppose, to the contrary, that say v has degree one in G_2. Since $\langle N(x) - \{v_1\}\rangle$ is complete, we have that $N(v) = \{w, x, u\}$ and $N(x) = \{u, v, w, v_1\}$. Define $G'_1 = \langle\{v, y, x, v_1\}\rangle$ and $G'_2 = G - G'_1$. By Lemma 14 we have that $\delta(G'_2) \leq 1$. Since v is not adjacent to any vertex in $V(G) - N[x] - \{y\}$ we have, by Claim 1, that no neighbor of y has degree at most one in G'_2. Furthermore, Claim 2 implies, without loss of generality, that $\deg_{G'_2}(w) = 1$ and so $N(w) = \{v, u, x\}$. Define the graph $G' = G - xu$. The graph G' has minimum degree at least two and is connected. If G' is claw-free then, by Observation 11, we are done.

Hence, G' has a claw $\langle\{w_4, w_3, w_2, w_1\}\rangle$. Clearly, $w_2 = x$ and $w_3 = u$. Furthermore, since $N(x) = \{u, v, w, v_1\}$ we have that $w_1 \in \{v, w\}$. If $w_1 = v$ ($w_1 = w$ respectively) then since $N(w) = \{v, u, x\}$ ($N(v) = \{w, x, u\}$ respectively) we have that $w_4 = w$ ($w_4 = v$ respectively). In both cases w_4 is adjacent to w_3, a contradiction. □

Let u' and v' be two vertices of $N(y) - \{v_1\}$. Let z be an arbitrary vertex in $\{u, v, w\}$ and let $z' \in \{u, v, w\} - \{z\}$.

Case 1.1. $\deg(y) = 3$ and neither u' nor v' is adjacent to a vertex in $V(G) - N[x] - N[y]$.

Let $G'_1 = \langle\{u', v', y, v_1\}\rangle$ and $G'_2 = G - G'_1$. The graph G'_2 has minimum degree at least two, is claw-free and is connected. Furthermore, $G'_2 \notin \mathcal{K}$. By Observation 7 we are done.

Case 1.2. $\deg(y) \geq 4$ or $\deg(y) = 3$ and at least one vertex in $\{u', v'\}$ is adjacent to a vertex in $V(G) - N[x] - N[y]$.

Suppose first that $\deg(y) = 3$. If the graph $G - \langle\{z, y, x, v_1\}\rangle$ has no degree one vertex in $N(y) - \{v_1\}$ for every $z \in \{u, v, w\}$, then neither $G - \langle\{u, y, x, v_1\}\rangle$ nor $G - \langle\{v, y, x, v_1\}\rangle$ has a degree one vertex in $N(y) - \{v_1\}$. If the graph $G - \langle\{z, y, x, v_1\}\rangle$ has a degree one vertex in $N(y) - \{v_1\}$ then, without loss of generality, u' has degree one in $G - \langle\{z, y, x, v_1\}\rangle$ and so $N(u') = \{z, y, v'\}$. This implies that u' has no neighbor in $V(G) - N[x] - N[y]$. Hence, v' must have a neighbor in $V(G) - N[x] - N[y]$ and $\{y, u'\} \subset N(v')$. It follows that for all $z' \in \{u, v, w\} - \{z\}$, we have that the graph $G - \langle\{z', y, x, v_1\}\rangle$ has no degree one vertex in $N(y) - \{v_1\}$. Hence we may assume, without loss of generality, that neither $G - \langle\{u, y, x, v_1\}\rangle$ nor $G - \langle\{v, y, x, v_1\}\rangle$ has a degree one vertex in $N(y) - \{v_1\}$. If $\deg(y) \geq 4$ then neither $G - \langle\{u, y, x, v_1\}\rangle$ nor $G - \langle\{v, y, x, v_1\}\rangle$ has a degree one vertex in $N(y) - \{v_1\}$.

By Claims 2 and 3, we have that u is adjacent to two adjacent degree two vertices u'' and u''' in $V(G) - N[x] - N[y]$. If u is adjacent to some vertex w' in $V(G) - N[x] - \{y\}$, then $\langle\{w', x, u'', u\}\rangle$ is a claw which is a contradiction. Hence, $N(u) = \{u'', u'''\} \cup N[x] - \{v_1, u\}$. By the same argument, v is adjacent to two adjacent degree two vertices v'' and v''' in $V(G) - N[x] - N[y] - \{u'', u'''\}$. Furthermore, $N(v) = \{v'', v'''\} \cup N[x] - \{v, v_1\}$.

Let $G_1' = \langle\{u, v, u'', u''', v'', v'''\}\rangle$ and $G_2' = G - G_1'$. If $\delta(G_2') \geq 2$ then note that $G_2' \notin \mathcal{K}$. Hence, G_2' has a *TRDS* S of cardinality at most $\frac{4(n-6)}{7}$. The set $S \cup \{u, v\}$ is a *TRDS* of G and so $\gamma_{\text{tr}}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{24}{7} + \frac{14}{7} \leq \frac{4n}{7}$, a contradiction. Thus, $\delta(G_2') \leq 1$. Clearly, $\deg_{G_2'}(w) \leq 1$ and so $N(w) = \{v, x, u\}$. Note that w has degree one in $G - \langle\{u, y, x, v_1\}\rangle$ and, by Claim 1, we have that no neighbor of y has degree one in $G - \langle\{u, y, x, v_1\}\rangle$. This contradicts Claim 3.

Case 2. $\deg(x) = 3$.

By symmetry it follows that $\deg(y) = 3$. Let $u, v \in N(x) - \{v_1\}$ and $u', v' \in N(y) - \{v_1\}$. Furthermore, $N(x) = \{u, v, v_1\}$ and $N(y) = \{u', v', v_1\}$.

Claim 4. Every vertex in $N(x) \cup N(y) - \{v_1\}$ has at least one neighbor in $V(G) - N[x] - N[y]$.

Proof. Suppose, to the contrary, that say u is adjacent to no vertex in $V(G) - N[x] - N[y]$. By Claim 1 and the fact that $\langle N(x) - \{v_1\}\rangle$ is complete, we have that $\{x, v\} \subset N(u) \subseteq \{v, x, v', u'\}$. Suppose that u is adjacent to say z, where $z \in \{u', v'\}$. If z is adjacent to some vertex w in $V(G) - N[x] - N[y]$, then $\langle\{u, y, w, z\}\rangle$ is a claw which is a contradiction. Hence, z is adjacent to no vertex in $V(G) - N[x] - N[y]$.

Case 1. u is adjacent to u' and v'.

It immediately follows that neither u' nor v' is adjacent to any vertex in $V(G) - N[x] - N[y]$. Let $G_1' = \langle\{u', v', y, v_1\}\rangle$ and $G_2' = G - G_1'$. The graph G_2' has minimum degree at least two, is claw-free and is connected. If $G_2' \notin \mathcal{K}$ then, by Observation 7, we are done. It follows that $G_2' \in \mathcal{K}$. Since x lies on a triangle, we have that $G_2' \in \{C_3, \mathcal{B}\}$.

If $G_2' = C_3$ then $n(G) = 7$ and the set $\{x, y, v_1\}$ is a *TRDS* of G and so $\gamma_{\text{tr}}(G) \leq 3 < \frac{4.7}{7} = \frac{4n}{7}$, which is a contradiction.

If $G_2' = \mathcal{B}$ then $n(G) = 9$ and the set $\{u_1, u_2, u_3, y, v_1\}$ is a *TRDS* of G and so $\gamma_{\text{tr}}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$, which is a contradiction.

Case 2. u is adjacent to say u' and not to v'.

Clearly, $N(u) = \{u', x, v\}$ and by symmetry, we have, by the previous case, that $N(u') = \{u, y, v'\}$. Let $G' = G - uu'$. The graph G' has minimum degree at least two and is connected. If G' is claw-free then, by Observation 11, we are done. Hence, G' has a claw $\langle\{w_4, w_3, w_2, w_1\}\rangle$. Clearly, $w_2 = u$ and $w_3 = u'$. Hence, $w_1 \in \{x, y, v, v'\}$. But then w_1 cannot be adjacent to both w_2 and w_3, a contradiction. □

Claim 5. Every vertex in $N(x) \cup N(y) - \{v_1\}$ has exactly one neighbor in $V(G) - N[x] - N[y]$.

Proof. By Claim 4 we have that every vertex in $N(x) \cup N(y) - \{v_1\}$ has at least one neighbor in $V(G) - N[x] - N[y]$. Suppose that u has at least two neighbors in $V(G) - N[x] - N[y]$. Define $G_1' = \langle\{v, x, y, v_1\}\rangle$ and $G_2' = G - G_1'$. Note that the graph G_2' has no degree one vertex in $N(y) - \{v_1\}$. As u has at least two neighbors in $V(G) - N[x] - N[y]$, we have that u does not have degree one in G_2' and so, by Claim 2, we have that v is adjacent to two adjacent degree two vertices v'' and v''' in $V(G) - N[x] - N[y]$. Since v has two neighbors in $V(G) - N[x] - N[y]$, we have, by the same argument, that u is adjacent to two degree two vertices u'' and u''' in $V(G) - N[x] - N[y]$. Furthermore, if u (v respectively) is adjacent to a vertex w in $V(G) - N[x] - \{y, v'', v''', u'', u'''\}$ then $\langle\{u'', w, x, u\}\rangle$ ($\langle\{v'', w, x, v\}\rangle$ respectively) induces a claw, a contradiction. Hence, $N(u) = \{u'', u''', v, x\}$ and $N(v) = \{v'', v''', u, x\}$.

We define $G_1'' = \langle\{v'', v''', u'', u''', u, v, x, v_1\}\rangle$ and $G_2'' = G - G_1''$. The graph G_2'' is connected, claw-free and has minimum degree at least two. Furthermore, $G_2'' \notin \mathcal{K}$. Hence, G_2'' has a *TRDS* S of cardinality at most $\frac{4(n-8)}{7}$. The set $\{v_1, x, v, u\} \cup S$ is a *TRDS* G and so $\gamma_{\text{tr}}(G) \leq |S| + 4 \leq \frac{4n}{7} - \frac{32}{7} + \frac{28}{7} \leq \frac{4n}{7}$, which is a contradiction. □

Claim 6. Exactly one of the following holds:

1. There are two distinct degree two vertices w and w' in $V(G) - N[x] - N[y]$, such that w is adjacent to u and w' is adjacent to v. Furthermore, w and w' have no common neighbors.
2. There is exactly one degree two vertex w in $V(G) - N[x] - N[y]$, such that w is adjacent to exactly one vertex in $\{u, v\}$.

Proof. Define $G_1' = \langle\{v_1, u, v, x\}\rangle$ and $G_2' = G - G_1'$.

Case 1. $\delta(G_2') = 0$.

Suppose that w is an isolated vertex in G_2'. Then w has degree two and $N(w) = \{u, v\}$. The path $P' : u, w, v$ is a 2-path of length two such that x is adjacent to both u and v. This contradicts Lemma 17.

Case 2. $\delta(G_2') = 1$.

If a vertex of G_2' has degree one, then it must have degree at most three in G. Suppose that w is a degree one vertex of G_2'. Clearly, $w \in V(G) - N[x] - N[y]$. Assume first that w has degree three in G. Hence, w is adjacent to a vertex $w' \in V(G) - N[x] - \{y, w\}$. Thus, $N(w) = \{w', u, v\}$. Define $G' = G - xv$. The graph G' has minimum degree at least two and is connected. If G' is claw-free then, by Observation 11, we are done. Hence, G' has a claw $\langle\{w_4, w_3, w_2, w_1\}\rangle$ where $w_2 = x$ and $w_3 = v$. Since $N(x) = \{v_1, u, v\}$ we have that $w_1 = u$. If $w_4 = w$ then w_3 and w_4 are adjacent which is a contradiction. By Claim 5 we have that $w_4 \in N(y) - \{v_1\}$ and so the graph $G - \langle\{v, y, x, v_1\}\rangle$ has minimum degree at least two. This contradicts Lemma 14. We may conclude that $\deg(w) = 2$ and that w is adjacent to exactly one vertex in the set $\{u, v\}$, say u. If no degree two vertex in $V(G) - N[x] - N[y] - \{w\}$ is adjacent to v then we are done. We may assume that there is a degree two vertex w' in $V(G) - N[x] - N[y] - \{w\}$, such that w' is adjacent to v.

By Claim 5, u (v respectively) is not adjacent to w' (w respectively). Let $t \in N(w) - \{u\}$ ($t' \in N(w') - \{v\}$ respectively). The fact that G has no 2-paths of length greater than two implies that $\deg(t) \geq 3$ and $\deg(t') \geq 3$. If $t = t'$ then since $N(w) = \{u, t\}$ and $N(w') = \{v, t'\}$ we have that G must contain a claw, which is a contradiction. Hence, w and w' have no common neighbors.

Case 3. $\delta(G_2') \geq 2$.

If G_2' has no components in $\mathcal{K}$ then, by Observation 7, we are done. Hence, G_2' has a component $\mathcal{U}$ in $\mathcal{K}$. Clearly, $N[\mathcal{U}] \subseteq V(\mathcal{U}) \cup \{v, u, v_1\}$. In addition, also note that if $N[y] \cap V(\mathcal{U}) \neq \emptyset$ and $y \notin V(\mathcal{U})$ then there will be a vertex of $V(\mathcal{U})$ that is adjacent to y and this will contradict the fact that $N[\mathcal{U}] \subseteq V(\mathcal{U}) \cup \{v, u, v_1\}$. Clearly, $y \in V(\mathcal{U})$ if $N[y] \cap V(\mathcal{U}) \neq \emptyset$.

Suppose that $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$. If $N[y] \cap V(\mathcal{U}) \neq \emptyset$ then, without loss of generality, we have that say $u_1 = y$. By Observation 8 we have that $\deg(v_1) \geq 3$ which is a contradiction. Hence, $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y]$. Furthermore, u_1 is adjacent to say u. By Observation 8, u is adjacent to two consecutive vertices of $\mathcal{U}$. This contradicts Claim 5. Hence, $\mathcal{U} \in \{\mathcal{B}, C_3\}$.

Suppose first that $\mathcal{U} = C_3$. If $N[y] \cap V(\mathcal{U}) \neq \emptyset$ then, without loss of generality, we have that $u_1 = y$. Hence, u_2 is say u' and u_3 is v'. But then neither u' nor v' is adjacent to a vertex in $V(G) - N[x] - N[y]$, contradicting Claim 4. It follows that $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y]$. Clearly, u_1 is adjacent to say u. If u is adjacent to a vertex $w \in N(y)$, then $\langle\{w, x, u_1, u\}\rangle$ is a claw which is a contradiction. By Claim 5 we have that $N(u) = \{v, u_1, x\}$. If v is adjacent to no vertex of $V(\mathcal{U})$ then v has a neighbor in $V(G) - V(\mathcal{U}) - N[x] - N[y]$. The graph $G - \langle V(\mathcal{U}) \cup \{u\}\rangle$ has minimum degree at least two and has no components in $\mathcal{K}$. By Observation 7, we are done. Hence, v has exactly one neighbor in $V(\mathcal{U})$. By using the same argument that was used for u we can deduce that $N(v) = \{x, u, z\}$, where $z \in V(\mathcal{U})$. If v is adjacent to say u_2, then $P' : u_1, u_3, u_2$ is a 2-path where $u_1u_2 \in E(G)$ and this contradicts our earlier assumption. Hence, v is adjacent to u_1 and so $N(v) = \{x, u, u_1\}$. Let $G' = G - xv$. The graph G' has minimum degree at least two, is connected and claw-free. By Observation 11, we are done.

We may assume that $\mathcal{U} = \mathcal{B}$. Suppose first that $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y]$. By Observation 8, u is adjacent to say u_1. If $\deg(u_2) = 2$ then the path $P' : u_1, u_2, u_3$ is a 2-path of length two such that the end vertices are adjacent, a contradiction. Hence, $\deg(u_2) \geq 3$. Furthermore, if u is adjacent to u_2 then Claim 5 will be contradicted. Hence, v is adjacent to u_2. If u is adjacent to a vertex $w \in N(y)$ then $\langle\{w, x, u_1, u\}\rangle$ is a claw which is a contradiction. Hence, $N(u) = \{u_1, v, x\}$. By the same argument $N(v) = \{u_2, u, x\}$. Let $G' = G - vu_2$. The graph G' has minimum degree at least two, is connected and claw-free. By Observation 11, we are done. Hence, $N[y] \cap V(\mathcal{U}) \neq \emptyset$.

If $u_3 = y$ then the second part of Observation 8 implies that v_1 is adjacent to say u_1 which is a contradiction, as $\deg(v_1) = 2$. Hence, without loss of generality, we have that $u_1 = y$. Hence, $u_2 = v'$ and $u_3 = u'$. But then u_3 has two neighbors, u_4 and u_5, in $V(G) - N[x] - N[y]$ and this contradicts Claim 5. □

Claim 7. The set $V(G) - N[x] - N[y]$ cannot have two distinct degree two vertices w and w', such that w is adjacent u and w' adjacent to v.

Proof. Suppose, to the contrary, that w and w' are two distinct degree two vertices in $V(G) - N[x] - N[y]$, such that w is adjacent to u and w' is adjacent to v. Let $G_1' = \langle\{x, v_1, u, v, w, w'\}\rangle$ and $G_2' = G - G_1'$. By the first part of Claim 6 we have that $\delta(G_2') \geq 2$.

Suppose first that G_2' has no component in $\mathcal{K}$. Then G_2' has a *TRDS* S of cardinality at most $\frac{4(n-6)}{7}$. Constructing a *TRDS* of G of cardinality $|S| + 3$ will suffice since then $\gamma_{\mathrm{tr}}(G) \leq |S| + 3 \leq \frac{4n}{7} - \frac{24}{7} + \frac{21}{7} \leq \frac{4n}{7}$, which is a contradiction. Consider, arbitrarily, vertices v_1 and w from the set $\{v_1, w, w'\}$. If v_1 and w have a neighbor in S then $\{w', v\} \cup S$ is a *TRDS* of G. If v_1 and w both have a neighbor in $V(G_2') - S$ and w' is adjacent to a vertex in S, then $\{u, x\} \cup S$ is a *TRDS* of G. Hence, every vertex from the set $\{v_1, w, w'\}$ has a neighbor in $V(G_2') - S$. The set $\{u, v, x\} \cup S$ is a *TRDS* of G. Hence, G_2' has a component $\mathcal{U}$ in $\mathcal{K}$. If $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$, then, without loss of generality, u_1 is adjacent to say w and, by the first part of Observation 8, we get that $\deg(w) \geq 3$ which is a contradiction. Hence, $\mathcal{U} \in \{\mathcal{B}, C_3\}$. Note that $N[\mathcal{U}] \subseteq V(\mathcal{U}) \cup \{v, u, v_1, w, w'\}$ and if $N[y] \cap V(\mathcal{U}) \neq \emptyset$ then $y \in V(\mathcal{U})$.

Suppose that $\mathcal{U} = C_3$. Suppose first that $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y] - \{w, w'\}$. Without loss of generality, u_1 is adjacent to say w. If $\deg(u_2) = \deg(u_3) = 2$, then we may form $G_1'' = \langle V(\mathcal{U}) \cup \{w\}\rangle$ and $G_2'' = G - G_1''$. The graph G_2'' has degree at least two, is connected and $G_2'' \notin \mathcal{K}$. By Observation 7 we are done. Hence, u_2 is adjacent to say w'. It follows that $\deg(u_3) = 2$. The path $P' : u_1, u_3, u_2$ is a 2-path with adjacent end vertices which is a contradiction. Hence, $N[y] \cap V(\mathcal{U}) \neq \emptyset$. Suppose that $u_1 = y$. Hence, $u_2 = v'$ and $u_3 = u'$. By Claim 4, v' (u' respectively) must be adjacent to w (w' respectively). Hence, $n(G) = 9$. The set $\{x, u, v, w, v'\}$ is a *TRDS* of G and so $\gamma_{\mathrm{tr}}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$, which is a contradiction.

Hence, $\mathcal{U} = \mathcal{B}$. Suppose first that $N[y] \cap V(\mathcal{U}) \neq \emptyset$. Hence, without loss of generality, $u_1 = y$. Furthermore, $u_2 = u'$ and $u_3 = v'$. But then u_3 will be adjacent to two vertices, u_5 and u_4, in $V(G) - N[x] - N[y]$. This contradicts Claim 5. Hence, $V(\mathcal{U}) \subseteq V(G) - N[x] - N[y] - \{w, w'\}$. Note that if either w or w', say w, is adjacent to u_3 then, by the second part of Observation 8, we get that $\deg(w) \geq 3$ which is a contradiction. Hence, u_1 is adjacent to say w. If $\deg(u_2) = 2$ then G has a 2-path of length two with adjacent end vertices, a contradiction. Hence, without loss of generality, w' is adjacent to u_2. Let $G_1'' = \langle V(\mathcal{U}) \cup \{w, w'\}\rangle$ and $G_2'' = G - G_1''$. The graph G_2'' has degree at least two, is connected and $G_2'' \notin \mathcal{K}$. Hence, G_2'' has a

TRDS S of cardinality at most $\frac{4(n-7)}{7}$. If, without loss of generality, $u \in S$ then $\{u_3, u_2, w'\} \cup S$ is a *TRDS* of G. If $u, v \notin S$ then $\{u_3, u_2, u_1\} \cup S$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \le |S| + 3 \le \frac{4n}{7} - \frac{28}{7} + \frac{21}{7} \le \frac{4n}{7}$ which is a contradiction. □

By Claims 6 and 7 and by symmetry, we may assume that say u (u' respectively) is adjacent to a degree two vertex w (w' respectively) in $V(G) - N[x] - N[y]$. Note that if w and w' are adjacent, then G will have a 2-path of length three which is a contradiction. Furthermore, v (v' respectively) has exactly one neighbor in $V(G) - N[x] - N[y]$ and this neighbor has degree at least three. If v is adjacent to v' then the graph $G - \langle\{v_1, x, u, w\}\rangle$ will have minimum degree at least two. This will contradict Lemma 13. Hence, v' is not adjacent to v. Let $G_1' = \langle N[y] \cup N[x]\rangle$. We form the graph $G_2' = G - G_1'$.

We first claim that no components of G_2' are in $\mathcal{K}$. Suppose, to the contrary, that $\mathcal{U}$ is a component of G_2' that is in $\mathcal{K}$. Note that either v or v', say v, is adjacent to a vertex of $\mathcal{U}$. If $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$ then, by the first part of Observation 8, v is adjacent to two vertices of $\mathcal{U}$. This contradicts Claim 5. Hence, $\mathcal{U} \in \{\mathcal{B}, C_3\}$. Let $G_1'' = \langle\{v\} \cup V(\mathcal{U})\rangle$ and $G_2'' = G - G_1''$. Note that $\delta(G_2'') \ge 2$ and G_2'' has no components in $\mathcal{K}$. If $\mathcal{U} = C_3$ ($\mathcal{U} = \mathcal{B}$ respectively) then we contradict Observation 7 (second part of Observation 10 respectively). Hence, G_2' has no components in $\mathcal{K}$.

We also claim that w and w' are the only possible vertices of degree at most one in G_2'. Suppose that there is a vertex w'' in $V(G_2') - \{w, w'\}$ that has degree at most one. Note that $\deg(w'') = 3$ and w'' must be adjacent to both v' and v. Since v and v' are not adjacent and by Claim 5, we have that G has a claw which is a contradiction.

Case 1. $w = w'$.

Let $G_1'' = \langle N[y] \cup N[x] \cup \{w\}\rangle$ and $G_2'' = G - G_1''$. Clearly, the fact that G_2' has no components in $\mathcal{K}$ implies that G_2'' has no components in $\mathcal{K}$. Since w and w' are the only possible vertices of degree at most one in G_2', we have that $\delta(G_2'') \ge 2$. Hence, G_2'' has a *TRDS* S of cardinality at most $\frac{4(n-8)}{7}$. The set $\{u, v, v', y\} \cup S$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \le |S| + 4 \le \frac{4n}{7} - \frac{32}{7} + \frac{28}{7} \le \frac{4n}{7}$ which is a contradiction.

Case 2. $w \ne w'$.

Form G_2'', from G_2', by joining w and w'. Note that the component of G_2'' that contains the edge ww' has a 2-path of length three. Hence, this component cannot be in $\mathcal{K}$. Furthermore, the fact that G_2' has no components in $\mathcal{K}$ implies that G_2'' has no components in $\mathcal{K}$. In addition, $\delta(G_2'') \ge 2$. Hence, G_2'' has a *TRDS* S of cardinality at most $\frac{4(n-7)}{7}$. If $w, w' \in S$ ($w, w' \notin S$ respectively) then $\{v', y, u', u\} \cup S$ ($\{v', y, v_1, x\} \cup S$ respectively) is a *TRDS* of G. If, without loss of generality, $w' \in S$ and $w \notin S$ then $\{x, u, v, u'\} \cup S$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \le |S| + 4 \le \frac{4n}{7}$ which is a contradiction. □

Proof of Lemma 19. Suppose, to the contrary, that $\delta(G) \ge 3$. Let u be a vertex of G of minimum degree. Let $G_1 = \langle N[u]\rangle$ and $G_2 = G - G_1$.

Claim 1. $\langle N(u)\rangle$ contains no isolated vertices and G has no bridges.

Proof. Let x be an isolate in $\langle N(u)\rangle$. Let $G' = G - xu$. The graph G' has minimum degree at least two and has no components isomorphic to C_3. If G' has a claw then the center of this claw must be in $N(u) - \{x\}$. This contradicts the fact that x is an isolate in $\langle N(u)\rangle$. By Observation 11, we are done. If G has a bridge, then we may form G' by removing, from G, this bridge. The graph G' is claw-free, $\delta(G') \ge 2$ and G' has no components in $\mathcal{K}$. By Observation 11, we are done. □

Claim 2. We may assume that G_2 has no components in $\mathcal{K}$.

Proof. Suppose, to the contrary, that G_2 has a component $\mathcal{U}$ in $\mathcal{K}$. Note that each degree two vertex of $\mathcal{U}$ has $\delta - 2$ neighbors in $N(u)$.

Suppose first that $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$. Assume first that either $\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3, C_5\}$ or $\mathcal{U} = C_5$ and, without loss of generality, u_5 has at least two neighbors in $N(u)$. Let $G_1' = \langle\{u_1, u_2, u_3, u_4\}\rangle$ and $G_2' = G - G_1'$. The graph G_2' has minimum degree at least two, is connected, and $G_2' \notin \mathcal{K}$. By Observation 7, we are done. Hence, $\mathcal{U} = C_5$ and every vertex of $\mathcal{U}$ has exactly one neighbor in $N(u)$. Furthermore, $\delta = 3$. Let $x, y, z \in N(u)$. Suppose, without loss of generality, that u_1 is adjacent to say x. By Observation 8, x is adjacent to say u_2. If u_3 is adjacent to x then $\langle\{u, u_3, u_1, x\}\rangle$ is a claw which is a contradiction. Hence, suppose that u_3 is adjacent to y. By Observation 8, y is adjacent to say u_4. If u_5 is adjacent to x (y respectively) then $\langle\{u, u_5, u_2, x\}\rangle$ ($\langle\{u, u_5, u_3, y\}\rangle$ respectively) is a claw which is a contradiction. By the pigeonhole principle, u_5 is adjacent to z and so z is adjacent to either u_1 or u_4 which is a contradiction. Hence, $\mathcal{U} \in \{\mathcal{B}, C_3\}$. Let x be a vertex of $N(u)$ that is adjacent to say u_1.

Suppose that $\mathcal{U} = \mathcal{B}$. Let $G_1' = \langle V(\mathcal{U})\rangle$ and $G_2' = G - G_1'$. The graph G_2' is connected and has minimum degree at least two. Note that if $G_2' \in \mathcal{K}$ then $G_2' = \mathcal{B}$. Hence, if $G_2' \in \mathcal{K}$ ($G_2' \notin \mathcal{K}$ respectively) then G_2' has a *TRDS* S of cardinality at most $\frac{4(n-5)}{7} + \frac{1}{7}$. If, without loss of generality, u_1 has a neighbor in $V(G_2') - S$ then $\{u_2, u_3\} \cup S$ is a *TRDS* of G. If u_1 and u_2 have a neighbor in S then $\{u_4, u_5\} \cup S$ is a *TRDS* of G. Hence, $\gamma_{\mathrm{tr}}(G) \le |S| + 2 \le \frac{4(n-5)}{7} + \frac{1}{7} + \frac{14}{7} \le \frac{4n}{7}$ which is a contradiction.

Hence, $\mathcal{U} = C_3$. Note that if x is adjacent to a vertex $w \in V(G_2) - V(\mathcal{U})$, then $\langle\{u, w, u_1, x\}\rangle$ is a claw. Hence, $N(x) \cap V(G_2) \subseteq V(\mathcal{U})$. Let $G_1' = \langle\{u_1, u_2, u_3, x\}\rangle$ and $G_2' = G - G_1'$. The graph G_2' is connected. Assume first that $\delta(G_2') \ge 2$. By Observation 7, $G_2' \in \mathcal{K}$. Let $u_1', \ldots, u_i'$ be the vertices of the Hamiltonian path of G_2'. If $G_2' \in \mathcal{K} - \{\mathcal{B}, C_3\}$ then, without loss of generality, let $u = u_1'$. Furthermore, $u_i', u_2' \in N(u) - \{x\}$ and $V(G_2') - \{u_i', u_1', u_2'\} \subset V(G_2)$. But then, by the pigeonhole principle and the fact that $\delta \ge 3$, we have that every vertex of $V(G_2') - \{u_i', u_1', u_2'\}$ is adjacent to x. This contradicts the fact that $N(x) \cap V(G_2) \subseteq V(\mathcal{U})$. Hence, $G_2' \in \{\mathcal{B}, C_3\}$.

If $G_2' = \mathcal{B}$ then $n(G) = 9$ and the set $\{u_1', u_2', u_3', x, u_1\}$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq 5 < \frac{4.9}{7} = \frac{4n}{7}$ which is a contradiction.

If $G_2' = C_3$ then $n(G) = 7$ and the set $\{u, x, u_1\}$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq 3 < \frac{4.7}{7} = \frac{4n}{7}$ which is a contradiction.

Hence, $\delta(G_2') \leq 1$ and so there is a vertex $y \in N(u) - \{x\}$ such that $N(y) \subseteq V(\mathcal{U}) \cup \{x, u\}$. By Claim 1, y is not isolated in $\langle N(u) \rangle$ and so y is adjacent to x. Furthermore, the pigeonhole principle implies that y has a neighbor in $V(\mathcal{U})$. If we re-label x as y in G_2', we have, by using the argument of the previous paragraph, that x has degree one in G_2'. Hence, we may assume that $N(x) \subseteq V(\mathcal{U}) \cup \{y, u\}$. If $\delta = 3$ then $\langle N(u) \rangle$ has an isolate, contradicting Claim 1. Hence, $\delta \geq 4$. By the pigeonhole principle, x and y have a common neighbor in $V(\mathcal{U})$, say u_1. Let $G_1'' = \langle \{u_1, u_2, u_3, x, y\} \rangle$ and $G_2'' = G - G_1''$. Since $\delta \geq 4$ and by Claim 1, we have that G_2'' has a triangle. Furthermore, G_2'' is connected and $\delta(G_2'') \geq 2$. Suppose first that $G_2'' \notin \mathcal{K}$. Hence, G_2'' has a *TRDS* S of cardinality at most $\frac{4(n-5)}{7}$. If $u \notin S$ or ($u \in S$ respectively) then $\{y, u_1\} \cup S$ ($\{u_2, u_3\} \cup S$ respectively) is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{20}{7} + \frac{14}{7} \leq \frac{4n}{7}$ which is a contradiction. Hence, $G_2'' \in \{\mathcal{B}, C_3\}$.

Let $u_1', \ldots, u_i'$ be the vertices of the Hamiltonian path of G_2''. If $G_2'' = C_3$ then set $u = u_1'$. If $G_2'' = \mathcal{B}$ then let $u \in \{u_1', u_3'\}$. If $G_2'' = \mathcal{B}$ ($G_2'' = C_3$ respectively) then $n(G) = 10$ ($n(G) = 8$ respectively) and so $\{u_1', u_2', u_3', u_2, u_3\}$ ($\{u_1', x, u_1, y\}$ respectively) is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq \frac{n}{2} < \frac{4n}{7}$ which is a contradiction. □

Suppose that $\delta(G_2) \geq 2$ or $V(G_2) = \emptyset$. By Claim 2 we may assume that G_2 has no components in $\mathcal{K}$. It follows that G_2 has a *TRDS* S of cardinality at most $\frac{4(n-\delta-1)}{7}$. Let x be an arbitrary vertex of $N(u)$. If $\{x, u\} \cup S$ is a *TRDS* of G, then $\gamma_{tr}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{4(\delta+1)}{7} + \frac{14}{7} \leq \frac{4n}{7}$ which is a contradiction. Hence, there is a vertex $y \in N(u) - \{x\}$ such that $\{u, x\} \subset N(y) \subseteq S \cup \{u, x\}$. By switching the roles of y and x we have, by the same argument, that $\{u, y\} \subset N(x) \subseteq S \cup \{u, y\}$. Since x is arbitrary, we have that every vertex of $N(u)$ has a neighbor in S. Hence, $\{x\} \cup S$ is a *TRDS* of G and so $\gamma_{tr}(G) < |S| + 2 \leq \frac{4n}{7} - \frac{4(\delta+1)}{7} + \frac{14}{7} \leq \frac{4n}{7}$ which is a contradiction. Hence, $\delta(G_2) \leq 1$. If $\delta(G_2) = 0$ then there is a vertex $x \in V(G_2)$ such that $N(x) = N(u)$. Furthermore, if $y \in N(u)$ is adjacent to a vertex $w \in V(G_2) - \{x\}$ then $\langle \{w, x, u, y\} \rangle$ induces a claw. Thus, $V(G_2) - \{x\} = \emptyset$. Hence, $n(G) = \delta + 2$ and the set $\{u, y\}$ is a *TRDS* of G. Thus, $\gamma_{tr}(G) = 2 < \frac{4(\delta+2)}{7} = \frac{4n}{7}$ which is a contradiction. Thus, $\delta(G_2) = 1$.

Let $\deg_{G_2}(x) = 1$ and let $y \in N(u)$, where x is adjacent to every vertex of $N(u) - \{y\}$. Note that x is adjacent to exactly one vertex in $V(G_2) - \{x\}$, say x'. Furthermore, we claim if z' is an arbitrary vertex in $N(u) - \{y\}$, then z' is adjacent to no vertex in $V(G_2) - \{x, x'\}$. Suppose, to the contrary, that there is a vertex $w \in V(G_2) - \{x, x'\}$ that is adjacent to z'. The graph $\langle \{x, u, w, z'\} \rangle$ is a claw. Hence, $N[z'] \subseteq \{x', x\} \cup N[u]$ for every $z' \in N(u) - \{y\}$.

Let $G_1' = \langle N[u] \cup \{x\} \rangle$ and $G_2' = G - G_1'$. By Claim 1 we have that y is adjacent to a vertex in $N(u) - \{y\}$, say z.

Suppose first that $\delta(G_2') \geq 2$. If G_2' has no components in $\mathcal{K}$ then G_2' has a *TRDS* S of cardinality at most $\frac{4(n-\delta-2)}{7}$. The set $\{z, x\} \cup S$ is a *TRDS* of G and so $\gamma_{tr}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{4(\delta+2)}{7} + \frac{14}{7} \leq \frac{4n}{7}$ which is a contradiction. Hence, G_2' has a component $\mathcal{U}$ in $\mathcal{K}$. Now if x is adjacent to no vertex of $V(\mathcal{U})$ then $\mathcal{U}$ will be a component of G_2, contradicting Claim 2. We have, by the second part of Observation 8, that $x' = u_1$. It follows that the remaining degree two vertices of $\mathcal{U}$ must be adjacent to y.

If $\mathcal{U} = \mathcal{B}$ ($\mathcal{U} \in \mathcal{K} - \{\mathcal{B}, C_3\}$ respectively) then $\langle \{u, u_4, u_2, y\} \rangle$ induces a claw. Hence, $\mathcal{U} = C_3$ and if y is adjacent to a vertex $w \in V(G_2') - V(\mathcal{U})$ then $\langle \{u, u_2, w, y\} \rangle$ induces a claw. Hence, $n(G) = \delta + 5$. The set $\{z, x, y\}$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq 3 < \frac{4(\delta+5)}{7} = \frac{4n}{7}$ which is a contradiction.

Hence, $\delta(G_2') \leq 1$. Let w be a vertex of G_2' that has degree at most one. If w is not adjacent to x then $w \neq x'$ and, by the pigeonhole principle, w is adjacent to a vertex in $N(u) - \{y\}$. This contradicts the fact that no vertex in $N(u) - \{y\}$ is adjacent to a vertex in $V(G_2) - \{x, x'\}$. Hence, $x' = w$.

Case 1. $\deg_{G_2'}(x') = 0$.

Hence, $N(x') \subseteq N(u) \cup \{x\}$. Let z' be a neighbor of x' in $N(u) - \{y\}$. The only vertex in $\langle N[u] \cup \{x\} \rangle$ that can be adjacent to a vertex of $V(G_2) - \{x, x'\}$ is y. Let $G_1'' = \langle N[u] \cup \{x, x'\} \rangle$ and $G_2'' = G - G_1''$. Note that $\delta(G_2'') \geq 2$ or $V(G_2'') = \emptyset$. If G_2'' has a component $\mathcal{U}$ in $\mathcal{K}$, then $\mathcal{U}$ will also be a component of G_2. This contradicts Claim 2. Hence, G_2'' has no components in $\mathcal{K}$. It follows that G_2'' has a *TRDS* S of cardinality at most $\frac{4(n-\delta-3)}{7}$. The set $\{u, y, z'\} \cup S$ is a *TRDS* of G. Thus, $\gamma_{tr}(G) \leq |S| + 3 \leq \frac{4n}{7} - \frac{4(\delta+3)}{7} + \frac{21}{7} \leq \frac{4n}{7}$ which is a contradiction.

Case 2. $\deg_{G_2'}(x') = 1$.

Note that x' has exactly one neighbor in $V(G_2') - \{x'\}$, say x''. Furthermore, $N(x') \subseteq \{x, x''\} \cup N(u)$. Suppose first that y is not adjacent to x'. Then $N(x') \subseteq \{x, x''\} \cup N(u) - \{y\}$. Let $G' = G - x'x''$. Clearly, $\delta(G') \geq 2$ and G' has no components isomorphic to C_3. If G' has a claw, then the center of this claw must be in $N(u) \cup \{x\} - \{y\}$. But no vertex in $N(u) \cup \{x\} - \{y\}$ is adjacent to a vertex of $V(G_2) - \{x, x'\}$. Hence, G' is claw-free and so, by Observation 11, we are done. Hence, y is adjacent to x'.

If y is not adjacent to x'' then if y is adjacent to a vertex $w' \in V(G_2) - \{x, x', x''\}$ then $\langle \{u, x', w', y\} \rangle$ induces a claw. Hence, $N[y] \subseteq N[u] \cup \{x', x\}$. It follows that $x'x''$ is a bridge. Hence, y is adjacent to x''. Let $G_1'' = \langle N[u] \cup \{x\} - \{y\} \rangle$ and $G_2'' = G - G_1''$. The graph G_2'' is connected, $\delta(G_2'') \geq 2$ and G_2'' has no component in $\mathcal{K}$. Hence, G_2'' has a *TRDS* S of cardinality at most $\frac{4(n-\delta-1)}{7}$. The set $\{x, z\} \cup S$ is a *TRDS* of G. Hence, $\gamma_{tr}(G) \leq |S| + 2 \leq \frac{4n}{7} - \frac{4(\delta+1)}{7} + \frac{14}{7} \leq \frac{4n}{7}$ which is a contradiction. □

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