ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 427 (2007) 42–54
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On the null-spaces of acyclic and unicyclic singular graphs

Milan Nath [a], Bhaba Kumar Sarma [b,*]

[a] *Department of Mathematics, Handique Girls' College, Guwahati 781 001, India*
[b] *Department of Mathematics, IIT Guwahati, Guwahati 781 039, India*

Received 3 January 2006; accepted 17 June 2007
Available online 27 June 2007
Submitted by S. Fallat

Abstract

For acyclic and unicyclic graphs we determine a necessary and sufficient condition for a graph G to be singular. Further, it is shown that this characterization can be used to construct a basis for the null-space of G.

AMS classification: 05C05; 05C50; 15A18

Keywords: Adjacency matrix; Singular graph; Unicyclic graph; Matching; Null-eigenvector; Null-space

1. Introduction

Let G be a finite undirected graph without loops or multiple edges on vertices $v_1, v_2, \ldots, v_n$. The *adjacency matrix* of G is defined as the $n \times n$ matrix $A(G) = (a_{ij})$, with $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge and 0 otherwise. Since $A(G)$ is a real symmetric matrix, all its eigenvalues are real and their algebraic multiplicities are same as their respective geometric multiplicities. The graph G is said to be *singular* (*nonsingular*) if $A(G)$ is singular (nonsingular). It is clear that G is singular if and only if G has a connected component which is singular. In particular, if G has an isolated vertex, then G is singular.

* Corresponding author. Tel.: +91 361 2582609; fax: +91 361 2582649.
E-mail address: bks@iitg.ernet.in (B.K. Sarma).

0024-3795/$ - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2007.06.017

The multiplicity of 0 as an eigenvalue of $A(G)$ is called the *nullity* of G and is denoted by $\eta(G)$. The eigenvectors of $A(G)$ corresponding to the eigenvalue 0 of a singular graph G and the null-space of $A(G)$ are called the *null-eigenvectors* and the *null-space* of G, respectively.

The problem of characterizing a singular graph by its graph theoretic properties was first posed by Collatz and Sinogowitz [1] almost 50 years back. The problem is relevant in many disciplines of science (see [2,4]). Some recent discussions on singularity of graphs in specific situations can be seen in [5,7].

A connected graph with a unique cycle is called a *unicyclic* graph. A *matching* M in a graph G is a set of mutually nonadjacent edges in G. We denote the set of vertices of the edges in M by $V(M)$. A matching M in G is *perfect*, if every vertex of G is in $V(M)$. The following well-known results are helpful in computing $\eta(G)$, particularly when G is acyclic or unicyclic.

Theorem 1.1 [3]. *Let v be a pendent vertex of a graph G and u be the vertex in G adjacent to v. Then, $\eta(G) = \eta(G - u - v)$, where $G - u - v$ is the induced subgraph of G obtained by deleting u and v.*

Theorem 1.2 [3]. *If q is the maximum number of mutually nonadjacent edges in a tree T having n vertices, then $\eta(T) = n - 2q$. In particular, T is nonsingular if and only if T has a perfect matching.*

In Section 2 of this paper, we derive a necessary condition for a graph to be singular in terms of its graph properties. We also give a sufficient condition for a graph to be singular. In Section 3, we show that our necessary condition is also sufficient for acyclic and unicyclic graphs. In Section 4, we show how this characterization can be used to construct bases for the null-spaces of trees and unicyclic graphs.

2. Singularity of a graph

Let $V(G)$ and $E(G)$ denote the vertex set $\{v_1, v_2, \dots, v_n\}$ and the edge set of a graph G, respectively. The *neighborhood* of a vertex $v \in V$ in G is defined to be $N(v) = \{u \in V(G) | \{u, v\} \in E(G)\}$.

A nonzero vector $(\alpha_1, \alpha_2, \dots, \alpha_n)^{\mathrm{t}}$ is a null-eigenvector of G if and only if for each $v_i \in V(G)$ we have $\sum_{v_j \in N(v_i)} \alpha_j = 0$.

Let $A(G) = [C_1, C_2, \dots, C_n]$, where C_j is the jth column vector of $A(G)$. If G is singular and $(\alpha_1, \alpha_2, \dots, \alpha_n)^{\mathrm{t}}$ is a null-eigenvector of $A(G)$, then the relation

$$\alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n = 0$$

is called a *kernel relation* of G.

Definition 2.1. A pair V_1, V_2 of subsets of $V(G)$ is said to satisfy the *property* (N) if (a) V_1 and V_2 are nonempty and disjoint, and (b) $\bigcup\{N(v) | v \in V_1\} = \bigcup\{N(v) | v \in V_2\}$. Further, such a pair is said to be *minimal satisfying the property* (N) if for any pair U_1, U_2 of $V(G)$ satisfying the property (N) with $U_1 \subseteq V_1,\ U_2 \subseteq V_2$, we have $U_1 = V_1,\ U_2 = V_2$.

Theorem 2.2. *Let G be a connected graph on $n \geqslant 2$ vertices. If G is singular, then $V(G)$ has a pair of subsets satisfying the property* (N).

Proof. Let G be singular and

$$\alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n = 0$$

be a kernel relation of G. Let $V_1 = \{v_j | \alpha_j > 0\}$ and $V_2 = \{v_j | \alpha_j < 0\}$. Since $A(G)$ is nonnegative and has no zero columns, V_1 and V_2 are nonempty. Clearly, $V_1 \cap V_2 = \emptyset$, and we have

$$\sum_{v_j \in V_1} \alpha_j C_j + \sum_{v_j \in V_2} \alpha_j C_j = 0. \tag{2.1}$$

Let $X = \bigcup\{N(v) | v \in V_1\}$ and $Y = \bigcup\{N(v) | v \in V_2\}$. Let $v_k \in V_1$ and $v_i \in N(v_k)$. Then $a_{ik} = 1$. This implies that the ith entry of the vector $\sum_{v_j \in V_1} \alpha_j C_j$ is positive. Therefore, in view of (2.1), the ith entry of the vector $\sum_{v_j \in V_2} \alpha_j C_j$ must be negative. Consequently, $a_{il} = 1$, that is, $v_i \in N(v_l)$ for some $v_l \in V_2$. This shows that $X \subseteq Y$. Similarly, we have $Y \subseteq X$. □

Definition 2.3. A pair V_1, V_2 of subsets of $V(G)$ is said to satisfy the *property* (S) if it satisfies the property (N) and for all pairs u, v in $V_i,\ i = 1, 2$, we have $N(u) \cap N(v) = \emptyset$.

Theorem 2.4. *Suppose that $V(G)$ has a pair of subsets V_1 and V_2 satisfying the property (S). Then G is singular.*

Proof. Let $X = \bigcup\{N(v) | v \in V_1\} = \bigcup\{N(v) | v \in V_2\}$. For $v_i \in X$ we have a unique $v_j \in V_1$, such that $v_i \in N(v_j)$. Consequently, $a_{ij} = 1$ and $a_{it} = 0$ for all other $v_t \in V_1$. Similarly, there exists a unique $v_k \in V_2$ such that $a_{ik} = 1$. On the other hand, if $v_i \notin X$, then $a_{is} = 0$ for all $v_s \in V_1 \cup V_2$. Consequently,

$$\sum_{v_j \in V_1} C_j = \sum_{v_j \in V_2} C_j, \tag{2.2}$$

which shows that the columns of $A(G)$ are linearly dependent. □

From the kernel relation (2.2) we have the following.

Corollary 2.5. *Let V_1, V_2 be a pair in $V(G)$ satisfying the property (S). Let α_j be defined by*

$$\alpha_j = \begin{cases} 1, & if\ v_j \in V_1, \\ -1, & if\ v_j \in V_2, \\ 0, & otherwise. \end{cases} \tag{2.3}$$

Then $(\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{t}}$ is a null-eigenvector of G.

Theorem 2.2 gives a necessary condition for G to be singular. In general the condition is not sufficient. For example, consider the complete graph K_4 on the vertex set $\{1, 2, 3, 4\}$. Then $V_1 = \{1, 2\},\ V_2 = \{3, 4\}$ is a minimal pair in $V(K_4)$ satisfying the property (N), though K_4 is nonsingular. However, in the next section we will show that for acyclic and unicyclic graphs the condition in Theorem 2.2 is also sufficient for the graph to be singular.

Theorem 2.4 gives a sufficient condition for G to be singular. However, a singular graph may not satisfy the condition as can be seen from the following example.

Example 2.6. The graph G in Fig. 1 is singular, $V_1 = \{2, 6\}$, $V_2 = \{4, 7\}$ is the only pair in $V(G)$ satisfying the property (N). However, $1 \in N(2) \cap N(6)$ and therefore G does not satisfy the condition of Theorem 2.4.

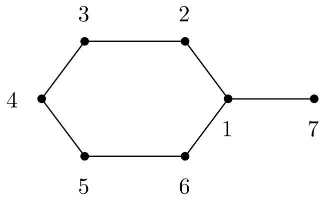

Fig. 1. A singular graph.

Before ending this section, we have two results which will be useful for the next section.

Proposition 2.7. *Suppose V_1, V_2 be a minimal pair in $V(G)$ satisfying the property (N). Let u and v be distinct vertices in V_1(or in V_2) and $x \in N(u) \cap N(v)$. Then u, v and x lie on a cycle in G.*

Proof. Let $u, v \in V_1$ and $x \in N(u) \cap N(v)$. Let $w \in V_2$ such that $x \in N(w)$. Clearly, u, v, w and x are all distinct. Suppose, if possible, u, v and x do not lie on any cycle in G. Then u and v are in different components of $G - x$. Without any loss of generality, we assume that w is not a vertex of the component G_1 containing u. Let $U_1 = V_1 - V(G_1)$ and $U_2 = V_2 - V(G_1)$. Then U_1 and U_2 are disjoint and contain v and w, respectively. We set $X = \bigcup\{N(z)|z \in U_1\}$ and $Y = \bigcup\{N(z)|z \in U_2\}$.

Let $u_1 \in U_1$ and $y \in N(u_1)$. If $y = x$, then $y \in N(w)$. Suppose that $y \neq x$. Since $u_1 \in V_1$, we have $u_2 \in V_2$ such that $y \in N(u_2)$. Since G_1 is a component of $G - x$, we must have $u_2 \in U_2$, otherwise both u_1 and u_2 would be vertices of G_1 contrary to our choice of u_1. Thus $X \subseteq Y$. Similarly, $Y \subseteq X$. This shows that U_1, U_2 satisfy the property (N), contradicting the minimality of the pair V_1, V_2. □

Proposition 2.8. *Let G be a unicyclic graph and V_1, V_2 be a minimal pair in $V(G)$ satisfying the property (N). If v is a vertex in $V_1 \cup V_2$ not on the cycle of G, then $N(v)$ and $V_1 \cup V_2$ are disjoint.*

Proof. Let, if possible, $u \in N(v) \cap (V_1 \cup V_2)$. Assume that $u \in V_1$ (the case $u \in V_2$ is similar). Because $v \in N(u)$, there is $u' \in V_2$ (and therefore $u' \in V_1 \cup V_2$) such that $v \in N(u')$. As V_1 and V_2 are disjoint, we have $u \neq u'$. Since v is not on the cycle, at least one of u and u' is not on the cycle of G. Moreover, because u and u' are distinct and adjacent to v, the unique path from v to the cycle cannot have both u and u' as its vertices. We choose one of them which does not lie on the path and rename it as u_1. Now, the vertices of the path $P_2 = [v = v_0, u_1]$ (of length one) are in $V_1 \cup V_2$ and none of them is on the cycle.

Suppose that vertices $v = v_0, u_1, \ldots, u_{k-1}$, $k \geqslant 2$, are chosen in $V_1 \cup V_2$ such that none of them is on the cycle and they form a path $P_k = [v = v_0, u_1, \ldots, u_{k-1}]$ (of length $k - 1$) in G. Since $u_{k-2} \in V_1 \cup V_2$ and $u_{k-1} \in N(u_{k-2})$, there is a vertex u_k different from u_{k-2} in $V_1 \cup V_2$ such that $u_{k-1} \in N(u_k)$. Since none of the vertices on P_k is on the cycle, u_k is not on P_k. Moreover, since the unique path from u_1 to the cycle passes through v, u_k is not on the cycle. Thus, the vertices of the path $P_{k+1} = [v = v_0, u_1, \ldots, u_k]$ (of length k) are in $V_1 \cup V_2$ and none of them is on the cycle.

By induction, we get a path of arbitrary length n in G. However, this is not possible, since G is finite. Hence the result follows. □

3. On singularity of trees and unicyclic graphs

In this section, we show that the converse of Theorem 2.2 is true in case G is acyclic or unicyclic. Thereby, we have a characterization of singular trees and singular unicyclic graphs.

By a *trivial tree* we mean a tree with a single vertex. Any other tree is *nontrivial*.

Theorem 3.1. *Let T be a nontrivial tree. Then, the following statements are equivalent*:

(a) *T is singular*.
(b) *There exist subsets V_1 and V_2 of $V(T)$ satisfying the property* (N).
(c) *There exist subsets V_1 and V_2 of $V(T)$ satisfying the property* (S).

Proof. (a) $\Rightarrow$ (b). Follows from Theorem 2.2.

(b) $\Rightarrow$ (c). Choose a minimal pair V_1, V_2 of subsets of $V(T)$ satisfying the property (N). Since T is acyclic, in view of Proposition 2.7, V_1 and V_2 must satisfy the property (S).

(c) $\Rightarrow$ (a). Follows from Theorem 2.4. $\square$

We need two lemmas before proving the result for unicyclic graphs.

Lemma 3.2. *Let G be a unicyclic graph. Let V_1, V_2 be a minimal pair in $V(G)$ satisfying the property* (N). *Then, for any distinct $u, v \in V_i$ ($i = 1$ or 2), $|N(u) \cap N(v)| \leqslant 1$.*

Proof. If possible, let $|N(u) \cap N(v)| > 1$ for some distinct $u, v \in V_1$. Let $x, y \in N(u) \cap N(v)$, $x \neq y$. Then u, v, x, y are all distinct and they form a four cycle in G. This must be the unique cycle C of G. Let T_1 be the component (a tree) of $G - x - y$ containing u. Let w_1, w_2 be vertices in V_2 such that $x \in N(w_1)$ and $y \in N(w_2)$. Clearly, $w_i \notin V(C)$, otherwise G will have another cycle (of length three). Since x, u and w_1 do not lie on a cycle, $w_1 \notin V(T_1)$. Similarly, $w_2 \notin V(T_1)$. We put $U_1 = V_1 - V(T_1)$ and $U_2 = V_2 - V(T_1)$. Then U_1 and U_2 are disjoint. Moreover, U_1 contains v and U_2 contains w_1 and w_2. Let $X = \bigcup\{N(w)|w \in U_1\}$ and $Y = \bigcup\{N(w)|w \in U_2\}$.

Let $z \in N(u_1)$ for some $u_1 \in U_1$. If z is x or y, then z is in $N(w_1)$ or $N(w_2)$. Suppose that z is neither of x and y. Since $u_1 \in V_1$, we have $u_2 \in V_2$ such that $z \in N(u_2)$. Since T_1 is a component of $G - x - y$, we must have $u_2 \in U_2$, otherwise both u_1 and u_2 would be vertices of T_1 contrary to our choice of u_1. Thus $X \subseteq Y$. Similarly, $Y \subseteq X$. This shows that U_1, U_2 satisfy the property (N), contradicting the minimality of the pair V_1, V_2. $\square$

Lemma 3.3. *Let G be a unicyclic graph and V_1, V_2 be a minimal pair in $V(G)$ satisfying the property* (N). *Let u, v be a pair of vertices in V_1 such that $N(u) \cap N(v) \neq \emptyset$. Then $N(u')$ and $N(v')$ are disjoint for all pairs $u', v' \in V_2$. Moreover, if u', v' is a pair in V_1 different from u, v, then $N(u)$ and $N(v)$ are disjoint.*

Proof. In view of Lemma 3.2, $N(u) \cap N(v) = \{x\}$ for some $x \in V(G)$. Suppose, if possible, there is another pair u', v' in either V_1 or V_2 such that $N(u') \cap N(v') = \{x'\}$, $x' \in V(G)$. In case u', v' are in V_1, we assume that u, v, u' are distinct. Now, in view of Proposition 2.7, the vertices u, v, u', x, x' lie on the cycle C of G. Since the vertices u, v, u' are all distinct, $x \neq x'$. Moreover, x' must be distinct from at least one of u and v; so let $x' \neq u$. Let T be the component (a tree) of $G - x - x'$ containing u. Consider the sets $U_1 = V_1 - V(T)$ and $U_2 = V_2 - V(T)$. It can be seen that U_1 and U_2 satisfy the property (N), contradicting the minimality of V_1, V_2. $\square$

Theorem 3.4. *A unicyclic graph G is singular if and only if there is a pair of subsets V_1 and V_2 of $V(G)$ satisfying the property* (N).

Proof. It is enough to show that the condition is sufficient. We choose a minimal pair V_1, V_2 of subsets of $V(G)$ satisfying the property (N). If V_1, V_2 satisfy the property (S), then we are done. Otherwise, there is a pair u, v of vertices in V_1 (say) such that $N(u) \cap N(v) \neq \emptyset$. Then, by Lemma 3.2, $N(u) \cap N(v) = \{x\}$ for some $x \in V(G)$. In view of Proposition 2.7, u, v and x are vertices of the cycle C of G. Moreover, in view of Lemma 3.3, $N(u') \cap N(v') = \emptyset$ for all other pairs u', v' in V_1 and all pairs u', v' in V_2. Let $x \in N(w),\ w \in V_2$. Clearly, $w \notin V(C)$. Let T be the component (a tree) of $G - x$ containing w.

Define a function α on $V(G) = \{v_1, v_2, \ldots, v_n\}$ as follows:

$$\alpha(v_i) = \begin{cases} 2, & \text{if } v_i \in V_1 \cap V(T), \\ -2, & \text{if } v_i \in V_2 \cap V(T), \\ 1, & \text{if } v_i \in V_1 - V(T), \\ -1, & \text{if } v_i \in V_2 - V(T), \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

We show that $(\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_n))^{\mathrm{t}}$ is a null-eigenvector for G, that is, for each $v_i \in V(G)$

$$\sum_{z \in N(v_i)} \alpha(z) = 0. \tag{3.5}$$

If $v_i = x$, then

$$\sum_{z \in N(v_i)} \alpha(z) = \alpha(u) + \alpha(v) + \alpha(w) = 0.$$

Next, let $v_i = w$. Since $w \notin V(C)$, we have $N(w) \cap (V_1 \cup V_2) = \emptyset$, by Proposition 2.8, and (3.5) is satisfied. Finally, let v_i be different from x and w. If

$$v_i \notin X = \bigcup\{N(z) | z \in V_1\} = \bigcup\{N(z) | z \in V_2\},$$

then $\alpha(z) = 0$ for all $z \in N(v_i)$ and therefore (3.5) is satisfied. On the other hand, if $v_i \in X$, then $v_i \in N(u_1),\ v_i \in N(u_2)$ for unique vertices $u_1 \in V_1,\ u_2 \in V_2$. Clearly, either both of u_1 and u_2 are in T or both are outside T, and therefore (3.5) is satisfied. This completes the proof. □

4. On the null-spaces of singular trees and unicyclic graphs

In this section, we show how a basis for the null-space of a graph G can be obtained, when G is either a tree or a unicyclic graph. We note that for each minimal pair V_1, V_2 of subsets of $V(G)$ satisfying the property (N) a null-eigenvector is obtained using (2.3), if the pair satisfies the property (S), and using (3.4), otherwise. Moreover, the null-eigenvector will have entries in $\{0, \pm 1\}$ and $\{0, \pm 1, \pm 2\}$, respectively, in the two cases.

Definition 4.1. A graph which is either a cycle or obtained by attaching some pendants to a cycle is called an *elementary unicyclic* graph.

The following result follows from Theorem 1.1.

Proposition 4.2. *Let G be an elementary unicyclic graph on n vertices having a pendant. Then $\eta(G) = n - 2q$, where q is the maximum number of mutually nonadjacent edges in G.*

Definition 4.3. A matching M_0 in a unicyclic graph G is called an *outer matching* in G if $G - V(M_0)$ is the disjoint union of an elementary unicyclic graph and a set of isolated vertices (possibly empty). (Note that $M_0 = \emptyset$, if G is elementary.) A path P in a graph G is an *alternating path* relative to a matching M in G if alternate edges in P are in M (terminating edges may or may not be in M).

Remark 4.4. (a) For a unicyclic graph G which is not elementary, we construct an outer matching M_0 as follows. Let u_1 be a (pendent) vertex which is at a maximum distance from the cycle C in G and v_1 the vertex adjacent to u_1. Then v_1 is not on C, since G is not elementary. We choose $e_1 = \{u_1, v_1\}$ as an edge in M_0. Clearly, $G - u_1 - v_1$ is a disjoint union of a unicyclic graph G_1 and a set of isolated vertices (possibly empty). If G_1 is not elementary, we can choose another edge for M_0 by the same process, and then proceed recursively. The process must terminate and an outer matching M_0 of G is obtained.

(b) Let T be a nontrivial tree. Using a similar recursive process for T, choosing the vertex u_1 to be at a maximum distance from the *center* of T (see [6] for definition), we obtain a matching M_0 of T such that $V(T) - V(M_0) = \Lambda_0$ is either empty or a set of vertices inducing an empty subgraph in T. In this case, we have $\eta(T) = |\Lambda_0|$ by Theorem 1.1 and therefore M_0 must be a maximal matching of T, in view of Theorem 1.2.

Example 4.5. Consider the unicyclic graph G in Fig. 2. Here, the set M_0 of edges in bold face in the figure of G is an outer matching of G. The corresponding elementary unicyclic graph is G_0 (depicted in the figure) and the set of isolated vertices of $G - V(M_0)$ is $\{7, 12, 19\}$.

For the rest of this section G denotes either a tree or a unicyclic graph. We fix an outer matching (respectively a maximal matching) M_0 of G constructed as in Remark 4.4, if G is unicyclic (respectively acyclic). We denote the set of isolated vertices and the elementary unicyclic component (if G is unicyclic) of $G - V(M_0)$ by Λ_0 and G_0, respectively.

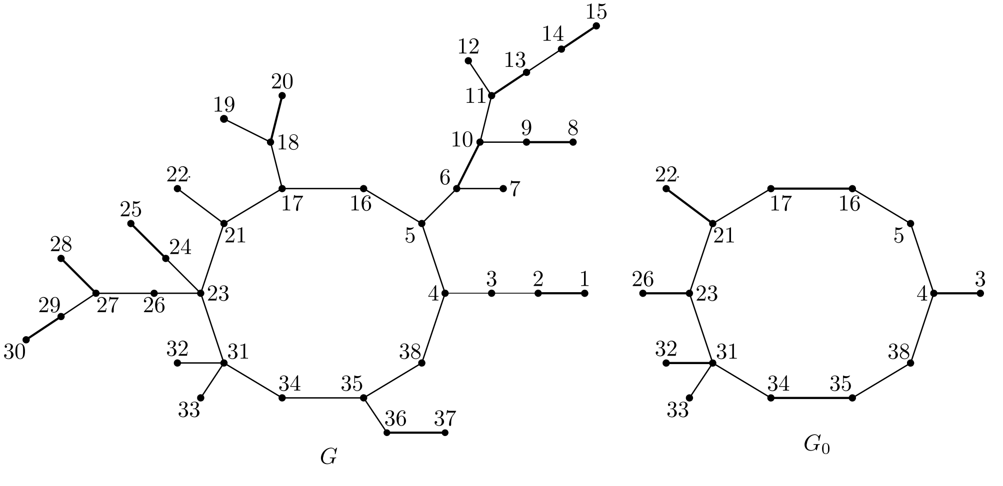

Fig. 2. An outer matching and the resulting elementary component.

Lemma 4.6. *Let $x \in \Lambda_0$ and $T_{(x)}$ be the subgraph of G induced by x and the vertices v in G for which there are alternating x–v paths, relative to M_0. Then*

(a) *The vertices in $T_{(x)}$ other than x are in $V(M_0)$.*
(b) *$T_{(x)}$ is a tree.*
(c) *If y in $T_{(x)}$ is at an even distance from x, then the degree of y in $T_{(x)}$ is the same as the degree of y in G.*
(d) *If y in $T_{(x)}$ is at an odd distance from x, then the degree of y in $T_{(x)}$ is two.*

Proof. (a) Let y be any vertex on $T_{(x)}$, $y \neq x$. First, suppose that y is adjacent to x. Then $y \notin \Lambda_0$, otherwise there would be an edge in the graph induced by Λ_0. Moreover, if G is unicyclic, then $y \notin G_0$, since x and G_0 are distinct components of $G - V(M_0)$. Therefore, $y \in V(M_0)$. Next, suppose that y is not adjacent to x and $P = [x, u_1, u_2, \ldots, u_r, y]$, $r \geqslant 1$, is an alternating x–y path in G. If r is odd, then $y \in V(M_0)$, since in that case the edges $\{u_1, u_2\}, \{u_3, u_4\}, \ldots, \{u_r, y\}$ are in M_0.

Suppose that r is even so that $\{u_1, u_2\}, \{u_3, u_4\}, \ldots, \{u_{r-1}, u_r\}$ are edges in M_0. Let C denote the cycle (if G is unicyclic) or the center (if G is acyclic) of G. It follows from our construction of M_0 that for any edge $\{z, w\}$ in M_0 there is a subset M of M_0 such that the vertices z, w lie in the component G_1 containing C of $G - V(M)$ and one of these vertices is at a maximum distance from C in G_1. Now, because $x, u_1, \ldots, u_r$ are not on C and x and y are the terminating vertices of the path P, either the distance of x from C is larger than the distances of u_1 and u_2 from C, or the distance of y from C is larger than the distances of u_{r-1} and u_r from C. If $y \notin V(M_0)$, then x and y must be vertices of the component G_1 containing C of $G - V(M)$ for any $M \subseteq M_0$ such that G_1 contains the edges $\{u_1, u_2\}, \{u_3, u_4\}, \ldots, \{u_{r-1}, u_r\}$. Therefore, for any such M none of the edges $\{u_1, u_2\}, \{u_3, u_4\}, \ldots, \{u_{r-1}, u_r\}$ has a vertex which is at a maximum distance from C in G_1. Hence, we must have $y \in V(M_0)$.

(b) If the path $P = [x, u_1, u_2, \ldots, u_r, y]$ is alternating, then so are $[x, u_1, u_2, \ldots, u_i]$, $1 \leqslant i \leqslant r$, and therefore the path P is in $T_{(x)}$. Thus, $T_{(x)}$ is connected. Moreover, from (a) it follows that $T_{(x)}$ does not contain any vertex of G_0 in case G is unicyclic.

(c) Let $y \in V(T_{(x)})$ be at an even distance from x. Let z be a vertex in G adjacent to y. If $y = x$, then $[x, z]$ is an alternating path and therefore $z \in V(T_{(x)})$. Suppose $y \neq x$ and $[x, u_1, u_2, \ldots, u_r, y]$ is an alternating x–y path in G. Then r is odd and $\{u_r, y\}$ is an edge in M_0. If $z = u_r$, then $z \in V(T_{(x)})$. Suppose $z \neq u_r$. Then $\{y, z\}$ is not an edge of M_0 and therefore $[x, u_1, u_2, \ldots, u_r, y, z]$ is an alternating path in G. Hence, $z \in V(T_{(x)})$.

(d) Let $y \in V(T_{(x)})$ be at an odd distance from x and $[x, u_1, u_2, \ldots, u_r, y]$ be an alternating x–y path in G. Then r is even and $\{u_r, y\}$ is not an edge in M_0. Since $y \in V(M_0)$, there is a unique $z \in V(G)$ such that $\{y, z\}$ is an edge in M_0. Consequently, $[x, u_1, u_2, \ldots, u_r, y, z]$ is an alternating path in G. Moreover, since $T_{(x)}$ is a tree, the alternating x–y path in G is unique. Hence, u_r and z are the only vertices adjacent to y in $T_{(x)}$. □

Example 4.7. For the graph G in Example 4.5 we have $\Lambda_0 = \{7, 12, 19\}$. It is easy to see that, T_{19} is the path [19, 18, 20], T_{12} is the path [12, 11, 13, 14, 15] and T_7 is the tree induced by the vertices 6, 7, 8, 9, 10, 11, 13, 14 and 15.

Proposition 4.8. *Let V_1, V_2 be a minimal pair of subsets of $V(G_0)$ satisfying the property (N) in G_0. Let $W_1 = V_1 \cup U_1$, $W_2 = V_2 \cup U_2$, where U_1 and U_2 are subsets of $V(M_0)$ defined as follows: $y \in U_1$ if from y there is either an alternating path (relative to M_0) of length*

0 (mod 4) *from a vertex in* V_1 *or an alternating path* (*relative to* M_0) *of length* 2 (mod 4) *from a vertex in* V_2; $y \in U_2$ *if from y there is either an alternating path* (*relative to* M_0) *of length* 2 (mod 4) *from a vertex in* V_1 *or an alternating path* (*relative to* M_0) *of length* 0 (mod 4) *from a vertex in* V_2. *Then* W_1, W_2 *is a minimal pair of subsets of* $V(G)$ *satisfying the property* (N) *in* G.

Proof. If $U_1 \cup U_2 = \emptyset$, then the result is obvious. Suppose that $U_1 \cup U_2 \neq \emptyset$ and for $y \in U_1 \cup U_2$ let u be the vertex in G_0 nearest to y. Then u is the unique vertex in G_0 such that the u–y path is alternating relative to M_0. Therefore, U_1 and U_2 are disjoint. Since $V_1 \cup V_2$ and $V(M_0)$ are disjoint, we have $W_1 \cap W_2 = \emptyset$.

Let $X = \bigcup\{N(y)|y \in W_1\},\ Y = \bigcup\{N(y)|y \in W_2\}$. First, let $y_1 \in V_1$ and $z \in N(y_1)$. If $z \in V(G_0)$, then $z \in N(y_2)$ for some $y_2 \in V_2$. If $z \notin V(G_0)$, then there is an alternating path $[y_1, z, y]$ in G. Consequently, $y \in U_2$ and $z \in N(y)$.

Next, let $y_1 \in U_1$ and $z \in N(y_1)$. Let u be the vertex in $V_1 \cup V_2$ such that the u–y_1 path $[u, u_1, u_2, \ldots, u_r, y_1]$ is alternating in G relative to M_0. Suppose $z = u_r$. If $r = 1$, then $u \in V_2$ and we put $y_2 = u$. If $r > 1$, then $u_{r-1} \in U_2$ and we put $y_2 = u_{r-1}$. Suppose that $z \neq u_r$. Then $[u, u_1, u_2, \ldots, u_r, y_1, z]$ is alternating and therefore $z \in V(M_0)$. Let y_2 be the vertex in $V(M_0)$ such that $\{z, y_2\} \in M_0$. Then the path $[u, u_1, u_2, \ldots, u_r, y_1, z, y_2]$ is alternating in G and $y_2 \in U_2$. Thus, we have $y_2 \in W_2$ such that $z \in N(y_2)$. This shows that $X \subseteq Y$. The reverse inclusion is similar.

Suppose, if possible, there are subsets W_1' and W_2' of W_1 and W_2, respectively, such that W_1', W_2' satisfy the property (N) in G and at least one of the inclusions is proper. First, we show that $W_1' \cap V_1$ and $W_2' \cap V_2$ are nonempty. If $W_i' \cap U_i = \emptyset,\ i = 1, 2$, then $\emptyset \neq W_i' \subseteq V_i$ and the result follows. Otherwise, we choose a vertex $w \in \left(W_1' \cap U_1\right) \cup \left(W_2' \cap U_2\right)$ with minimum distance from G_0. Without any loss of generality, we assume that $w \in W_1'$. Let u be the vertex in $V_1 \cup V_2$ for which the u–w path $P = [u = u_0, \ldots, u_r, w]$ is alternating relative to M_0. Then P is of even length and $r \geqslant 1$. Since $u_r \in N(w)$, there is a vertex w' in W_2' adjacent to u_r. However, all the vertices in $N(u_r)$ other than u_{r-1} have the same distance $d(u, w')$ from u. Therefore, we must have $u_{r-1} = w' \in W_2'$. Now, the choice of w forces us to have $u_{r-1} = u \in V_2$. Hence, we have $W_2' \cap V_2 \neq \emptyset$. Next, $N(u)$ has a vertex u' not on P which must be adjacent to a vertex in W_1'. Therefore, we must have $W_1' \cap V_1 \neq \emptyset$.

Since $N(y) \cap V(G_0) = \emptyset$ for $y \in U_1 \cup U_2$, the subsets $W_1' \cap V_1$ and $W_2' \cap V_2$ of $V(G_0)$ must satisfy the property (N) in G_0. Therefore, we must have $V_1 \subseteq W_1'$ and $V_2 \subseteq W_2'$. We choose a vertex $y \in (W_1 \cup W_2) - \left(W_1' \cup W_2'\right)$ with minimum distance from G_0. Let u be the vertex in $V_1 \cup V_2$ for which the u–y path $[u = u_0, u_1, \ldots, u_r, y]$ is alternating. Since y and u_{r-1} are the only vertices in $W_1 \cup W_2$ adjacent to u_r, we must have $u_{r-1} \notin W_1' \cup W_2'$. This contradicts our choice of y in case $r > 1$ and the fact that $V_i \subseteq W_i',\ i = 1, 2$, in case $r = 1$. Hence W_1, W_2 form a minimal pair satisfying the property (N) in G. □

Definition 4.9. The minimal pair W_1, W_2 (in Proposition 4.8) satisfying the property (N) in G is said to be *generated by* the minimal pair V_1, V_2 in G_0.

Example 4.10. Consider the unicyclic graph G in Example 4.5. Note that $V_1 = \{3, 17\}$, $V_2 = \{5, 22\}$ is a minimal pair of subsets of $V(G_0)$ satisfying the property (N). The minimal pair satisfying the property (N) generated by V_1, V_2 is

$$W_1 = \{3, 10, 15, 17\}, \quad W_2 = \{1, 5, 8, 13, 20, 22\}.$$

We now present a systematic approach for finding a basis for the null-space of G. The following result gives an overview of our approach.

Proposition 4.11. *Let $\mathscr{V}$ be a collection of minimal pairs V_1, V_2 of subsets of $V(G)$ satisfying the property (N). Suppose $\mathscr{V}$ has the property that for each pair V_1, V_2 in $\mathscr{V}$ the set $V_1 \cup V_2$ contains a vertex which is not in $U_1 \cup U_2$ for any other pair U_1, U_2 in $\mathscr{V}$. Then, the null-eigenvectors obtained from the pairs in $\mathscr{V}$ by (2.3) and (3.4) are linearly independent.*

Proof. Let V_1, V_2 be a minimal pair of subsets of $V(G)$ satisfying the property (N). The coordinate corresponding to a vertex v in the null-eigenvector obtained from V_1, V_2 by (2.3) or (3.4) is nonzero if and only if $v \in V_1 \cup V_2$. Therefore, with the given property of $\mathscr{V}$, the null-eigenvector obtained by any pair V_1, V_2 in $\mathscr{V}$, has a nonzero coordinate such that the corresponding coordinate in each of the null-eigenvectors obtained by the other pairs in $\mathscr{V}$ is zero. Hence, the result follows. □

To obtain a basis for the null-space of a tree or a unicyclic graph G it is enough to find a collection $\mathscr{V}$ of size $\eta(G)$ consisting of pairs of subsets of $V(G)$ as in Proposition 4.11. Then, the vectors obtained using (2.3) and (3.4) will form a basis for the null-space of G.

Theorem 4.12. *Let G be a singular elementary unicyclic graph.*

(a) *If G has no pendant and $G = C_n = (v_0, v_1, \ldots, v_n = v_0)$, then $n = 0 \pmod 4$ and $\eta(G) = 2$. The two pairs V_1, V_2 and V_1', V_2' given by*

$$V_1 = \{v_i \mid i = 2 \pmod 4\}, \quad V_2 = \{v_i \mid i = 0 \pmod 4\},$$
$$V_1' = \{v_i \mid i = 1 \pmod 4\}, \quad V_2' = \{v_i \mid i = 3 \pmod 4\},$$

are minimal satisfying the property (S).

(b) *If G has exactly one pendent vertex, then n is odd and $\eta(G) = 1$. Moreover, if w is the pendent vertex of G attached to the vertex v_1 of the cycle $C_{n-1} = (v_0, v_1, \ldots, v_{n-1} = v_0)$, then (V_1, V_2) as defined below is a minimal pair satisfying the property (N).*

(i) *In case $n = 1 \pmod 4$,*

$$V_1 = \{v_i \mid i = 2 \pmod 4\}, \quad V_2 = \{v_i \mid i = 0 \pmod 4\}.$$

(*The pair satisfies the property (S) in this case.*)

(ii) *In case $n = 3 \pmod 4$,*

$$V_1 = \{v_i \mid d(v_i, w) = 2 \pmod 4\}, \quad V_2 = \{u\} \cup \{v_i \mid d(v_i, w) = 0 \pmod 4\}.$$

(*The pair does not satisfy the property (S). The vertices v_2, v_{n-2} are in V_1 and $v_1 \in N(v_2) \cap N(v_{n-2})$.*)

(c) *Suppose that G has more than one pendent vertex and the pendants are attached at the vertices $u_1, \ldots, u_k$ of the cycle C of G. Choose a pendent vertex w_i attached at u_i, $1 \leqslant i \leqslant k$. Let M_1 be a maximal matching in G containing the edges $\{u_i, w_i\}$, $1 \leqslant i \leqslant k$. Then $\eta(G) = |\Lambda_1|$, where $\Lambda_1 = V(G) - V(M_1)$. For each $v \in \Lambda_1$, (V_1, V_2) as defined below is a minimal pair satisfying the property (N).*

(i) *In case v is a pendant attached to u_i, $V_1 = \{v\}$, $V_2 = \{w_i\}$. (The pair satisfies the property (S).)*

(ii) *In case* v *is on the cycle and* $k = 1$, *consider the subgraph induced by* C *and* w_1, *and set* V_1, V_2 *as in* (*b*).

(iii) *In case* v *is on the cycle and* $k \geqslant 2$, *choose* w_i *and* w_j *such that the* w_i–w_j *path* P *passing through* v *is of minimum length. Then*

$$V_1 = \{u \in P | d(w_i, u) = 0 \ (\mathrm{mod}\ 4)\}, \quad V_2 = \{u \in P | d(w_i, u) = 2 \ (\mathrm{mod}\ 4)\}.$$

(*The pair satisfies the property* (*S*).)

Moreover, $V_1 \cup V_2$ *contains no vertex from* Λ_1 *other than* v.

Proof. The first assertion of (a) follows from the fact that the spectrum of C_n, the cycle of order n, is $\{2\cos\frac{2k\pi}{n} | 1 \leqslant k \leqslant n\}$ ([2], p. 53). Those in (b) and (c) follows from Propositions 4.2. Moreover, for the case (iii) of (c), the path P has exactly one vertex from Λ_1, namely v, and therefore $V_1 \cup V_2$ contains no vertex from Λ_1 other than v. The rest of the assertions can be easily verified. □

We note that in each of the cases of Theorem 4.12, the collection $\mathscr{V}$ of minimal pairs (V_1, V_2) satisfies the condition of Proposition 4.11 and therefore give rise to a basis for the null-space of the elementary unicyclic graph G. The final result of this paper is the following, which produces a basis for the null-space of an arbitrary acyclic or unicyclic graph.

Theorem 4.13. *Let* G *be an acyclic* (*respectively a unicyclic*) *graph*, *and let* Λ_0 *be the set of isolated vertices remaining after deletion of a maximal* (*respectively an outer*) *matching of* G.

(a) *If* G *is a tree*, *then* $\eta(G) = |\Lambda_0|$. *If* G *is unicyclic*, *then*

$$\eta(G) = \eta(G_0) + |\Lambda_0|.$$

(b) *For each* $x \in \Lambda_0$, *consider the tree* $T_{(x)}$ *as defined in Lemma* 4.6. *Then*

$$V_1^{(x)} = \{v \in V(T_{(x)}) | d(v, x) = 0 \ (\mathrm{mod}\ 4)\},$$
$$V_2^{(x)} = \{v \in V(T_{(x)}) | d(v, x) = 2 \ (\mathrm{mod}\ 4)\}$$

are minimal pair of subsets of $V(G)$ *satisfying the property* (*S*). *Moreover*, $V_1^{(x)} \cup V_2^{(x)}$ *contains no vertex from* Λ_0 *other than* x. *If* G *is a tree*, *then the vectors obtained from these pairs using* (2.3) *form a basis for the null-space of* G.

(c) *Let* G *be unicyclic and* $\left\{\left(V_1^{(i)}, V_2^{(i)}\right) \middle| 1 \leqslant i \leqslant \eta(G_0)\right\}$ *be a collection of pairs of subsets of* $V(G_0)$ *satisfying the property* (*N*) *in* G_0 *giving rise to a basis for the null-space of* G_0. *If* $\left(W_1^{(i)}, W_2^{(i)}\right)$ *is the pair in* G *generated by the pair* $\left(V_1^{(i)}, V_2^{(i)}\right)$ *as in Proposition* 4.8, *then the vectors obtained from the pairs in*

$$\mathscr{V} = \left\{\left(W_1^{(i)}, W_2^{(i)}\right) \middle| 1 \leqslant i \leqslant \eta(G_0)\right\} \bigcup \left\{\left(V_1^{(x)}, V_2^{(x)}\right) \middle| x \in \Lambda_0\right\}$$

using (2.3) *and* (3.4) *form a basis for the null-space of* G.

Proof. (a) Follows from the reduction formula of Theorem 1.1.

(b) Since G is connected, x is not an isolated vertex in G. Therefore, $T_{(x)}$ is nontrivial and $V_i^{(x)}$ are nonempty. Clearly, the two sets are disjoint. Let $X = \bigcup \left\{N(z) \middle| z \in V_1^{(x)}\right\}$, $Y =$

$\bigcup \left\{ N(z) \,\middle|\, z \in V_2^{(x)} \right\}$. A vertex z in $V_1^{(x)} \cup V_2^{(x)}$ is at even distance from x and therefore have same degree in $T_{(x)}$ and G, by Lemma 4.6(c). Consequently, $N(z)$ in G is a subset of $V(T_{(x)})$ and therefore X and Y are subsets of $V(T_{(x)})$. Let y be any vertex in X (or in Y). Being adjacent to a vertex in the tree $T_{(x)}$ which is at an even distance from x, y is at an odd distance from x. In view of Lemma 4.6(d), there are exactly two vertices in $T_{(x)}$ adjacent to y. Clearly, one of them is in $V_1^{(x)}$ and the other is in $V_2^{(x)}$. This implies that $y \in X \cap Y$, that is, $X = Y$.

Suppose $U_1 \subseteq V_1^{(x)}$ and $U_2 \subseteq V_2^{(x)}$ is a pair satisfying the property (N). Let $y \in V_1^{(x)} \cup V_2^{(x)}$, $y \neq x$. Let $P = [x, u_1, \ldots, u_{2r} = y]$ be the alternating x–y path in G. For odd i, u_i are vertices of degree two in $T_{(x)}$. Therefore, if $y \notin U_1 \cup U_2$, then none of the vertices $u_{2r-2}, \ldots, u_2, x$ is in $U_1 \cup U_2$. Similarly, if $x \notin U_1$, then $y \notin U_1 \cup U_2$. Hence, we must have $U_1 = V_1^{(x)}$ and $U_2 = V_2^{(x)}$. This proves the first assertion.

The second assertion follows from Lemma 4.6(a). If G is a tree, then the collection

$$\left\{ \left(V_1^{(x)}, V_2^{(x)} \right) \,\middle|\, x \in \Lambda_0 \right\}$$

satisfies the condition of Proposition 4.11, and the third assertion follows.

(c) It is easy to see that $\mathscr{V}$ satisfies the condition of Proposition 4.11. □

We illustrate some of the features presented in the last two theorems by two examples.

Example 4.14. Each of the elementary unicyclic graphs in Fig. 3 has a single pendant and is of odd order. By Theorem 4.12(b), each of them is of nullity one and we get a minimal pair of subsets for each of these graphs satisfying the property (N). For the first graph the pair $V_1 = \{2, 6\}$, $V_2 = \{4, 7\}$ satisfying the property (N) is obtained using (ii) of Theorem 4.12(b). The pair does not satisfy the property (S) and the corresponding null-eigenvector is obtained by (3.4). For the second graph we use (i) of Theorem 4.12(b) and obtain the pair $V_1 = \{2, 6\}$, $V_2 = \{4, 8\}$ satisfying the property (S). The corresponding null-eigenvector is obtained by (2.3). The two null-eigenvectors are depicted in the respective figure. The coordinate of a vertex in the null-eigenvector is shown as the suffix of the vertex.

Example 4.15. Consider the graph G in Example 4.5 (Fig. 2). The three trees $T_{(19)}, T_{(12)}, T_{(7)}$ corresponding to the vertices in $\Lambda_0 = \{19, 12, 7\}$ give rise to minimal pairs

$$(\{19\}, \{20\}), \quad (\{12, 15\}, \{13\}) \quad \text{and} \quad (\{7, 8, 13\}, \{10, 15\})$$

in G, respectively.

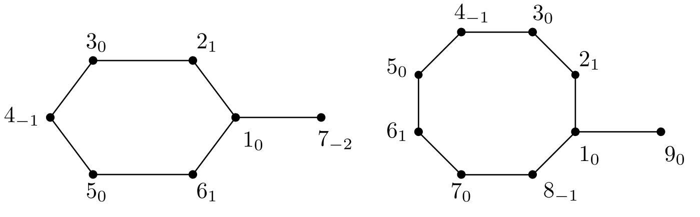

Fig. 3. Null-eigenvector of elementary unicyclic graphs with a single pendant.

Next, we fix a maximal matching M_1 of the elementary unicyclic component G_0 of $G - V(M_0)$ as one consisting of the edges in bold face in Fig. 2. We have $\Lambda_1 = V(G_0) - V(M_1) = \{5, 33, 38\}$. Consequently, using Theorem 4.12(c), we get three minimal pairs

$$(\{3, 17\}, \{5, 22\}), \quad (\{33\}, \{32\}) \quad \text{and} \quad (\{3, 34\}, \{38, 32\})$$

in G_0. Using Proposition 4.8, we get minimal pairs

$$(\{3, 10, 15, 17\}, \{1, 5, 8, 13, 20, 22\}), \quad (\{33\}, \{32\}) \quad \text{and} \quad (\{3, 34\}, \{1, 32, 38\})$$

of G generated by the above minimal pairs in G_0.

Using the notation used in Example 4.14 and omitting the vertices with zero coordinates, we can now write down the basis for the null-space of G with vectors given by these minimal pairs in G as follows:

$(19_1, \; 20_{-1});$
$(12_1, \; 13_{-1}, \; 15_1);$
$(7_1, \; 8_1, \; 10_{-1}, \; 13_1, \; 15_{-1});$
$(1_{-1}, \; 3_1, \; 5_{-1}, \; 8_{-1}, \; 10_1, \; 13_{-1}, \; 15_1, \; 17_1, \; 20_{-1}, \; 22_{-1});$
$(32_{-1}, \; 33_1);$
$(1_{-1}, \; 3_1, \; 32_{-1}, \; 34_1, \; 38_{-1}).$

Acknowledgments

The authors sincerely thank the referee for careful reading and many valuable suggestions which greatly improved the presentation of the article. The first author thanks University Grant Commission, India, for financial assistance.

References

[1] L. Collatz, U. Sinogowitz, Spektren endlicher Grafen, Abh. Math. Sem. Univ. Hamburg 21 (1957) 63–77.
[2] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, Academic, New York, 1980.
[3] D.M. Cvetković, I. Gutman, The algebraic multiplicity of the number zero in the spectrum of a bipartite graph, Mat. Vesnik 9 (1972) 141–150.
[4] D.M. Cvetković, I. Gutman, N. Trinajstić, Graph theory and molecular orbitals, II, Croat. Chem. Acta 44 (1972) 365–374.
[5] S. Fiorini, I. Gutman, I. Sciriha, Trees with maximum nullity, Linear Algebra Appl. 397 (2005) 245–251.
[6] F. Harary, Graph Theory, Addition-Wesley, 1969.
[7] Tan Xuezhong, Bolian Liu, On the nullity of unicyclic graphs, Linear Algebra Appl. 408 (2005) 212–220.