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Relative perturbation bounds for weighted polar decomposition

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ABSTRACT

In this paper, we obtain the relative perturbation bounds for weighted unitary polar factors of the weighted polar decomposition in the weighted unitarily invariant norm, the weighted spectral norm, and the weighted Frobenius norm. As special cases, we also derive the new bounds for subunitary and unitary polar factors of (generalized) polar decomposition. These special bounds improve the corresponding results published recently to some extent.

1. Introduction

Let $\mathbb{C}^{m\times n}$, $\mathbb{C}_r^{m\times n}$, $\mathbb{C}_{\geq}^{m}$, and $\mathbb{C}_{>}^{m}$ denote the set of $m \times n$ complex matrices, the subset of $\mathbb{C}^{m\times n}$ consisting of matrices with rank r, the set of Hermitian positive semidefinite matrices of order m, and the subset of $\mathbb{C}_{\geq}^{m}$ comprising positive definite matrices, respectively. Let I_r be the identity matrix of order r. Given $A \in \mathbb{C}^{m\times n}$, the symbols A^*, $A_{MN}^{\#}$, $R(A)$, $A^{\dagger}$, $A_{MN}^{\dagger}$, $\|A\|_2$, $\|A\|_F$, and $\|A\|$ stand for the conjugate transpose, weighted conjugate transpose, range, Moore–Penrose inverse, weighted Moore–Penrose inverse, spectral norm, Frobenius norm, and unitarily invariant norm of A, respectively. The concepts and symbols of $A_{MN}^{\#}$ and $A_{MN}^{\dagger}$ can be found in detail in, e.g., [1,2]. Moreover, without specification, here we always assume that $m > n > r$ and the given weight matrices $M \in \mathbb{C}_{>}^{m}$, $N \in \mathbb{C}_{>}^{n}$.

For a matrix $A \in \mathbb{C}_r^{m\times n}$, there are an (M, N) weighted partial isometric matrix Q [3,4] and a generalized Hermitian positive semidefinite matrix H satisfying $NH \in \mathbb{C}_{\geq}^{n}$ such that

$$A = QH. \tag{1.1}$$

This decomposition is called the (M, N) weighted polar decomposition [4,5] (MN-WPD) of A, and Q and H are called the (M, N) weighted unitary polar factor and generalized nonnegative polar factor, respectively, of this decomposition.

In general, MN-WPD is not unique, while it has been proved that it is unique if the decomposition satisfies

$$R(Q_{MN}^{\#}) = R(H). \tag{1.2}$$

The condition was given by Yang and Li [5]. In this condition, MN-WPD (1.1) can be calculated from the (M, N) singular value decomposition (MN-SVD) [2,6]

$$A = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^* = U_1\Sigma V_1^* \tag{1.3}$$

by

$$Q = U_1V_1^* \quad \text{and} \quad H = N^{-1}V_1\Sigma V_1^*, \tag{1.4}$$

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where $U = (U_1, U_2) \in \mathbb{C}^{m\times m}$ and $V = (V_1, V_2) \in \mathbb{C}^{n\times n}$ satisfy $U^*MU = I_m$ and $V^*N^{-1}V = I_n$, $U_1 \in \mathbb{C}_r^{m\times r}$, $V_1 \in \mathbb{C}_r^{n\times r}$, $\Sigma = diag(\sigma_1, \ldots, \sigma_r)$, and $\sigma_1 \geq \cdots \geq \sigma_r > 0$ are the nonzero (M, N) singular values of A. In this paper, we assume that the condition (1.2) is always satisfied.

The MN-WPD is reduced to the generalized polar decomposition (see, e.g., [7,8]), and Q and H are reduced to the subunitary polar factor and nonnegative polar factor when $M = I_m$ and $N = I_n$. If, in addition, $rank(A) = n$, then decomposition (1.1) is the polar decomposition, and Q and H are the unitary polar factor and positive polar factor.

The problem of estimating the perturbation bounds for subunitary and unitary polar factors has been studied by many authors for additive perturbation in different norms (see, e.g., [8–21]), where the additive perturbation refers to the situation when the perturbed matrix $\widetilde{A}$ is represented as $A + E$. In view of the significance of MN-WPD [22], we studied the absolute perturbation bounds for weighted polar decomposition under additive perturbation in [5,22]. In the present paper, we focus our attention on the relative perturbation bounds for weighted unitary polar factors. That is, the bounds are involved with the weighted Moore–Penrose inverse of the original matrix. Listed are some published relative bounds for subunitary and unitary polar factors under additive perturbation.

Let $A = QH$ and $\widetilde{A} = A + E = \widetilde{Q}\widetilde{H}$ be the (generalized) polar decompositions of A and $\widetilde{A}$, respectively. For in the unitarily invariant norm, for the subunitary polar factor, i.e., when $A, \widetilde{A} \in \mathbb{C}_r^{m\times n}$, the following bound was obtained by Chen and Li [10]:

$$\left\|\widetilde{Q} - Q\right\| \leq \frac{\sigma_1}{\sigma_r + \widetilde{\sigma}_r}\left(\left\|A^\dagger E\right\| + \left\|EA^\dagger\right\|\right) + \left\|EA^\dagger\right\| + \left\|A^\dagger E\right\|, \tag{1.5}$$

where σ_1 is the biggest singular value of A, and $\sigma_r, \widetilde{\sigma}_r$ are the smallest singular values of $A, \widetilde{A}$, respectively. Moreover, Chen and Li [10] also gave two similar bounds for the unitary polar factor, i.e., when $A, \widetilde{A} \in \mathbb{C}_n^{m\times n}$ and $A, \widetilde{A} \in \mathbb{C}_n^{n\times n}$.

For if the unitarily invariant norm is replaced with the Frobenius norm, a bound for the unitary polar factor, i.e., when $A, \widetilde{A} \in \mathbb{C}_n^{m\times n}$, was derived by Chen et al. [12]:

$$\left\|\widetilde{Q} - Q\right\|_F \leq \sqrt{\frac{1}{\eta^2}(\|A^\dagger E\|_F + \|EA^\dagger\|_F)^2 + \left(1 - \frac{1}{\eta^2}\right)\|EA^\dagger\|_F^2} \tag{1.6}$$

$$\leq \frac{1}{\eta}(\|A^\dagger E\|_F + \|EA^\dagger\|_F) + \|EA^\dagger\|_F\sqrt{1 - \frac{1}{\eta^2}}, \tag{1.7}$$

where $\eta = \frac{\sigma_1 + \widetilde{\sigma}_n}{\sigma_1}$, and $\widetilde{\sigma}_n$ is the smallest singular value of $\widetilde{A}$. Li [17] extended the bound (1.6) and presented a bound for the subunitary polar factor:

$$\left\|\widetilde{Q} - Q\right\|_F \leq \sqrt{\frac{1}{\eta^2}(\|A^\dagger E\|_F + \|EA^\dagger\|_F)^2 + \left(1 - \frac{1}{\eta^2}\right)(\|EA^\dagger\|_F^2 + \|A^\dagger E\|_F^2)}, \tag{1.8}$$

where $\eta = \frac{\sigma_1 + \widetilde{\sigma}_r}{\sigma_1}$.

Furthermore, Chen and Li [11] also obtained an alternative relative perturbation bound involved in both the original matrix A and the perturbed matrix $\widetilde{A}$ for the subunitary polar factor in Frobenius norm without the restriction that the ranks of A and $\widetilde{A}$ are the same.

In order to make this paper more self-contained, we now introduce some preliminaries which include the definitions of the weighted norms (see Definition 1.1) and two lemmas needed later in this paper, where Lemma 1.2 can be found in [23] and Lemma 1.3 can be found in [24].

Definition 1.1. Let $A \in \mathbb{C}_r^{m\times n}$. The norms $\|A\|_{(MN)} = \left\|M^{1/2}AN^{-1/2}\right\|$, $\|A\|_{2(MN)} = \left\|M^{1/2}AN^{-1/2}\right\|_2$, and $\|A\|_{F(MN)} = \left\|M^{1/2}AN^{-1/2}\right\|_F$ are called the weighted unitarily invariant norm, weighted spectral norm, and weighted Frobenius norm of A, respectively.

It is worth pointing out that the weighted spectral norm of A is synonymous with the weighted norm of A defined as $\|A\|_{MN} = \left\|M^{1/2}AN^{-1/2}\right\|_2$ in [1] and the weighted unitarily invariant norm is equivalent to the (M, N)-invariant norm defined by Rao and Rao [25] in essence. Furthermore, combining the properties of the Frobenius norm with Definition 1.1, we can show that

$$\|A\|_{F(MN)} = (tr(A^\#_{MN}A))^{1/2}. \tag{1.9}$$

Lemma 1.2. *Let $\Omega \in \mathbb{C}^{s\times s}$ and $\Gamma \in \mathbb{C}^{t\times t}$ be two Hermitian matrices, and $S \in \mathbb{C}^{s\times t}$, and*

$$\Delta = [\alpha, \beta] \subset R, \qquad \Delta' = R \setminus [\alpha - \delta, \beta + \delta], \quad \delta > 0.$$

Let $\lambda(\Omega)$ and $\lambda(\Gamma)$ denote the eigenvalue sets of Ω and Γ, respectively. If

$$\lambda(\Omega) \subset \Delta,\ \lambda(\Gamma) \subset \Delta',$$

then the equation $\Omega X - X\Gamma = S$ has a unique solution $X \in \mathbb{C}^{s\times t}$, and moreover, $\|X\| \leq \frac{\|S\|}{\delta}$ for any unitarily invariant norm.

Lemma 1.3. *Let $\Omega \in \mathbb{C}^{s\times s}$ and $\Gamma \in \mathbb{C}^{t\times t}$ be two Hermitian matrices, and let $\lambda(\Omega)$ and $\lambda(\Gamma)$ denote the sets of eigenvalues of Ω and Γ, respectively. If $\lambda(\Omega)\cap\lambda(\Gamma)=\emptyset$, then for any $E, F \in \mathbb{C}^{s\times t}$ the equation $\Omega X - X\Gamma = \Omega E + F\Gamma$ has a unique solution $X \in \mathbb{C}^{s\times t}$, and moreover,*

$$\|X\|_F \le \frac{1}{\eta}\sqrt{\|E\|_F^2 + \|F\|_F^2},$$

where $\eta = \min_{\omega\in\lambda(\Omega),\gamma\in\lambda(\Gamma)} \frac{|\omega-\gamma|}{\sqrt{|\omega|^2+|\gamma|^2}}$. If, in addition, $F = 0$, we have a better bound

$$\|X\|_F \le \frac{1}{\eta}\|E\|_F\,,$$

where $\eta = \min_{\omega\in\lambda(\Omega),\gamma\in\lambda(\Gamma)} \frac{|\omega-\gamma|}{|\omega|}$.

2. Main results

Let us have the perturbed matrix $\widetilde{A} \in \mathbb{C}_r^{m\times n}$. Like for (1.1) and (1.3), we can give the MN-WPD and MN-SVD of $\widetilde{A}$ as follows:

$$\widetilde{A} = \widetilde{Q}\widetilde{H} \quad \text{and} \quad \widetilde{A} = \widetilde{U}\begin{pmatrix}\widetilde{\Sigma} & 0\\ 0 & 0\end{pmatrix}\widetilde{V}^* = \widetilde{U}_1\widetilde{\Sigma}\widetilde{V}_1^*, \tag{2.1}$$

in which

$$\widetilde{Q} = \widetilde{U}_1\widetilde{V}_1^* \quad \text{and} \quad \widetilde{H} = N^{-1}\widetilde{V}_1\widetilde{\Sigma}\widetilde{V}_1^*, \tag{2.2}$$

where $\widetilde{U} = (\widetilde{U}_1, \widetilde{U}_2) \in \mathbb{C}^{m\times m}$ and $\widetilde{V} = (\widetilde{V}_1, \widetilde{V}_2) \in \mathbb{C}^{n\times n}$ satisfy $\widetilde{U}^*M\widetilde{U} = I_m$ and $\widetilde{V}^*N^{-1}\widetilde{V} = I_n$, $\widetilde{U}_1 \in \mathbb{C}_r^{m\times r}$, $\widetilde{V}_1 \in \mathbb{C}_r^{n\times r}$, $\widetilde{\Sigma} = diag(\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_r)$, and $\widetilde{\sigma}_1 \ge \cdots \ge \widetilde{\sigma}_r > 0$ are the nonzero (M, N) singular values of $\widetilde{A}$.

Next, we first present the relative perturbation bounds for weighted unitary polar factors in the weighted norms introduced in Definition 1.1.

Theorem 2.1. *Let $A, \widetilde{A} = A + E \in \mathbb{C}_r^{m\times n}$ with the MN-WPDs as in (1.1) and (2.1), respectively. Then*

$$\left\|\widetilde{Q} - Q\right\|_{(MN)} \le \frac{\sigma_1}{\sigma_r + \widetilde{\sigma}_r}\left(\left\|A_{MN}^\dagger E\right\|_{(NN)} + \left\|EA_{MN}^\dagger\right\|_{(MM)}\right) + \left\|EA_{MN}^\dagger\right\|_{(MM)} + \left\|A_{MN}^\dagger E\right\|_{(NN)}, \tag{2.3}$$

where σ_1 is the biggest (M, N) singular value of A, and σ_r, $\widetilde{\sigma}_r$ are the smallest nonzero (M, N) singular values of A, $\widetilde{A}$, respectively.

Proof. From the MN-SVD of A in (1.3) and the two facts $U^*MU = I_m$ and $V^*N^{-1}V = I_n$, we know the weighted Moore–Penrose inverse of A can be written as

$$A_{MN}^\dagger = N^{-1}V\begin{pmatrix}\Sigma^{-1} & 0\\ 0 & 0\end{pmatrix}U^*M = N^{-1}V_1\Sigma^{-1}U_1^*M \tag{2.4}$$

and

$$U_1^*MU_1 = V_1^*N^{-1}V_1 = I_r, \qquad \widetilde{U}_1^*M\widetilde{U}_1 = \widetilde{V}_1^*N^{-1}\widetilde{V}_1 = I_r. \tag{2.5}$$

According to (2.4), (1.3) and (2.1), we have

$$\begin{aligned} A_{MN}^\dagger(\widetilde{A} - A) &= N^{-1}V_1\Sigma^{-1}U_1^*M(\widetilde{U}_1\widetilde{\Sigma}\widetilde{V}_1^* - U_1\Sigma V_1^*)\\ &= N^{-1}V_1\Sigma^{-1}U_1^*M\widetilde{U}_1\widetilde{\Sigma}\widetilde{V}_1^* - N^{-1}V_1V_1^*, \end{aligned} \tag{2.6}$$

$$\begin{aligned} (\widetilde{A} - A)A_{MN}^\dagger &= (\widetilde{U}_1\widetilde{\Sigma}\widetilde{V}_1^* - U_1\Sigma V_1^*)N^{-1}V_1\Sigma^{-1}U_1^*M\\ &= \widetilde{U}_1\widetilde{\Sigma}\widetilde{V}_1^*N^{-1}V_1\Sigma^{-1}U_1^*M - U_1U_1^*M. \end{aligned} \tag{2.7}$$

The equalities (2.6) and (2.7) imply that

$$V_1^*A_{MN}^\dagger EN^{-1}\widetilde{V}_1 = \Sigma^{-1}U_1^*M\widetilde{U}_1\widetilde{\Sigma} - V_1^*N^{-1}\widetilde{V}_1, \tag{2.8}$$

$$\widetilde{U}_1^*MEA_{MN}^\dagger U_1 = \widetilde{\Sigma}\widetilde{V}_1^*N^{-1}V_1\Sigma^{-1} - \widetilde{U}_1^*MU_1. \tag{2.9}$$

Subtracting (2.9) from the conjugate transpose of (2.8) leads to

$$\begin{aligned} &\widetilde{V}_1^*N^{-1}(A_{MN}^\dagger E)^*V_1 - \widetilde{U}_1^*MEA_{MN}^\dagger U_1\\ &\quad = (\widetilde{\Sigma}\widetilde{U}_1^*MU_1\Sigma^{-1} - \widetilde{V}_1^*N^{-1}V_1) - (\widetilde{\Sigma}\widetilde{V}_1^*N^{-1}V_1\Sigma^{-1} - \widetilde{U}_1^*MU_1). \end{aligned} \tag{2.10}$$

Thus, postmultiplying (2.10) by Σ, we have

$$\widetilde{\Sigma}(\widetilde{U}_1^*MU_1 - \widetilde{V}_1^*N^{-1}V_1) + (\widetilde{U}_1^*MU_1 - \widetilde{V}_1^*N^{-1}V_1)\Sigma = \left(\widetilde{V}_1^*N^{-1}(A_{MN}^{\dagger}E)^*V_1 - \widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right)\Sigma. \tag{2.11}$$

Applying Lemma 1.2 to (2.11) with $\Omega = \widetilde{\Sigma}$, $\Gamma = -\Sigma$, and

$$\begin{aligned} X &= \widetilde{U}_1^*MU_1 - \widetilde{V}_1^*N^{-1}V_1, \\ S &= \left(\widetilde{V}_1^*N^{-1}(A_{MN}^{\dagger}E)^*V_1 - \widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right)\Sigma \end{aligned} \tag{2.12}$$

gives

$$\|X\| \le \frac{\left\|\left(\widetilde{V}_1^*N^{-1}(A_{MN}^{\dagger}E)^*V_1 - \widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right)\Sigma\right\|}{\delta}, \tag{2.13}$$

where $\delta = \sigma_r + \widetilde{\sigma}_r$. Note that

$$\begin{aligned} \left\|\left(\widetilde{V}_1^*N^{-1}(A_{MN}^{\dagger}E)^*V_1 - \widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right)\Sigma\right\| &\le \left\|\widetilde{V}_1^*N^{-1}(A_{MN}^{\dagger}E)^*V_1 - \widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right\| \|\Sigma\|_2 \\ &\le \sigma_1\left(\left\|\widetilde{V}_1^*N^{-1}(A_{MN}^{\dagger}E)^*V_1\right\| + \left\|\widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right\|\right). \end{aligned} \tag{2.14}$$

Then, (2.13) and (2.14) together reveal that

$$\|X\| \le \frac{\sigma_1}{\sigma_r + \widetilde{\sigma}_r}\left(\left\|\widetilde{V}_1^*N^{-1}(A_{MN}^{\dagger}E)^*V_1\right\| + \left\|\widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right\|\right). \tag{2.15}$$

Since

$$\widetilde{U}^*M(\widetilde{Q} - Q)N^{-1}V = \begin{pmatrix} \widetilde{V}_1^*N^{-1}V_1 - \widetilde{U}_1^*MU_1 & \widetilde{V}_1^*N^{-1}V_2 \\ -\widetilde{U}_2^*MU_1 & 0 \end{pmatrix}, \tag{2.16}$$

observing (2.12), we can obtain

$$\left\|\widetilde{U}^*M(\widetilde{Q} - Q)N^{-1}V\right\| \le \|X\| + \left\|\widetilde{V}_1^*N^{-1}V_2\right\| + \left\|\widetilde{U}_2^*MU_1\right\|. \tag{2.17}$$

Thus, it follows from (2.15) and (2.17) that

$$\left\|\widetilde{U}^*M(\widetilde{Q} - Q)N^{-1}V\right\| \le \frac{\sigma_1}{\sigma_r + \widetilde{\sigma}_r}\left(\left\|\widetilde{V}_1^*N^{-1}(A_{MN}^{\dagger}E)^*V_1\right\| + \left\|\widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right\|\right) + \left\|\widetilde{V}_1^*N^{-1}V_2\right\| + \left\|\widetilde{U}_2^*MU_1\right\|. \tag{2.18}$$

From (2.6) and (2.7), it is seen that

$$\left\|V_1^*N^{-1}\widetilde{V}_2\right\| = \left\|-V_1^*A_{MN}^{\dagger}EN^{-1}\widetilde{V}_2\right\|, \qquad \left\|\widetilde{U}_2^*MEA_{MN}^{\dagger}U_1\right\| = \left\|-\widetilde{U}_2^*MU_1\right\|. \tag{2.19}$$

Note that $V^*N^{-1}\widetilde{V}$ is unitary. Then

$$\left\|V_1^*N^{-1}\widetilde{V}_2\right\| = \left\|\widetilde{V}_2^*N^{-1}V_1\right\| = \left\|\widetilde{V}_1^*N^{-1}V_2\right\| = \left\|-V_1^*A_{MN}^{\dagger}EN^{-1}\widetilde{V}_2\right\|. \tag{2.20}$$

Furthermore, from Definition 1.1 and the properties of the unitarily invariant norm, we have

$$\left\|\widetilde{U}^*M(\widetilde{Q} - Q)N^{-1}V\right\| = \left\|M^{1/2}(\widetilde{Q} - Q)N^{-1/2}\right\| = \left\|\widetilde{Q} - Q\right\|_{(MN)}, \tag{2.21}$$

$$\left\|\widetilde{V}_1^*N^{-1}(A_{MN}^{\dagger}E)^*V_1\right\| \le \left\|N^{-1/2}(A_{MN}^{\dagger}E)^*N^{1/2}\right\| = \left\|A_{MN}^{\dagger}E\right\|_{(NN)}, \tag{2.22}$$

$$\left\|\widetilde{U}_1^*MEA_{MN}^{\dagger}U_1\right\| \le \left\|M^{1/2}EA_{MN}^{\dagger}M^{-1/2}\right\| = \left\|EA_{MN}^{\dagger}\right\|_{(MM)}, \tag{2.23}$$

$$\left\|V_1^*A_{MN}^{\dagger}EN^{-1}\widetilde{V}_2\right\| \le \left\|N^{1/2}A_{MN}^{\dagger}EN^{-1/2}\right\| = \left\|A_{MN}^{\dagger}E\right\|_{(NN)}, \tag{2.24}$$

$$\left\|\widetilde{U}_2^*MEA_{MN}^{\dagger}U_1\right\| \le \left\|M^{1/2}EA_{MN}^{\dagger}M^{-1/2}\right\| = \left\|EA_{MN}^{\dagger}\right\|_{(MM)}. \tag{2.25}$$

Therefore, the proof follows from (2.18)–(2.25). □

If the weighted spectral norm, i.e., the weighted norm, is chosen as the specific weighted unitarily invariant norm in Theorem 2.1, we have the following smaller perturbation bound.

Theorem 2.2. *Assume that the conditions of Theorem 2.1 hold. Then*

$$\left\|\widetilde{Q}-Q\right\|_{MN} \le \frac{\sigma_1}{\sigma_r+\widetilde{\sigma}_r}\left(\left\|A^\dagger_{MN}E\right\|_{NN}+\left\|EA^\dagger_{MN}\right\|_{MM}\right)+\max\left\{\left\|EA^\dagger_{MN}\right\|_{MM},\left\|A^\dagger_{MN}E\right\|_{NN}\right\}.$$

Proof. Observe that (2.16) can be rewritten as

$$\widetilde{U}^*M(\widetilde{Q}-Q)N^{-1}V=\begin{pmatrix}\widetilde{V}_1^*N^{-1}V_1-\widetilde{U}_1^*MU_1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & \widetilde{V}_1^*N^{-1}V_2\\ -\widetilde{U}_2^*MU_1 & 0\end{pmatrix}.$$

Then, noting (2.12), we have

$$\begin{aligned}\left\|\widetilde{U}^*M(\widetilde{Q}-Q)N^{-1}V\right\|_2 &= \left\|\widetilde{Q}-Q\right\|_{MN}\le \|X\|_2+\left\|\begin{pmatrix}0 & \widetilde{V}_1^*N^{-1}V_2\\ -\widetilde{U}_2^*MU_1 & 0\end{pmatrix}\right\|_2\\ &\le \|X\|_2+\max\left\{\left\|\widetilde{V}_1^*N^{-1}V_2\right\|_2,\left\|-\widetilde{U}_2^*MU_1\right\|_2\right\},\end{aligned}$$

which together with (2.15), (2.19), (2.20), (2.24) and (2.25) implies the proof. □

If we take the weighted Frobenius norm as the specific weighted unitarily invariant norm in Theorem 2.1, an alternative perturbation bound can be derived as follows.

Theorem 2.3. *Assume that the conditions of Theorem 2.1 hold and set $\eta=\frac{\sigma_1+\widetilde{\sigma}_r}{\sigma_1}\ge\sqrt{2-\epsilon}$, where σ_1 is the biggest (M,N) singular value of A, $\widetilde{\sigma}_r$ is the smallest nonzero (M,N) singular value of $\widetilde{A}$, and $0\le\epsilon\le 1$. Then*

$$\left\|\widetilde{Q}-Q\right\|_{F(MN)}\le\sqrt{\left(1+\frac{\epsilon}{\eta^2}\right)\left(\left\|A^\dagger_{MN}E\right\|^2_{F(NN)}+\left\|EA^\dagger_{MN}\right\|^2_{F(MM)}\right)} \tag{2.26}$$

$$\le\sqrt{\frac{2}{2-\epsilon}\left(\left\|A^\dagger_{MN}E\right\|^2_{F(NN)}+\left\|EA^\dagger_{MN}\right\|^2_{F(MM)}\right)}. \tag{2.27}$$

Proof. Applying Lemma 1.3 to (2.11) with $\Omega=\widetilde{\Sigma}$, $\Gamma=-\Sigma$, $E=0$, X as in (2.12), and

$$F=-\widetilde{V}_1^*N^{-1}(A^\dagger_{MN}E)^*V_1+\widetilde{U}_1^*MEA^\dagger_{MN}U_1$$

leads to

$$\|X\|_F^2\le\frac{1}{\eta^2}\left\|-\widetilde{V}_1^*N^{-1}(A^\dagger_{MN}E)^*V_1+\widetilde{U}_1^*MEA^\dagger_{MN}U_1\right\|_F^2\le\frac{1}{\eta^2}\left(\left\|\widetilde{V}_1^*N^{-1}(A^\dagger_{MN}E)^*V_1\right\|_F+\left\|\widetilde{U}_1^*MEA^\dagger_{MN}U_1\right\|_F\right)^2, \tag{2.28}$$

where $\eta=\min_{1\le i,j\le r}\frac{|\sigma_i+\widetilde{\sigma}_j|}{|\sigma_i|}=\frac{\sigma_1+\widetilde{\sigma}_r}{\sigma_1}\ge\sqrt{2-\epsilon}$. Then, it follows from Definition 1.1, (2.16), (2.21), (2.28), (2.19) and (2.20) that

$$\begin{aligned}\left\|\widetilde{U}^*M(\widetilde{Q}-Q)N^{-1}V\right\|_F^2 &= \left\|M^{1/2}(\widetilde{Q}-Q)N^{-1/2}\right\|_F^2=\left\|\widetilde{Q}-Q\right\|^2_{F(MN)}\\ &\le\frac{1}{\eta^2}\left(\left\|\widetilde{V}_1^*N^{-1}(A^\dagger_{MN}E)^*V_1\right\|_F+\left\|\widetilde{U}_1^*MEA^\dagger_{MN}U_1\right\|_F\right)^2\\ &\quad+\left\|\widetilde{V}_2^*N^{-1}(A^\dagger_{MN}E)^*V_1\right\|_F^2+\left\|\widetilde{U}_2^*MEA^\dagger_{MN}U_1\right\|_F^2\\ &\le\frac{2-\epsilon}{\eta^2}\left(\left\|(\widetilde{V}_1,\widetilde{V}_2)^*N^{-1}(A^\dagger_{MN}E)^*V_1\right\|_F^2+\left\|(\widetilde{U}_1,\widetilde{U}_2)^*MEA^\dagger_{MN}U_1\right\|_F^2\right)\\ &\quad+\frac{\epsilon}{\eta^2}\left(\left\|\widetilde{V}_1^*N^{-1}(A^\dagger_{MN}E)^*V_1\right\|_F^2+\left\|\widetilde{U}_1^*MEA^\dagger_{MN}U_1\right\|_F^2\right)\\ &\quad+\left(1-\frac{2-\epsilon}{\eta^2}\right)\left(\left\|\widetilde{V}_2^*N^{-1}(A^\dagger_{MN}E)^*V_1\right\|_F^2+\left\|\widetilde{U}_2^*MEA^\dagger_{MN}U_1\right\|_F^2\right)\\ &\le\frac{2-\epsilon}{\eta^2}\left(\left\|N^{-1/2}(A^\dagger_{MN}E)^*N^{1/2}\right\|_F^2+\left\|M^{1/2}EA^\dagger_{MN}M^{-1/2}\right\|_F^2\right)\\ &\quad+\frac{\epsilon}{\eta^2}\left(\left\|N^{-1/2}(A^\dagger_{MN}E)^*N^{1/2}\right\|_F^2+\left\|M^{1/2}EA^\dagger_{MN}M^{-1/2}\right\|_F^2\right)\end{aligned}$$

$$+\left(1-\frac{2-\epsilon}{\eta^2}\right)\left(\left\|N^{-1/2}(A_{MN}^{\dagger}E)^*N^{1/2}\right\|_F^2+\left\|M^{1/2}EA_{MN}^{\dagger}M^{-1/2}\right\|_F^2\right)$$
$$=\left(1+\frac{\epsilon}{\eta^2}\right)\left(\left\|A_{MN}^{\dagger}E\right\|_{F(NN)}^2+\left\|EA_{MN}^{\dagger}\right\|_{F(MM)}^2\right)$$
$$\le\frac{2}{2-\epsilon}\left(\left\|A_{MN}^{\dagger}E\right\|_{F(NN)}^2+\left\|EA_{MN}^{\dagger}\right\|_{F(MM)}^2\right).$$

Therefore, we obtain the desired results. □

When $A, \widetilde{A} \in \mathbb{C}_n^{m\times n}$, the MN-SVDs of A and $\widetilde{A}$ are reduced to

$$A = U\begin{pmatrix}\Sigma\\0\end{pmatrix}V^* = U_1\Sigma V^* \quad\text{and}\quad \widetilde{A} = \widetilde{U}\begin{pmatrix}\widetilde{\Sigma}\\0\end{pmatrix}\widetilde{V}^* = \widetilde{U}_1\widetilde{\Sigma}\widetilde{V}^*.$$

In this case, the MN-WPDs of A and $\widetilde{A}$ can be computed using

$$Q = U_1V^*,\quad H = N^{-1}V\Sigma V^* \quad\text{and}\quad \widetilde{Q} = \widetilde{U}_1\widetilde{V}^*,\quad \widetilde{H} = N^{-1}\widetilde{V}\widetilde{\Sigma}\widetilde{V}^*,$$

and (2.4) is reduced to

$$A_{MN}^{\dagger} = N^{-1}V(\Sigma^{-1},0)U^*M = N^{-1}V\Sigma^{-1}U_1^*M.$$

Thus, X, F, and $\widetilde{U}^*M(\widetilde{Q}-Q)N^{-1}V$ appearing in the proof of Theorems 2.1 and 2.3 are changed to

$$X = \widetilde{U}_1^*MU_1 - \widetilde{V}^*N^{-1}V,$$
$$F = -\widetilde{V}^*N^{-1}(A_{MN}^{\dagger}E)^*V + \widetilde{U}_1^*MEA_{MN}^{\dagger}U_1,$$
$$\widetilde{U}^*M(\widetilde{Q}-Q)N^{-1}V = \begin{pmatrix}\widetilde{V}^*N^{-1}V - \widetilde{U}_1^*MU_1\\ -\widetilde{U}_2^*MU_1\end{pmatrix}.$$

In terms of the above discussions and the proofs of Theorems 2.1 and 2.3, we have the following two theorems.

Theorem 2.4. *Let $A, \widetilde{A} = A+E \in \mathbb{C}_n^{m\times n}$ with the MN-WPDs as in (1.1) and (2.1), respectively. Then*

$$\|\widetilde{Q}-Q\|_{(MN)} \le \frac{\sigma_1}{\sigma_n+\widetilde{\sigma}_n}\left(\left\|A_{MN}^{\dagger}E\right\|_{(NN)}+\left\|EA_{MN}^{\dagger}\right\|_{(MM)}\right)+\left\|EA_{MN}^{\dagger}\right\|_{(MM)},$$

where σ_1 is the biggest (M,N) singular value of A, and σ_n, $\widetilde{\sigma}_n$ are the smallest nonzero (M,N) singular values of A, $\widetilde{A}$, respectively.

Theorem 2.5. *Assume that the conditions of Theorem 2.4 hold and set $\eta = \frac{\sigma_1+\widetilde{\sigma}_n}{\sigma_1} \ge \sqrt{2-\epsilon}$, where σ_1 is the biggest (M,N) singular value of A, $\widetilde{\sigma}_n$ is the smallest nonzero (M,N) singular value of $\widetilde{A}$, and $0\le\epsilon\le 1$. Then*

$$\|\widetilde{Q}-Q\|_{F(MN)} \le \sqrt{\frac{2}{\eta^2}\left\|A_{MN}^{\dagger}E\right\|_{F(NN)}^2+\left(1+\frac{\epsilon}{\eta^2}\right)\left\|EA_{MN}^{\dagger}\right\|_{F(MM)}^2} \tag{2.29}$$
$$\le \sqrt{\frac{2}{2-\epsilon}\left(\left\|A_{MN}^{\dagger}E\right\|_{F(NN)}^2+\left\|EA_{MN}^{\dagger}\right\|_{F(MM)}^2\right)}. \tag{2.30}$$

Two new relative perturbation bounds for subunitary or unitary polar factors can be obtained as follows when the weight matrices M and N in Theorems 2.3 and 2.5 are reduced to the identity matrices I_m and I_n, respectively.

Corollary 2.6. *Let $A, \widetilde{A} = A+E \in \mathbb{C}_r^{m\times n}$ with the generalized polar decompositions $A = QH$, $\widetilde{A} = \widetilde{Q}\widetilde{H}$. If $\eta = \frac{\sigma_1+\widetilde{\sigma}_r}{\sigma_1} \ge \sqrt{2-\epsilon}$, where σ_1 is the biggest singular value of A, $\widetilde{\sigma}_r$ is the smallest nonzero singular value of $\widetilde{A}$, and $0\le\epsilon\le 1$, then*

$$\|\widetilde{Q}-Q\|_F \le \sqrt{\left(1+\frac{\epsilon}{\eta^2}\right)(\|A^{\dagger}E\|_F^2+\|EA^{\dagger}\|_F^2)} \tag{2.31}$$
$$\le \sqrt{\frac{2}{2-\epsilon}(\|A^{\dagger}E\|_F^2+\|EA^{\dagger}\|_F^2)}. \tag{2.32}$$

Corollary 2.7. *Let $A, \widetilde{A} = A + E \in \mathbb{C}_n^{m\times n}$ with the polar decompositions $A = QH, \widetilde{A} = \widetilde{Q}\widetilde{H}$. If $\eta = \frac{\sigma_1 + \widetilde{\sigma}_n}{\sigma_1} \geq \sqrt{2-\epsilon}$, where σ_1 is the biggest singular value of A, $\widetilde{\sigma}_n$ is the smallest nonzero singular value of $\widetilde{A}$, and $0 \leq \epsilon \leq 1$, then*

$$\left\|\widetilde{Q} - Q\right\|_F \leq \sqrt{\frac{2}{\eta^2}\|A^\dagger E\|_F^2 + \left(1 + \frac{\epsilon}{\eta^2}\right)\|EA^\dagger\|_F^2} \tag{2.33}$$

$$\leq \sqrt{\frac{2}{2-\epsilon}(\|A^\dagger E\|_F^2 + \|EA^\dagger\|_F^2)}. \tag{2.34}$$

Remark 2.8. It is easy to find that if the conditions in Corollaries 2.6 and 2.7 hold, i.e., $\eta \geq \sqrt{2-\epsilon}$ and $0 \leq \epsilon \leq 1$, and the value of ϵ is set to a suitable one, the new bounds (2.31)–(2.34) may be smaller than the corresponding ones (1.6)–(1.8). In fact, for the bounds (1.8) and (2.31), if $0 \leq \epsilon \leq \frac{2\|A^\dagger E\|_F \|EA^\dagger\|_F}{\|A^\dagger E\|_F^2 + \|EA^\dagger\|_F^2}$ and $\eta \geq \sqrt{2-\epsilon}$, the bound (2.31) is less than or equal to that of (1.8). For the bounds (1.6) and (2.33), if $0 \leq \epsilon \leq \frac{2\|A^\dagger E\|_F \|EA^\dagger\|_F - \|A^\dagger E\|_F^2}{\|EA^\dagger\|_F^2}$ and $\eta \geq \sqrt{2-\epsilon}$, the bound (2.33) is not greater than that of (1.6). Moreover, the new bounds (2.31) and (2.32) are obviously better than that of (1.8) if $\epsilon = 0$, i.e., $\eta \geq \sqrt{2}$. Of course, it must be emphasized that if $\epsilon = 1$, i.e., without the restriction on singular values, the bounds (2.31) and (2.33) are worse than the corresponding ones (1.8) and (1.6). Furthermore, in comparison with the bounds (2.31) and (2.33), the bounds (2.32) and (2.34) are simpler in form although they are a little larger than the corresponding ones (2.31) and (2.33).

Remark 2.9. As we know, without assuming that $\|E\|_F$ is tiny, the current best absolute perturbation bound for additive perturbation in the Frobenius norm is [19]

$$\left\|\widetilde{Q} - Q\right\|_F \leq \frac{2}{\sigma_r + \widetilde{\sigma}_r}\left\|\widetilde{E}\right\|_F, \tag{2.35}$$

where σ_r and $\widetilde{\sigma}_r$ are the smallest nonzero singular values of A and $\widetilde{A}$, respectively. The bound (2.35) was obtained by Li [16] for $A, \widetilde{A} \in \mathbb{C}_n^{m\times n}$ and by Li and Sun [18] for $A, \widetilde{A} \in \mathbb{C}_r^{m\times n}$.

From the discussions in [12] and [17], we know that neither the relative perturbation bound nor the absolute perturbation bound for (generalized) polar decomposition in the Frobenius norm is uniformly better than the other in general. This rule is also applicable to the bounds (2.31)–(2.35). Two examples are given below, for which the relative perturbation bounds (2.33) and (2.31) are a little better than the corresponding ones (1.6) and (1.8), and not greater than that of (2.35) if we set $\epsilon = 0.2, 0.5, 0.8$, respectively. Moreover, it is easy to verify that the rule mentioned above is also suitable for the relative perturbation bounds obtained in this paper and absolute perturbation bounds derived in [5,22] for weighted unitary polar factors.

Example 2.10. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1.5 \\ 0.5 & 0 \end{pmatrix} \in \mathbb{C}_2^{3\times 2}, \qquad \widetilde{A} = A + \begin{pmatrix} 0 & 0 \\ 0 & 0.001 \\ 0 & 0.001 \end{pmatrix} \in \mathbb{C}_2^{3\times 2}.$$

Then, we can get $\eta = 1.7454$ and the perturbation bounds (1.6) and (2.35), respectively, as follows:

$$\left\|\widetilde{Q} - Q\right\|_F \leq 0.00125 \quad \text{and} \quad \left\|\widetilde{Q} - Q\right\|_F \leq 0.00126.$$

If the value of ϵ is set to be 0.2, 0.5, 0.8, respectively, then the corresponding bound (2.33) can be obtained as follows:

$$\left\|\widetilde{Q} - Q\right\|_F \leq 0.00116, 0.00120, 0.00123.$$

Furthermore, we have $\frac{2\|A^\dagger E\|_F \|EA^\dagger\|_F - \|A^\dagger E\|_F^2}{\|EA^\dagger\|_F^2} = 0.9692$. As a result, we can conclude that the bound (2.33) is always not bigger than that of (1.6) if $0 \leq \epsilon \leq 0.9692$ for this example.

Example 2.11. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0.75 & 0 \end{pmatrix} \in \mathbb{C}_2^{4\times 3}, \qquad \widetilde{A} = A + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0.001 & 0 \end{pmatrix} \in \mathbb{C}_2^{4\times 3}.$$

Then, we can obtain $\eta = 1.67$ and the perturbation bounds (1.8) and (2.35), respectively, as follows:

$$\left\|\widetilde{Q} - Q\right\|_F \leq 0.00159 \quad \text{and} \quad \left\|\widetilde{Q} - Q\right\|_F \leq 0.00155.$$

If we set $\epsilon = 0.2, 0.5, 0.8$, the bound (2.31) can be derived, respectively, as follows:

$$\left\| \widetilde{Q} - Q \right\|_F \leq 0.00141, 0.00148, 0.00155.$$

In addition, we can get $\frac{2\|A^{\dagger}E\|_F \|EA^{\dagger}\|_F}{\|A^{\dagger}E\|_F^2 + \|EA^{\dagger}\|_F^2} = 0.9897$ for this example. Therefore, the bound (2.31) is always not greater than that of (1.8) if $0 \leq \epsilon \leq 0.9897$.

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