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Recurrence Relations for the Super-Halley Method

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Abstract—In this paper we give sufficient conditions in order to assure the convergence of the super-Halley method in Banach spaces. We use a system of recurrence relations analogous to those given in the classical Newton-Kantorovich theorem, or those given for Chebyshev and Halley methods by different authors. © 1998 Elsevier Science Ltd. All rights reserved.

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INTRODUCTION

The study of third-order iterative processes is becoming more and more important in recent years. Among these methods, probably, the most famous are Chebyshev, Halley, and super-Halley methods. In that way, many papers on third-order methods have been published. So for Chebyshev method we have [1-3] for Halley method [3-5], and [6-8] for super-Halley method. Most authors follow the technique developed by Kantorovich for Newton's method [9,10]. In [11], a unified theory is presented for the above third-order methods. To sum up, in these papers the convergence of the sequence in Banach spaces follows from the convergence of a majorizing sequence, which is obtained by applying the real third-order method to a cubic polynomial.

Another way to get the convergence is by using recurrence relations. This technique has also been used by Kantorovich and other authors for Newton's method [12,13].

In the same way, Candela and Marquina are established in [1,4], the convergence for Halley and Chebyshev methods, respectively. Starting with two parameters, they construct a system of recurrence relations, consisting of four real sequences of positive real numbers, which yields an increasing convergent sequence that majorizes the sequence in Banach spaces. The use of these relations supposes some advantages, because we can reduce our initial problem in a Banach space to a simpler problem with real sequences and functions.

In this paper we present the recurrences for the super-Halley method, and we use them in order to prove the convergence of this method under Kantorovich-type assumptions.

Besides, the use of these recurrences allows us to obtain *a priori* error bounds. We finish this paper with some examples to illustrate the given results and to compare Chebyshev, Halley, and super-Halley methods. As we will see, the error bounds obtained for the super-Halley method improve the ones obtained with the other two methods.

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RECURRENCE RELATIONS

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain $\Omega_0 \subseteq \Omega$. The super-Halley method to solve the equation $F(x) = 0$ is defined as follows:

$$x_{n+1} = x_n - \left[I + \frac{1}{2}L_F(x_n)[I - L_F(x_n)]^{-1}\right] F'(x_n)^{-1}F(x_n), \tag{1}$$

where, for $x \in X$, $L_F(x)$ is the linear operator defined as follows:

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x).$$

Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega_0$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from Y into X.

Throughout this paper we assume the following.

(i) $\|F''(x)\| \le k_1$, $x \in \Omega_0$.
(ii) $\|F''(x) - F''(y)\| \le k_2\|x - y\|$, $x, y \in \Omega_0$.
(iii) $\|\Gamma_0\| \le B$.
(iv) $\|\Gamma_0 F(x_0)\| \le \eta$.

Let us denote

$$a = k_1 B\eta, \qquad b = k_2 B\eta^2. \tag{2}$$

Then, we define the sequences

$$\begin{aligned}
a_0 = b_0 = 1; \qquad c_0 &= a; \qquad d_0 = \frac{2-a}{2(1-a)};\\
a_{n+1} &= \frac{a_n}{1 - a a_n d_n};\\
b_{n+1} &= a_{n+1} d_n^3 \left[\frac{a^2 a_n(1-c_n)c_n}{(2-c_n)^3} + \frac{b}{6}\right];\\
c_{n+1} &= a a_{n+1} b_{n+1};\\
d_{n+1} &= \frac{2 - c_{n+1}}{2(1-c_{n+1})} b_{n+1}.
\end{aligned}$$

In that situation we prove the following.

(I_n) $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \le a_n B$.
(II_n) $\|\Gamma_n F(x_n)\| \le b_n\eta$.
(III_n) $\|L_F(x_n)\| \le c_n$.
(IV_n) $\|x_{n+1} - x_n\| \le d_n\eta$.

Notice that (I_0), (II_0), and (III_0) follow immediately from the hypothesis. If

$$\|L_F(x_0)\| \le c_0 = a < 1,$$

then $[I - L_F(x_0)]^{-1}$ exists and

$$\begin{aligned}
\|x_1 - x_0\| &\le \left[I + \frac{1}{2}\|L_F(x_0)\|\|(I - L_F(x_0))^{-1}\|\right] \|\Gamma_0 F(x_0)\|\\
&\le \left[1 + \frac{a}{2(1-a)}\right]\eta = d_0\eta,
\end{aligned}$$

and (IV_0) also holds.

Following an inductive procedure and assuming

$$x_n \in \Omega_0 \quad \text{and} \quad aa_nd_n < 1,$$

if $x_{n+1} \in \Omega_0$, we have

$$\|I - \Gamma_n F'(x_{n+1})\| \le \|\Gamma_n\| \|F'(x_n) - F'(x_{n+1})\| \le aa_nd_n < 1.$$

Then Γ_{n+1} is defined and

$$\|\Gamma_{n+1}\| \le \frac{\|\Gamma_n\|}{1 - \|\Gamma_n\|\|F'(x_n) - F'(x_{n+1})\|} \le \frac{a_nB}{1 - aa_nd_n} = a_{n+1}B.$$

On the other hand, we deduce from (1) and the Taylor's formula that (see [11] for more details)

$$F(x_{n+1}) = \frac{1}{8}F''(x_n)[L_F(x_n)[I - L_F(x_n)]^{-1}\Gamma_n F(x_n)]^2 + \int_{x_n}^{x_{n+1}} [F''(x) - F''(x_n)](x_{n+1} - x)\,dx, \quad (3)$$

and then

$$\begin{aligned}\|\Gamma_{n+1}F(x_{n+1})\| &\le \|\Gamma_{n+1}\|\|F(x_{n+1})\| \le a_{n+1}B\left[\frac{k_1b_n^2c_n^2\eta^2}{8(1-c_n)^2} + \frac{k_2d_n^3\eta^3}{6}\right] \\ &= \eta\left[\frac{ab_n^2c_n^2}{8(1-c_n)^2} + \frac{bd_n^3}{6}\right]a_{n+1}.\end{aligned}$$

As $b_n = 2d_n(1-c_n)/(2-c_n)$, we obtain

$$\|\Gamma_{n+1}F(x_{n+1})\| \le \eta a_{n+1}d_n^3\left[\frac{b}{6} + \frac{a^2a_nc_n(1-c_n)}{(-c_n)^32}\right] = b_{n+1}\eta.$$

Finally, it is easy to deduce that

$$\|L_F(x_{n+1})\| \le \|\Gamma_{n+1}\|\|F''(x_{n+1})\|\|\Gamma_{n+1}F(x_{n+1})\| \le aa_{n+1}b_{n+1} = c_{n+1}$$

and, as in the case $n = 0$, if it is assumed $c_{n+1} < 1$, we get

$$\|x_{n+2} - x_{n+1}\| \le d_{n+1}\eta.$$

So, to study the sequence $\{x_n\}$ defined in a Banach space, we must analyse the real sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$. To establish the convergence of $\{x_n\}$, we only have to prove that $\{d_n\}$ is a Cauchy sequence and the assumptions

$$\begin{aligned} c_n &< 1, \qquad n \in \mathbb{N}, \\ x_n &\in \Omega_0, \qquad n \in \mathbb{N}, \\ aa_nd_n &< 1, \qquad n \in \mathbb{N}. \end{aligned}$$

That is the aim of the following section.

CONVERGENCE STUDY

In this section, we are going to study the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ defined in the previous one to prove the convergence of $\{x_n\}$ defined in (1). First at all, we give a technical lemma including the results concerning one and two variable functions that we next need. We omit the proof expecting the reader could get it patiently but without any difficulty.

LEMMA 1. *Let us define the functions*

$$Q(x) = \frac{6(1-x)(1-2x)(x^2-6x+4)}{(2-x)^2}, \qquad f(x) = \frac{x^4}{2(1-4x+x^2)^2}$$
$$g(x,y) = f(x)\left[1 + \frac{b(2-x)^3}{6a^2xy(1-x)}\right], \qquad g_0(x) = g(x,1).$$

Then $Q(x)$ is a decreasing function in $[0,0.5]$; $f(x)$ increases in $[0,0.5]$, $f(0)=0$, $f(0.5)=0.5$ and $f(x) < x$ for $x \in (0,0.5)$; for a fixed $x \in (0,0.5)$, $g(x,y)$ decreases as a function of y; $g_0(x)$, and $g_0'(x)$ are increasing for $x \in [0,0.5)$.

Now we start with an easy lemma that gives us a recurrence relation for the sequence $\{c_n\}$. The proof follows from the definition of the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$.

LEMMA 2. *For the sequence $\{c_n\}$, the following recurrence is true:*

$$c_{n+1} = \frac{c_n^4}{2(2-4c_n+c_n^2)^3}\left[1 + \frac{b(2-c_n)^2}{6a^2a_nc_n(1-c_n)}\right].$$

LEMMA 3. *Let $0 < a < 0.5$ and $0 \le b \le Q(a)$. Then the sequence $\{c_n\}$ is decreasing. Moreover $c_n < 1$ for $n \in \mathbb{N}$.*

PROOF. From the hypothesis we deduce that $c_1 \le c_0$. Besides, $c_1 < c_0$ if $b < Q(a)$ and $c_1 = c_0$ if $b = Q(a)$. But in both cases, $c_2 < c_1$ and, in general, $c_{n+1} < c_n$ for $n \ge 1$. From the previous lemma, $c_2 < c_1$ if and only if

$$\frac{b(2-c_1)^2}{6a^2a_1(1-c_1)} < \frac{(1-2c_1)(c_1^2-6c_1+4)}{c_1^2}. \tag{4}$$

Then, (4) is equivalent to

$$\frac{b}{a^2a_1} < \frac{Q(c_1)}{c_1^2}. \tag{5}$$

As $a_1 > 1$, $c_1 \le c_0$ and Q decreases in $[0,0.5]$ (see Lemma 1), we get

$$\frac{b}{a^2a_1} < \frac{b}{a^2} \le \frac{Q(a)}{a^2} \le \frac{Q(c_1)}{c_1^2}.$$

Therefore, (5) holds and $c_2 < c_1$. In a similar way, it can be established that $c_{n+1} < c_n$ for $n \ge 1$.

LEMMA 4. *Under the hypothesis of the Lemma 3, we have $aa_nd_n < 1$ for $n \ge 0$ and $\{a_n\}$ is an increasing sequence.*

PROOF. First notice that

$$aa_nd_n = \frac{c_n(2-c_n)}{2(1-c_n)}$$

and then $aa_nd_n < 1$ because of $0 \le c_n < 0.5$.

On the other hand, $a_0 = 1$, $a_1 = a_0/(1-aa_0d_0) > a_0 = 1$ and, inductively, $a_{n+1} = a_n/(1 - aa_nd_n) > a_n > a_{n-1} > \cdots > a_1 > a_0 = 1$.

LEMMA 5. *With the previous notations, let $0 < a < 0.5$ and $0 \le b \le Q(a)$. Let us define $\gamma = c_2/c_1$, then*

$$c_{n+1} \le (\gamma^{1/2})^{3^n}\frac{c_1}{\gamma^{1/2}}.$$

Consequently, as $\gamma < 1$, $\lim_{n\to\infty} c_n = 0$. Furthermore, $\sum_{n=0}^{\infty} c_n < \infty$.

PROOF. First, notice that $c_0 = a < 0.5$ implies $a_1 > 1$. Then, by using the Lemma 1 and with the same notation, we obtain

$$c_2 = g(c_1, a_1) < g(c_1, 1) = g_0(c_1) \le g_0(c_0) \le g(c_0, 1) = c_1.$$

So we have $c_2 = \gamma c_1$, where $\gamma < 1$. Besides, if $c_k \le \gamma c_{k-1}$, then $c_{k-1} \le \gamma^3 c_k$. Indeed,

$$\begin{aligned} c_{k+1} &= \frac{c_k^4}{2\left(2-4c_k+c_k^2\right)^2}\left[1+\frac{b(2-c_k)^2}{6a^2a_kc_k(1-c_k)}\right] \\ &\le \frac{\gamma^3 c_{k-1}^3}{2\left(2-4\gamma c_{k-1}+\gamma^2 c_{k-1}^2\right)}\left[\frac{\gamma c_{k-1}}{2-4\gamma c_{k-1}+\gamma^2 c_{k-1}^2}+\frac{b(2-\gamma c_k)^2}{6a^2a_k(1-\gamma c_{k-1})\left(2-4\gamma c_{k-1}+\gamma^2 c_{k-1}^2\right)}\right] \\ &\le \frac{\gamma^3 c_{k-1}^3}{2\left(2-4c_{k-1}+c_{k-1}^2\right)}\left[\frac{c_{k-1}}{2-4c_{k-1}+c_{k-1}^2}+\frac{b(2-c_k)^2}{6a^2a_k(1-c_{k-1})\left(2-4c_{k-1}+c_{k-1}^2\right)}\right]. \end{aligned}$$

Since $a_k \ge a_{k-1}$, we get

$$c_{k+1} \le \gamma^3 g(c_{k-1}, a_{k-1}) = \gamma^3 c_k.$$

By using the above inequality, we obtain $c_{n+1} \le \gamma^{3^{n-1}} c_n$, and recursively,

$$c_{n+1} \le (\gamma^{1/2})^{3^n} c_1 \gamma^{-1/2}.$$

Therefore $c_n \to 0$.

On the other hand, let us define $g_0(x) = g(x, a_0) = g(x, 1)$. From Lemma 1 we know that $g_0(x)$ increases in $[0, 0.5)$, $g_0'(x) \ge 0$ in $[0, 0.5)$, and $g_0'(0) = 0$. As g_0' is continuous in $[0, 0.5)$ and $c_n \to 0$, there exists $n_0 \in \mathbb{N}$ and $\alpha \in [0, 1)$ such that

$$g_0'(c_n) \le \alpha < 1, \qquad \forall n \ge n_0.$$

Then by using the Mean Value Theorem and Lemma 1 again,

$$\begin{aligned} c_{n_0+k+1} &= g(c_{n_0+k}, a_{n_0+k}) \le g(c_{n_0+k}, a_0) = g_0(c_{n_0+k}) \\ &= g_0(c_{n_0+k}) - g_0(0) \le g_0'(c_{n_0+k})c_{n_0+k} \le \alpha c_{n_0+k}, \end{aligned}$$

and recurrently $c_{n_0+k} \le \alpha^k c_{n_0}$. Therefore,

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{n_0-1} c_n + \sum_{n=n_0}^{\infty} c_n \le \sum_{n=0}^{n_0-1} c_n + c_{n_0} \sum_{n=n_0}^{\infty} \alpha^{n-n_0} < \infty.$$

LEMMA 6. *The sequence $\{a_n\}$ is upper bounded, that is, there exists a constant $M > 0$ such that $a_n \le M$ for $n \ge 0$.*

PROOF. By the definition of the sequences, we get

$$a_{n+1} = a_n \frac{2(1-c_n)}{2-4c_n+c_n^2}.$$

Taking into account this equality and with the notations of the Lemma 1, we write

$$a_n = \prod_{k=0}^{n}\left[1+\frac{c_k(2-c_k)}{2-4c_k+c_k^2}\right],$$

and consequently, as $0 \le c_k \le 0.5$, $\forall k \in \mathbb{N}$,

$$\prod_{k=0}^{n}[1+c_k] \le a_n \le \prod_{k=0}^{n}[1+6c_k].$$

Therefore

$$\ln a_n \le \ln \prod_{k=0}^{n} [1+6c_k] = \sum_{k=0}^{n} \ln [1+6c_k] \le 6 \sum_{k=0}^{n} c_k < \infty.$$

LEMMA 7. *We have* $d_n \le 3c_1(\gamma^{1/2})^{3^{n-1}}/2a\gamma^{1/2}$. *Consequently,* $\sum_{n=0}^{\infty} d_n < \infty$ *and* $\{d_n\}$ *is a Cauchy sequence.*

PROOF. Observe that

$$d_n = \frac{c_n(2-c_n)}{2aa_n(1-c_n)}.$$

Then, by using the previous lemmas and taking into account $0 \le c_n \le 0.5$ and $a_n \ge 1$, $\forall n \in \mathbb{N}$, we get

$$\sum_{n=0}^{\infty} d_n \le \frac{3}{2a} \sum_{n=0}^{\infty} c_n < \infty,$$

and the proof is completed.

So we are ready to state the following result on the convergence of the iterative method defined in (1).

THEOREM 1. *Let* X, Y *be Banach spaces and* $F : \Omega \subseteq X \to Y$ *be a nonlinear twice Fréchet differentiable operator in an open convex domain* $\Omega_0 \subseteq \Omega$. *Let us assume that* $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ *exists at some* $x_0 \in \Omega_0$ *and*

(i) $\|F''(x)\| \le k_1$, $x \in \Omega_0$;
(ii) $\|F''(x) - F''(y)\| \le k_2\|x-y\|$, $x, y \in \Omega_0$;
(iii) $\|\Gamma_0\| \le B$;
(iv) $\|\Gamma_0 F(x_0)\| \le \eta$.

Let us denote $a = k_1 B\eta$, $b = k_2 B\eta^2$. *Suppose that* $0 < a \le 0.5$ *and* $0 \le b \le Q(a)$, *where* $Q(x)$ *is the function defined in the Lemma 1. Then, if* $\overline{B(x_0, r\eta)} = \{x \in X; \|x - x_0\| \le r\eta\} \subseteq \Omega_0$, *where* $r = \sum_{n=0}^{\infty} d_n$, *the sequence* $\{x_n\}$ *defined in (1) and starting at* x_0 *converges at least R-cubically to a solution* x^* *of the equation* $F(x) = 0$. *In that case, the solution* x^* *and the iterates* x_n *belong to* $\overline{B(x_0, r\eta)}$, *and* x^* *is the only solution of* $F(x) = 0$ *in* $B(x_0, (2/k_1 B) - r\eta) \cap \Omega_0$.

Furthermore, we can give the following error estimates in terms of the real sequence $\{d_n\}$ *(or* $\{c_n\}$*):*

$$\|x^* - x_{n+1}\| \le \sum_{k=n+1}^{\infty} d_k \eta \le \frac{d_0}{\gamma} \sum_{k=n}^{\infty} \gamma^{2^k}, \qquad \gamma = \frac{c_2}{c_1}. \tag{6}$$

PROOF. When $0 < a < 0.5$, the convergence of the sequences $\{x_n\}$ follows immediately from the previous lemmas. When $a = 0.5$, then $b = 0$ and we have the following sequences: $a_n = 4^n$, $b_n = 1/4^n$, $c_n = 1/2$ and $d_n = 6/4^{n+1}$. Now it is easy to prove that the conditions $aa_n d_n < 1$, $c_n < 1$, and $\{d_n\}$ is a Cauchy sequence also hold and consequently the sequence $\{x_n\}$ is convergent.

On the other hand, if x^* is the limit of the sequences $\{x_n\}$, then taking norms in (3), we have

$$\|F(x_{n+1})\| \le \frac{k_1 b_n^2 c_n^2 \eta^2}{8(1-c_n)} + \frac{k_2 d_n^3 \eta^3}{6}.$$

The limit of the expression appearing on the right side is cero so, by the continuity of F, we prove that $F(x^*) = 0$.

Besides we have $\|x_{n+1} - x_n\| \le d_n \eta$, and therefore, for $p \ge 0$,

$$\|x_p - x_0\| \le \|x_p - x_{p-1}\| + \cdots + \|x_1 - x_0\| \le (d_{p-1} + \cdots + d_0)\eta.$$

Then $x_p \in \Omega_0$ and by letting $p \to \infty$, we obtain the region where the solution is located, $\|x^* - x_0\| \le r\eta$, and the error estimates given by (6).

To show the unicity, suppose that $y^* \in B(x_0, (2/k_1B) - r\eta)$ is another solution of $F(x) = 0$. Then

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\, dt (y^* - x^*).$$

Using the estimate

$$\begin{aligned}\|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|\, dt &\le k_1 B \int_0^1 \|x^* + t(y^* - x^*) - x_0\|\, dt \\ &\le k_1 B \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)\, dt < \frac{k_1 B}{2}\left(r\eta + \frac{2}{k_1 B} - r\eta\right) = 1,\end{aligned}$$

we have that the operator $\int_0^1 F'(x^* + t(y^* - x^*))\, dt$ has an inverse. Consequently $y^* = x^*$.

Finally, from Lemma 7, we deduce that the R-order of convergence [14] of the sequence (1) is, at least, three when $0 < a < 0.5$:

$$\|x^* - x_n\| \le \eta \sum_{k=n}^{\infty} d_k \le \frac{3c_2\eta}{2a(1 - \gamma^{1/2})} \left[\gamma^{1/2}\right]^{3^n}, \qquad \gamma < 1.$$

When $a = 0.5$, we have $\|x^* - x_n\| \le 2\eta/4^n$ and the convergence is linear.

REMARK. In the previous theorem we have proved that the sequence (1) converges cubically, but for a kind of functions, the order is four. Let us consider the situation $b = 0$ (for instance, when F is a quadratic operator) and $0 < a < 0.5$. Then, in Lemma 2 we have

$$c_{n+1} = \frac{c_n^4}{2\,(2 - c_n + c_n^2)^2}.$$

Consequently, $c_{n+1} \le 8c_n^4$, $c_n \le (2a)^{4^n}/2$ and $d_{n+1} \le 3(2a)^{4^n}/4a$. Then it is easy to deduce that the R-order of convergence is, at least, four.

EXAMPLES

Finally we give three examples to illustrate the previous results. We take three functions used as a test in several papers. In these examples we compare the error bounds obtained for different third-order iterative processes.

EXAMPLE 1. [1,4,15] Let us consider $F(x) = x^3 - 10$, $x_0 = 2$ and denote x^* the positive root of $F(x) = 0$. We will give an upper bound M for $10^{11}\|x^* - x_2\|$, where x_2 is the second iterate of the super-Halley method. Starting from the interval $[1, 3]$, we have $B = 1/12$, $\eta = 1/6$, $k_1 = 18$, and $k_2 = 6$. So $a = 1/4$ and $b = 1/72$. Then

$$\|x^* - x_2\| \le \eta \sum_{k=2}^{\infty} d_k = \eta \left[d_2 + \sum_{k=3}^{\infty} d_k\right].$$

Thus, if we estimate the sum of the series by means of the complete geometric series, we get $M = 383.384$. For the same function, Candela and Marquina [1,4] have obtained that $M = 21\,561.183$ for the Halley method and $M = 142\,360.973$ for the Chebyshev method.

Starting from the interval $[1.73, 2.27]$, the value of M is 36.3862, whereas for the Halley method, Candela and Marquina obtained $M = 1\,615.941$, and Döring [15] obtained that M is approximately 60 000 for the Halley method.

EXAMPLE 2. [1,4,15] Let us consider $F : C[0,1] \to C[0,1]$ the operator defined by

$$F(x)(s) = x(s) - s + \frac{1}{2} \int_0^1 s \cos(x(t))\, dt,$$

where $C[0,1]$ is the space of all continuous functions defined on the interval $[0,1]$ with the sup norm $\|\cdot\| = \|\cdot\|_\infty$.

If we take $x_0 = x_0(s) = s$ as a starting point, we obtain the upper bound for $M = 4\,631\,410$ for $10^{11}\|x^* - x_2\|$, where x_2 is the second iterate of the super-Halley method. Candela and Marquina have obtained that $M = 14\,987\,029$ for the Halley method and $M = 137\,022\,683$ for the Chebyshev method. For the Halley method, Döring obtained that $M = 82\,500\,000$.

EXAMPLE 3. [16] Let $X = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$, with the max-norm and consider the integral equation $F(x) = 0$, where

$$F(x)(s) = \lambda x(s) \int_0^1 \frac{s}{s+t} x(t)\,dt - x(s) + 1,$$

with $s \in [0,1]$, $x \in C[0,1]$ and $0 < \lambda \leq 2$. Integral equations of this kind (called Chandrasekhar equations) arise in elasticity or neutron transport problems [10,16]. Notice that the above operator is quadratic. Then the results got from the super-Halley method really improve the results obtained by using other third-order methods.

For $\lambda = 1/4$, and starting at $x_0 = x_0(s) = 1$, we obtain [16], $\|\Gamma_0\| = 1.53039421 = B$. With the same notation, we have

$$\|\Gamma_0 F(x_0)\| \leq 0.2651971 = \eta, \qquad \|F''(x)\| \leq 0.3465735 = k_1, \qquad k_2 = 0.$$

So $a = k_1 B\eta = 0.140659$ and $b = 0$. To compare Chebyshev, Halley, and super-Halley methods, we give an upper bound M to the number $10^{16}\|x^* - x_2\|$, where x_2 is the second iterate obtained by using these methods. We have that M is $63\,858\,314$ for the Chebyshev method, $5\,596\,218$ for the Halley method, and 31 for the super-Halley method.

As we can see, the error bound that we have for super-Halley method is much better than the ones obtained for the other two methods.

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