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The connectivity of hierarchical Cayley digraphs

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Abstract

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An ordered generating set of a group is hierarchical when the group generated by the first k generators is a proper subgroup of the group generated by the first $k+1$ for each k. Hierarchical Cayley graphs were introduced by Akers and Krishnamurthy in [1] and other related papers as an interesting model to build symmetrical networks. In this paper we study the connectivity of hierarchical Cayley digraphs, and we show that they have maximum connectivity except in a special case. An example of this exceptional case is given. The result generalizes similar statements of Godsil [3], Akers [1] and Hamidoune [5].

1. Introduction

Only simple digraphs are considered in this paper. We identify undirected graphs with symmetrical digraphs. Notions used but not defined can be found in [2].

The problem of determining the connectivity of Cayley digraphs has been widely studied. In [3], Godsil proved that the connectivity of (undirected) Cayley graphs with respect to a minimal generating set is the degree of the graph. This result was generalized to the oriented case by Hamidoune in [5]. Akers and Krishnamurthy state the same result in [1] for Cayley graphs with respect to a hierarchical generating set with certain restrictions on the size of the group. In this paper we prove that hierarchical Cayley digraphs have maximum connectivity except in some particular cases in which the connectivity is one unit lesser. Examples of these situations are also given. In Section 2, we summarize definitions and results on atoms of a digraph. These results are used in Section 3 to determine the connectivity of hierarchical

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Cayley digraphs. It is also shown that the results can be extended to the case of infinite Cayley digraphs.

2. Atoms in Cayley digraphs

Let $G=(V,A)$ be a finite strongly connected digraph. If X is a subset of V, the *subdigraph of G generated by X* is $G_X=(X,A_X)$ where $A_X=\{[x,y]\in A: x,y\in X\}$. A subset T of V is called a *cutset* of G if G_{V-T} is not strongly connected. The *connectivity* of G is

$$\kappa(G)=\min\{|T|: T \text{ is a cutset or } |T|=|V|-1\}.$$

A cutset of minimal cardinality is called a *minimum* cutset. Given $F\subset V$, we set

$$\Gamma^+(F)=\{y\in V: [x,y]\in A \text{ and } x\in F\},$$

$$N^+(F)=\Gamma^+(F)-F,$$

$$\Gamma^-(F)=\{y\in V: [y,x]\in A \text{ and } x\in F\},$$

$$N^-(F)=\Gamma^-(F)-F.$$

Notice that $N^+(F)$ ($N^-(F)$) is a cutset of G unless $F\cup N^+(F)=V$ ($F\cup N^-(F)=V$). Hence

$$\begin{cases} |N^+(F)|\geq\min\{|V-F|,\kappa(G)\}, \\ |N^-(F)|\geq\min\{|V-F|,\kappa(G)\}. \end{cases} \tag{1}$$

A subset F of V is called a *positive* (respectively *negative*) *fragment* of G if $|N^+(F)|=\kappa(G)$ and $F\cup N^+(F)\neq V$ (respectively, $|N^-(F)|=\kappa(G)$ and $F\cup N^-(F)\neq V$). A fragment of minimal cardinality is called an *atom*, see [4–8]. Notice that any strongly connected noncomplete digraph contains either a positive or a negative atom. Since a negative atom of G is a positive atom of its converse G^{-1}, we will assume that our digraphs contain positive atoms. If G is d-regular and without loops, and A is a positive atom of G such that $|A|=1$, then $\kappa(G)=d$. We next summarize some results about atoms.

2.1. Proposition [4]. *Let G be a vertex-transitive digraph. If G contains a positive atom, then the set of positive atoms is a partition of $V(G)$.*

Let $G=\text{Cay}(\mathcal{G},S)$ be a Cayley digraph. The left translations in $\mathcal{G}$, $\gamma_a(x)=ax$, are easily seen to be automorphisms of $\text{Cay}(\mathcal{G},S)$. Hence $\text{Cay}(\mathcal{G},S)$ is vertex-transitive. In particular, by Proposition 2.1, for every vertex of $\text{Cay}(\mathcal{G},S)$ there exists a positive atom which contains it.

2.2. Theorem [5]. *Let $\mathcal{A}$ be a positive atom of* $\text{Cay}(\mathcal{G},S)$ *containing* 1. *Then $\mathcal{A}$ is the subgroup of $\mathcal{G}$ generated by $S\cap\mathcal{A}$.*

When $S=S^{-1}$, the Cayley digraph $\mathrm{Cay}(\mathscr{G},S)$ is symmetrical. The following result is a direct consequence of the main theorem in [7].

2.3. Proposition [7]. *Let A be an atom of a symmetrical Cayley digraph* G. *Then* $2|A|\le\kappa(G)$.

3. Connectivity in hierarchical Cayley digraphs

A set S of generators of a group $\mathscr{G}$ is said to be *hierarchical* if there exists an ordering of the elements in S, say $S=\{s_1,\dots,s_d\}$ such that, for every $i=1,\dots,d-1$, the group $\mathscr{G}_i$ generated by $\{s_1,\dots,s_i\}$ is a proper subgroup of the group $\mathscr{G}_{i+1}$ generated by $\{s_1,\dots,s_i,s_{i+1}\}$. Notice that a minimal set of generators is hierarchical, but the converse is not true. For instance, if t has an even order greater than 2, $\{t^2,t\}$ is hierarchical but not minimal. If S is hierarchical and $S\subseteq\bar{S}\subseteq S\cup S^{-1}$, $\mathrm{Cay}(\mathscr{G},\bar{S})$ is said to be a hierarchical Cayley digraph. In this section we prove that the hierarchical Cayley digraphs have maximum connectivity except in a single case. We first establish some preliminary results.

3.1. Proposition. *Let* $\mathscr{H}$ *be a proper subgroup of* $\mathscr{G}$, $\bar{S}$ *a generating set of* $\mathscr{H}$, *and suppose that there exists t in* $\mathscr{G}-\mathscr{H}$ *such that* $\bar{S}\cup\{t\}$ *generates* $\mathscr{G}$. If $|\mathscr{H}|\ge|\bar{S}|+|\Delta|$, *where* $\emptyset\ne\Delta\subseteq\{t,t^{-1}\}$, *then*

$$\kappa(\mathrm{Cay}(\mathscr{G},\bar{S}\cup\Delta))\ge\kappa(\mathrm{Cay}(\mathscr{H},\bar{S}))+|\Delta|.$$

Proof. Set $X=\mathrm{Cay}(\mathscr{G},\bar{S}\cup\Delta)$ and $Y=\mathrm{Cay}(\mathscr{H},\bar{S})$. If X is the complete digraph, $\kappa(X)=|\bar{S}\cup\Delta|$. If not, let $\mathscr{A}$ be a positive atom of X containing 1. By Theorem 2.2, $\mathscr{A}$ is the subgroup of $\mathscr{G}$ generated by $(\bar{S}\cup t)\cap\mathscr{A}$.

First suppose that $t\notin\mathscr{A}$. Then $\mathscr{A}$ is a subgroup of $\mathscr{H}$ and $N^+(\mathscr{A})=N_Y^+(\mathscr{A})\cup\mathscr{A}\Delta$, where $N_Y^+(\mathscr{A})=N^+(\mathscr{A})\cap\mathscr{H}$. Since $\mathscr{A}\Delta\cap\mathscr{H}=\emptyset$, using (1),

$$\kappa(X)=|N^+(\mathscr{A})|=|N_Y^+(\mathscr{A})|+|\mathscr{A}\Delta|\ge\min\{|\mathscr{H}-\mathscr{A}|,\kappa(Y)\}+|\mathscr{A}\Delta|.$$

If $|\mathscr{H}-\mathscr{A}|\ge\kappa(Y)$, then

$$\kappa(X)\ge\kappa(Y)+|\mathscr{A}\Delta|\ge\kappa(Y)+|\Delta|.$$

On the other hand, if $|\mathscr{H}-\mathscr{A}|<\kappa(Y)$,

$$\kappa(X)\ge|\mathscr{H}-\mathscr{A}|+|\mathscr{A}\Delta|\ge|\mathscr{H}-\mathscr{A}|+|\mathscr{A}|=|\mathscr{H}|\ge|\bar{S}|+|\Delta|$$
$$\ge\kappa(Y)+|\Delta|.$$

Now suppose $t\in\mathscr{A}$. Let $s\in\bar{S}$ such that $s\notin\mathscr{A}$, and $\mathscr{U}$ the subgroup generated by $S\cap\mathscr{A}$. Then, $N_Y^+(\mathscr{U})\cup\mathscr{U}\Delta s\subseteq N_X^+(\mathscr{A})$. Since $\mathscr{H}\cap\mathscr{U}\Delta s=\emptyset$, we get,

$$\kappa(X)\ge|N_Y^+(\mathscr{U})|+|\mathscr{U}\Delta s|\ge\min\{|\mathscr{H}-\mathscr{U}|,\kappa(Y)\}+|\mathscr{U}\Delta|.$$

If $\kappa(Y) \leq |\mathcal{H} - \mathcal{U}|$ the result holds trivially, while if $\kappa(Y) > |\mathcal{H} - \mathcal{U}|$,

$$\begin{aligned}\kappa(X) &\geq |\mathcal{H} - \mathcal{U}| + |\mathcal{U}\Delta| \geq |\mathcal{H} - \mathcal{U}| + |\mathcal{U}| = |\mathcal{H}| \geq |\bar{S}| + |\Delta| \\ &\geq \kappa(Y) + |\Delta|. \quad \square\end{aligned}$$

Notice that we always have $|\mathcal{H}| \geq |\bar{S}| + 1$, where the equality holds iff $Y = \mathrm{Cay}(\mathcal{H}, \bar{S})$ is the complete digraph. Therefore, the hypothesis $|\mathcal{H}| \geq |\bar{S}| + |\Delta|$ in the last proposition always holds except when Y is the complete digraph and $|\Delta| = 2$. The next lemma characterizes the hierarchical Cayley digraphs which are complete. Let us introduce the following notation. Given a hierarchical generating set of $\mathcal{G}$, $S = \{s_1, \ldots, s_d\}$, let $S_k = \{s_1, \ldots, s_k\}$, $S_k \subseteq \bar{S}_k \subseteq S_k \cup S_k^{-1}$, $\emptyset \neq \Delta_k \subseteq \{s_k, s_k^{-1}\}$, $\mathcal{G}_k = \langle \bar{S}_k \rangle$ and $G_k = \mathrm{Cay}(\mathcal{G}_k, \bar{S}_k)$, $1 \leq k \leq d$. For $x \in \mathcal{G}$, o(x) denotes the order of x in $\mathcal{G}$.

3.2. Lemma. *The only complete hierarchical Cayley digraphs are*
(i) $\mathrm{Cay}(\mathcal{G}_1, \{s_1\})$, $s_1^2 = 1$, $G_1 = K_2$,
(ii) $\mathrm{Cay}(\mathcal{G}_1, \{s_1, s_1^{-1}\})$, $s_1^3 = 1$, $G_1 = K_3$ *and*
(iii) $\mathrm{Cay}(\mathcal{G}_2, \{s_1, s_2, s_2^{-1}\})$, $s_2^2 = s_1$, $s_1^2 = 1$, $G_2 = K_4$.

Proof. Since $\mathcal{G}_{k-1}$ is a proper subgroup of $\mathcal{G}_k$ and $|\mathcal{G}_1| \geq 2$, we have $|\mathcal{G}_k| \geq 2^k$. Therefore, G_k is complete iff $2^k \leq |\mathcal{G}_k| = |\bar{S}_k| + 1 \leq 2k+1$, hence $k \leq 2$. When $k=1$, either $|\bar{S}_1| = 1$, $|\mathcal{G}_1| = 2$ and $s_1^2 = 1$ or $|\bar{S}_1| = 2$, $|\mathcal{G}_1| = 3$ and $s_1^2 = s_1^{-1}$. When $k=2$, we must have $|\mathcal{G}_1| = 2$ (otherwise $|\mathcal{G}_2| \geq 6$) and then, $|\mathcal{G}_2| = 4$ and $s_2 \neq s_2^{-1}$. Therefore, $\mathrm{o}(s_2) = 4$ and $s_2^2 = s_1$. $\square$

3.3. Lemma. *For* $k \leq 3$, $\kappa(G_k) = |\bar{S}_k|$ *unless* $k = 3$, $s_3^2 = s_2^2 = s_1$, $s_1^2 = 1$ *and* $|\bar{S}_3| = 5$, *in which case,* $\mathcal{A} = \{1, s_1\}$ *is an atom of* G_3 *and* $\kappa(G_3) = |\bar{S}_3| - 1$.

Proof. The result is obvious when $k=1$. Suppose that $\kappa(G_2) < |\bar{S}_2|$. Then, by the remarks following Proposition 3.1, G_1 is complete and $|\Delta_2| = 2$. In particular, G_2 is symmetrical and $s_2 \neq s_2^{-1}$. Let $\mathcal{A}$ be a positive atom of G_2. By Proposition 2.3, $2|\mathcal{A}| \leq \kappa(G_2) < |\bar{S}_2| \leq 4$, hence $|\mathcal{A}| = 1$, a contradiction. Similarly, suppose that $\kappa(G_3) < |\bar{S}_3|$. Then, G_2 is complete, $|\Delta_2| = 2$, G_3 is symmetrical and $s_3 \neq s_3^{-1}$. Let $\mathcal{A}$ be an atom of G_3. From $2|\mathcal{A}| \leq \kappa(G_2) < |\bar{S}_2| \leq 6$ we get $|\mathcal{A}| = 2$. Since $\mathrm{o}(s_3) > 2$, $s_3 \notin \mathcal{A}$. Moreover, G_2 is the complete digraph shown in Lemma 3.2(iii), so that $\mathrm{o}(s_2) > 2$ and $s_2 \notin \mathcal{A}$. Hence we should have $\mathcal{A} = \{1, s_1\}$. Then, $|N^+(\mathcal{A})| = |\mathcal{A}s_2 \cup \mathcal{A}s_3 \cup \mathcal{A}s_3^{-1}| \leq 5$. As S_3 is hierarchical, $\mathcal{A}s_2 \cap \mathcal{A}s_3 = \emptyset$. Hence, $\mathcal{A}s_3 = \mathcal{A}s_3^{-1}$, or $s_3^2 = s_1$. In this case, $|N^+(\mathcal{A})| = 4$, so that $\{1, s_1\}$ is an atom of G_3 and $\kappa(G_3) = |\bar{S}_3| - 1$. $\square$

3.4. Lemma. *Suppose that* $s_1^2 = 1$ *and let* $\mathcal{A}$ *be an atom of* G_k, $k > 3$. *Then,* $\mathcal{U} = \mathcal{A} \cap G_{k-1}$ *is a positive fragment of* G_{k-1}. *Moreover,* $|N^+(\mathcal{A})| \geq \kappa(G_{k-1}) + |\mathcal{U}\Delta_k|$.

Proof. If $s_k \in \mathcal{A}$, there exist $s \in S_k$ such that $s \notin \mathcal{A}$ and $N^+(\mathcal{A}) \supset N^+_{G_{k-1}}(\mathcal{U}) \cup \mathcal{U}\Delta_k s$, where these two sets are disjoint. If $s_k \notin \mathscr{A}$, then $\mathscr{A} = \mathscr{U}$ and $N^+(\mathscr{A}) \supseteq N^+_{G_{k-1}}(\mathscr{U}) \cup \mathscr{U}\Delta_k$, where these two sets are also disjoint. In any case,

$$|N^+(\mathscr{A})| \geq |N^+_{G_{k-1}}(\mathscr{U})| + |\mathscr{U}\Delta_k| \geq \min\{\kappa(G_{k-1}), |G_{k-1} - \mathscr{U}|\} + |\mathscr{U}\Delta_k|.$$

If $\kappa(G_{k-1}) \geq |G_{k-1} - \mathscr{U}|$, then

$$2i - 1 \geq |N^+(\mathscr{A})| \geq |G_{k-1}| - |\mathscr{U}| + |\mathscr{U}\Delta_k| \geq |G_{k-1}| \geq 2^{k-1}.$$

Therefore, we must have $|N^+_{G_{k-1}}(\mathscr{U})| = \kappa(G_{k-1})$, and $\mathscr{U}$ is a positive fragment of G_{k-1}. □

3.5. Theorem. *Let $S = \{s_1, \ldots, s_k\}$ be a hierarchical generating set of $\mathscr{G}$, $S \subseteq \bar{S} \subseteq S \cup S^{-1}$ and $G = \mathrm{Cay}(\mathscr{G}, \bar{S})$. Then, $\kappa(G) = |\bar{S}|$ unless $k \geq 3$, $s_i^2 = s_1$ for $2 \leq i \leq k$, $s_1^2 = 1$ and $|\bar{S}| = 2k - 1$, in which case, $\mathscr{A} = \{1, s_1\}$ is an atom of G, and $\kappa(G) = |\bar{S}| - 1$.*

Proof. The proof is by induction on k. By Proposition 2.3, the result holds for $k \leq 3$. Using the above notation, suppose that $k \geq 4$ and $\kappa(G_k) < |\bar{S}_k|$. Then, by Proposition 3.1, $\kappa(G_{k-1}) < |S_{k-1}|$, and, by the induction hypothesis, $s_i^2 = s_1$ for $2 \leq i \leq k-1$, $s_1^2 = 1$ and $|\bar{S}_{k-1}| = 2k - 3$. Let $\mathscr{A}$ be a positive atom of G_k containing 1. By Lemma 3.4, $\mathscr{U} = \mathscr{A} \cap G_{k-1}$ is a positive fragment of G_{k-1}. In particular, $|\mathscr{U}| \geq 2$. Moreover,

$$2k - 2 \geq |N^+(\mathscr{A})| \geq \kappa(G_{k-1}) + |\mathscr{U}\Delta_k| \geq 2k - 2$$

so that $|\mathscr{U}\Delta_k| = |\mathscr{U}| = 2$ (otherwise $|N^+(\mathscr{A})| > 2k - 2$) and $|\Delta_k| = 2$ (otherwise $\kappa(G_k) = |\bar{S}_k|$). Then, $\mathscr{U}$ is an atom of G_{k-1} and, by the induction hypothesis $\mathscr{U} = \{1, s_1\}$. Then, $|\mathscr{U}\Delta_k| = 2$ implies $\mathscr{U}s_k = \mathscr{U}s_k^{-1}$, or $s_k^2 = s_1$. Finally, we must show that $\mathscr{U} = \mathscr{A}$. Suppose on the contrary that $s_k \in \mathscr{A}$ and let $V = S_k - \{s_1, s_k\}$. Then, $N^+(\mathscr{A}) \supset N^+_{G_{k-1}}(\mathscr{U}) \cup \mathscr{U}s_k V$ and

$$2k - 2 \geq |N^+(\mathscr{A})| \geq \kappa(G_{k-1}) + |\mathscr{U}s_k V| \geq 2k - 2.$$

Therefore, $|V| = 1$ and $k = 3$. Hence, $s_k \notin \mathscr{A}$, $\mathscr{A} = \mathscr{U} = \{1, s_1\}$ and $\kappa(G_k) = \kappa(G_{k-1}) + |\mathscr{A}s_k| = 2k - 2 = |\bar{S}_k| - 1$. □

For $k \geq 3$, let $\mathscr{G}_k$ be the subgroup of $\mathrm{Sym}(2k)$, generated by $s_1 = (12)(34)$, $s_2 = (1324)$ and $s_i = s_2(2i-1, 2i)$, $3 \leq i \leq k$. Then, $S_k = \{s_1, \ldots, s_k\}$ is a hierarchical generating set of $\mathscr{G}_k$ satisfying $s_1^2 = 1$, $s_i^2 = s_1$, $2 \leq i \leq k$. Therefore, according to the singular case considered in the theorem above, when $\bar{S}_k = S_k \cup S_k^{-1}$, $\kappa(\mathrm{Cay}(\mathscr{G}_k, \bar{S}_k)) = |\bar{S}_k| - 1$.

In particular, Theorem 3.5 leads to the following results.

3.6. Corollary. *A hierarchical nonsymmetric Cayley digraph has maximum connectivity.*

3.7. Corollary [3]. *If S is a minimal generating set of G, then* $\kappa(\mathrm{Cay}(\mathscr{G}, S \cup S^{-1})) = |S \cup S^{-1}|$.

Proof. We cannot have $s_i^2 = s_1$. □

3.8. Corollary [1]. *If S_k is a hierarchical generating set of $\mathcal{G}_k$ and $|\mathcal{G}_i| \geq (i+1)!$, then $\kappa(\mathrm{Cay}(\mathcal{G}_k, S_k \cup S_k^{-1})) = |S_k \cup S_k^{-1}|$.*

Proof. Since $|\mathcal{G}_2| \geq 6$, either $s_1^2 \neq 1$ or $s_2^2 \neq s_1$. □

The above results can be extended to the nonfinite case by using the generalizations introduced in [6]. Let $\mathcal{G}$ be a nonfinite group and S a finite generating set of $\mathcal{G}$. The *outconnectivity* of $G = \mathrm{Cay}(\mathcal{G}, S)$ is defined as

$$\kappa^+(G) = \min\{|N^+(F)| \text{ s.t. } F \text{ is a nonempty finite subset of } \mathcal{G}\}.$$

A positive fragment of G is a finite subset F s.t. $N^+(F) = \kappa^+(G)$, and a positive fragment of minimum cardinality is a positive atom of G. It is proved in [6] that, with these definitions, Proposition 2.1 and Theorem 2.2 can be extended to the nonfinite (finitely generated) case. Then, we can prove the following theorem.

3.9. Theorem. *Let $\mathcal{G}$ be an infinite group, S a finite hierarchical set of generators of $\mathcal{G}$, $\bar{S} \subset S \cup S^{-1}$. Then, $\kappa^+(\mathrm{Cay}(\mathcal{G}, \bar{S})) = |\bar{S}|$.*

Note added in proof

After the revision of the paper we were aware of two results due to B. Alspach and M. Baumslig. The three methods are different. B. Alspach (Cayley graphs with optimal fault tolerance, IEEE Trans. Comput., to appear) obtained Theorem 3.6 for the undirected case. M. Baumslig (Ph.D. Thesis, City University of New York (1991)) gave another generalization of the results in [1] using a lemma similar to Proposition 3.1. Both Alspach and Baumslig were only interested in the undirected case.

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