

Existence and Uniqueness of Periodic Solutions for (2n + 1)th-Order Differential Equations

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Applying the Schauder fixed-point theorem and the continuation theorem of Mawhin, we give the existence and uniqueness results of periodic solutions for a class of (2n + 1)th-order nonlinear ordinary differential equations. © 2000 Academic Press

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We consider the (2n + 1)th-order nonlinear ordinary differential equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + f(t, x) = 0, \quad (1)$$

where $f: R \times R \rightarrow R$ is continuous and 2π -periodic with respect to t and $c_i, i = 1, 2, \dots, n - 1$, are constants.

During the past 20 years, there has been a great amount of work in periodic solutions for the high-order Duffing equation

$$y^{(2n)} + \sum_{i=1}^{n-1} c_i y^{(2i)} + (-1)^{(n+1)} f(t, y) = 0.$$



Among previous results [1–4], one fundamental nonresonance condition

$$\begin{aligned} 0 &\leq N^{2n} + \sum_{j=1}^{n-1} (-1)^{j-n} c_j N^{2j} \\ &< \lambda \leq f_y(t, y) \leq \mu \\ &< (N+1)^{2n} + \sum_{j=1}^{n-1} (-1)^{j-n} c_j (N+1)^{2j}, \end{aligned}$$

where N is a nonnegative integer, is required. However, the study of odd-order differential equation is rare.

For $c_i = 0$, $i = 1, 2, \dots, n-1$, and f satisfies

$$M \geq f_x(t, x) \geq \varepsilon_0 > 0 \quad (\text{or } -M \leq f_x(t, x) \leq -\varepsilon_0 < 0),$$

Li Yong and Wang Huaizhong [4] proved that (1) has a unique 2π -periodic solution.

In this note, using the Schauder fixed-point theorem and the continuation theorem of Mawhin, we give results on the existence and uniqueness of the 2π -periodic solution to (1). Assume that the following conditions hold:

(H1) f_x is continuous and there are positive constants m and M such that

$$m \leq |f_x(t, x)| \leq M \quad \text{for all } (t, x). \quad (2)$$

(H2) The linear equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} = 0 \quad (3)$$

has only the constants for its 2π -periodic solutions.

(H3) $c_j \geq 0$, $j = 0, 1, \dots, n-1$, satisfies the inequality

$$\sum_{j=0}^{n-1} (2\pi)^{2(n-j)} c_j < 1. \quad (4)$$

Our results are the following.

THEOREM A. *Under conditions (H1) and (H2) above, the differential equation (1) has at least one 2π -periodic solution.*

THEOREM B. *Under conditions (H1) and (H3) above, the differential equation (1) has a unique 2π-periodic solution.*

To prove the theorems, we need the following lemma.

LEMMA 1. *Let x ∈ C¹(R) and let it be a 2π-periodic function; and there exists t₀ ∈ R such that x(t₀) = 0. Then*

$$\int_0^{2\pi} x^2(t) dt \leq 4\pi^2 \int_0^{2\pi} [x'(t)]^2 dt.$$

Employing the Schwarz inequality, the proof of Lemma 1 is easy. Set

$$X = \{y \in C^{2n}(R) : y(t) = y(t + 2\pi), \quad \text{for all } t \in R\},$$

with the norm || · || defined by ||y|| = Σ_{i=0}²ⁿ max_[0, 2π] |y⁽ⁱ⁾(t)|, for each y ∈ X.

Rewrite (1) in the following form:

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + g(t, x)x = -f(t, 0), \tag{5}$$

$$g(t, x) = \int_0^1 f_x(t, \theta x) d\theta.$$

For each y ∈ X, we consider the linear differential equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + g(t, y)x = -f(t, 0). \tag{6}_y$$

LEMMA 2. *There is M₁ > 0 such that any 2π-periodic solution x_y(t) of (6)_y satisfies*

$$\|x_y\| \leq M_1, \quad \text{for all } y \in X.$$

Proof. Let x_y(t) be any 2π-periodic solution of (6)_y. By Rolle's theorem, we have ξ_j ∈ [0, 2π] satisfying

$$x_y^j(\xi_j) = 0, \quad j = 1, 2, \dots, 2n. \tag{7}$$

Applying Lemma 1,

$$\begin{aligned}
& \left(\int_0^{2\pi} (x_y^{(j)})^2 dt \right)^{1/2} \\
& \leq 2\pi \left(\int_0^{2\pi} (x_y^{(2j+1)})^2 dt \right)^{1/2} \\
& \leq (2\pi)^{2n-j} \left(\int_0^{2\pi} (x_y^{(2n)})^2 dt \right)^{1/2}, \quad j = 1, 2, \dots, 2n-1. \quad (8)
\end{aligned}$$

From (5),

$$m \leq |g(t, x)| \leq M, \quad \text{for all } (t, x). \quad (9)$$

Let $L = \sup_{t \in R} |f(t, 0)|$. Take the product (6)_y with $x_y(t)$ and then integrate over $[0, 2\pi]$. It follows that

$$\begin{aligned}
& \int_0^{2\pi} x_y^{(2n+1)} x_y dt + \sum_{i=0}^{n-1} c_i \int_0^{2\pi} x_y^{(2i+1)} x_y dt + \int_0^{2\pi} g(t, y) x^2 dt \\
& = - \int_0^{2\pi} f(t, 0) x_y dt.
\end{aligned}$$

Noting that $\int_0^{2\pi} x_y^{(2i+1)} x_y dt = 0$ and using (9) and the Schwarz inequality, we derive

$$\left(\int_0^{2\pi} x_y^2 dt \right)^{1/2} \leq \frac{1}{m} \left(\int_0^{2\pi} |f(t, 0)|^2 dt \right)^{1/2} \leq \frac{\sqrt{2\pi}}{m} L. \quad (10)$$

From (6)_y and the Schwarz inequality, using Lemma 1, (10), and $2ab \leq \varepsilon a^2 + b^2/\varepsilon$, we have

$$\begin{aligned}
& \int_0^{2\pi} (x_y^{(2n)})^2 dt = - \int_0^{2\pi} (x_y^{(2n+1)})(x_y^{(2n-1)}) dt \\
& = \int_0^{2\pi} \left(\sum_{i=0}^{n-1} c_i x_y^{(2i+1)} x_y^{(2n-1)} \right) dt \\
& \quad + \int_0^{2\pi} g(t, y) x_y x_y^{(2n-1)} dt + \int_0^{2\pi} f(t, 0) x_y^{(2n-1)} dt \\
& \leq \sum_{i=0}^{n-1} c_i \left(\int_0^{2\pi} (x_y^{(2i+1)})^2 dt \right)^{1/2} \left(\int_0^{2\pi} (x_y^{(2n-1)})^2 dt \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
& + M \int_0^{2\pi} |x_y x_y^{(2n-1)}| dt + \int_0^{2\pi} L |x_y^{(2n-1)}| dt \\
& \leq \sum_{i=0}^{n-1} c_i (2\pi)^{2(n-i)} \int_0^{2\pi} (x_y^{(2n)})^2 dt + \frac{1}{2} M \varepsilon \int_0^{2\pi} (x_y^{(2n-1)})^2 dt \\
& \quad + \frac{1}{2\varepsilon} M \int_0^{2\pi} x_y^2 dt + \frac{1}{2} \varepsilon \int_0^{2\pi} (x_y^{(2n-1)})^2 dt + \frac{L^2}{2\varepsilon} \cdot 2\pi \\
& \leq \left(2\pi^2 M \varepsilon + 2\pi^2 \varepsilon + \sum_{i=0}^{n-1} c_i (2\pi)^{2(n-i)} \right) \int_0^{2\pi} (x_y^{(2n)})^2 dt \\
& \quad + \frac{\pi L^2}{\varepsilon} \left(1 + \frac{M}{m^2} \right).
\end{aligned}$$

Choose $\varepsilon > 0$ so that

$$1 - \sum_{j=0}^{n-1} (2\pi)^{2(n-j)} c_j - 2\pi^2 \varepsilon - 2\pi^2 M \varepsilon \triangleq A > 0;$$

then we get

$$\int_0^{2\pi} (x_y^{(2n)})^2 dt \leq \frac{\pi L^2}{A \varepsilon} \left(1 + \frac{M}{m^2} \right) \triangleq B^2 L^2. \quad (11)$$

On the basis of (7), (8), (11), and the Schwarz inequality, we obtain

$$\begin{aligned}
|x_y^{(j)}(t)| & = \left| \int_{\xi_j}^t x_y^{(j+1)} dt \right| \\
& \leq (2\pi)^{1/2} \left(\int_0^{2\pi} (x_y^{(j+1)})^2 dt \right)^{1/2} \\
& \leq (2\pi)^{2n-j+1/2} BL, \\
& \text{for all } t \in [0, 2\pi], \quad j = 1, 2, \dots, 2n-1. \quad (12)
\end{aligned}$$

Also by (6),

$$\int_0^{2\pi} g(t, y) x_y dt = - \int_0^{2\pi} f(t, 0) dt.$$

Using (9), there exists $\xi \in [0, 2\pi]$ satisfying

$$|x_y(\xi)| \leq \frac{1}{2\pi m} \int_0^{2\pi} |f(t, \mathbf{0})| dt \leq \frac{L}{m}. \quad (13)$$

From (8), (11), and (13),

$$\begin{aligned} |x_y(t)| &\leq |x_y(\xi)| + \left| \int_{\xi}^t x_y(t) dt \right| \\ &\leq \frac{L}{m} + (2\pi)^{2n-1/2} BL \triangleq LD, \quad \text{for all } t \in [0, 2\pi]. \end{aligned} \quad (14)$$

According to (6)_y, for all $t \in [0, 2\pi]$,

$$\begin{aligned} |x_y^{(2n)}(t)| &= \left| \int_{\xi_{2n}}^t x_y^{(2n+1)}(t) dt \right| \\ &\leq \sum_{i=0}^{n-1} c_i \int_0^{2\pi} |x_y^{(2i+1)}| dt + \int_0^{2\pi} |g(t, y)| |x_y| dt \\ &\quad + \int_0^{2\pi} |f(t, \mathbf{0})| dt. \end{aligned} \quad (15)$$

Using (12), (14), and (15), there is constant $E > 0$, such that

$$|x_y^{(2n)}(t)| \leq EL, \quad \text{for all } t \in [0, 2\pi],$$

which, with (12) and (14), completes the proof of Lemma 2.

THEOREM C. *Assume that $h(t) = h(t + 2\pi)$ is continuous and satisfies*

$$m \leq |h(t)| \leq M \quad \text{for all } t \in [0, 2\pi],$$

where m, M are constants, and $c_i, i = 0, 1, \dots, n-1$, satisfies the inequality (4). Then the equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + h(t)x = 0 \quad (16)$$

has only a zero solution.

The proof of Theorem C is similar to that of Lemma 2. We only need to note that in (12), (14), and (15), $L = 0$,

THEOREM D. Assume that $h(t)$, c_i , $i = 0, 1, \dots, n - 1$, satisfies the conditions in Theorem C. Then the linear differential equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + h(t)x = e(t), \tag{17}$$

where $e(t) = e(t + 2\pi)$ is continuous, has a unique 2π -periodic solution.

Proof. By Theorem C, (16) has only a zero solution. Since the equation (17) is linear, the uniqueness implies the existence. The proof is completed.

Proof of Theorem B. First prove the uniqueness. Let $x_1(t)$ and $x_2(t)$ be any two 2π -periodic solution of (1). Then $x(t) = x_1(t) - x_2(t)$ is a 2π -periodic solution of the following equation:

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + \int_0^1 f_x(t, x_2(t) + \theta x(t)) d\theta x = 0.$$

From the hypothesis (H1), we see

$$m \leq \left| \int_0^1 f_x(t, x_2(t) + \theta x(t)) d\theta \right| \leq M.$$

Hence by Theorem C, $x(t) \equiv 0$, on R .

We next prove the existence. For each $y \in X$, define

$$Py = x_y,$$

where x_y is a 2π -periodic solution of (6)_y. By Theorem D, P is well defined.

Let $x_n, x \in X$ such that $\|x_n - x\| \rightarrow 0$, as $n \rightarrow \infty$. We assert: $\|Px_n - Px\| \rightarrow 0(n \rightarrow \infty)$. If not, by Lemma 2, $\{Px_n\}$ would have a convergent subsequence, still denoting it by $\{Px_n\}$, such that $\|Px_n - y\| \rightarrow 0(n \rightarrow \infty)$ and $Px \neq y$. Set $u_n = Px_n - Px$. Then $u_n(t)$ satisfies

$$u_n^{(2n+1)} + \sum_{i=0}^{n-1} c_i u_n^{(2i+1)} + g(t, x_n)u_n + (g(t, x_n) - g(t, x))Px = 0,$$

and u_n is 2π -periodic. Hence by the continuity of g , as $n \rightarrow \infty$, we get

$$u_0^{(2n+1)} + \sum_{i=0}^{n-1} c_i u_0^{(2i+1)} + g(t, x)u_0 = 0, \tag{18}$$

where $u_0(t) = y(t) - Px(t)$. By Theorem C, (18) has only the zero solution, which contradicts $y \neq Px$. By (18) and the Arzela–Ascoli theorem, it follows that P is completely continuous. From Lemma 2, PX is bounded. Applying the Schauder fixed-point theorem, P has a fixed point $x(t)$ in X , which is a solution of (1). This completes the proof of Theorem B.

Now we prove Theorem A by applying the continuation theorem of Mawhin (see, e.g., [5] or [6]).

Consider the following auxiliary equations:

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + \lambda g(t, x)x = -\lambda f(t, 0), \quad 0 < \lambda \leq 1, \quad (19)_\lambda$$

using the notation in (5). Define the linear operator $L: C_{2\pi}^{2n} \rightarrow L^2(0, 2\pi)$ by

$$Lx = x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)}, \quad x \in C_{2\pi}^{2n}.$$

By (2) in Condition (H1), the operator $x(\cdot) \rightarrow f(\cdot, x(\cdot))$ maps L^2 into itself.

Multiplying each side of $(19)_\lambda$ by $x(t)$, integrating over $[0, 2\pi]$, and using the boundary conditions, we get

$$\lambda \int_0^{2\pi} g(t, x(t)) x^2(t) dt = -\lambda \int_0^{2\pi} f(t, 0) x(t) dt.$$

Dividing out λ and using $|g(t, x)| \geq m$ for all (t, x) , we get

$$\int_0^{2\pi} x^2(t) dt \leq M_0 \quad (20)$$

for a constant M_0 independent of x and λ . This gives an *a priori* bound in L^2 .

Instead of working in L^2 , one could work in $C_{2\pi}$, using (20) and the equation $(19)_\lambda$ to obtain a bound in $C_{2\pi}$.

Clearly, according to (H1) and the definition of topological degree, the mapping from the constants (the kernel of the operator L) to the constants, given by

$$\bar{x} \rightarrow \frac{1}{2\pi} \int_0^{2\pi} f(t, \bar{x}) dt, \quad \bar{x} \in R,$$

has nonzero topological degree on a large interval containing 0.

Employing the continuation theorem of Mawhin, we can complete the proof of Theorem A.

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