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Recurrence relations for a Newton-like method in Banach spaces☆

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Abstract

The convergence of iterative methods for solving nonlinear operator equations in Banach spaces is established from the convergence of majorizing sequences. An alternative approach is developed to establish this convergence by using recurrence relations. For example, the recurrence relations are used in establishing the convergence of Newton's method [L.B. Rall, Computational Solution of Nonlinear Operator Equations, Robert E. Krieger, New York, 1979] and the third order methods such as Halley's, Chebyshev's and super Halley's [V. Candela, A. Marquina, Recurrence relations for rational cubic methods I: the Halley method, Computing 44 (1990) 169–184; V. Candela, A. Marquina, Recurrence relations for rational cubic methods II: the Halley method, Computing 45 (1990) 355–367; J.A. Ezquerro, M.A. Hernández, Recurrence relations for Chebyshev-type methods, Appl. Math. Optim. 41 (2000) 227–236; J.M. Gutiérrez, M.A. Hernández, Third-order iterative methods for operators with bounded second derivative, J. Comput. Appl. Math. 82 (1997) 171–183; J.M. Gutiérrez, M.A. Hernández, Recurrence relations for the Super–Halley method, Comput. Math. Appl. 7(36) (1998) 1–8; M.A. Hernández, Chebyshev's approximation algorithms and applications, Comput. Math. Appl. 41 (2001) 433–445 [10]].

In this paper, an attempt is made to use recurrence relations to establish the convergence of a third order Newton-like method used for solving a nonlinear operator equation $F(x)=0$, where $F:\Omega\subseteq\mathbb{X}\to\mathbb{Y}$ be a nonlinear operator on an open convex subset Ω of a Banach space $\mathbb{X}$ with values in a Banach space $\mathbb{Y}$. Here, first we derive the recurrence relations based on two constants which depend on the operator F. Then, based on this recurrence relations a priori error bounds are obtained for the said iterative method. Finally, some numerical examples are worked out for demonstrating our approach.

MSC: 47H10; 41A25; 65Q05

Keywords: Nonlinear operator equations; Newton-like method; Cubic convergence; Recurrence relations; A priori error bounds

1. Introduction

The well-known Newton's method and it's variants are used to solve nonlinear operator equations $F(x)=0$. These methods are of second order and their convergence established by Kantorovich theorem [11,13], provide sufficient conditions to ensure convergence through a system of error bounds for the distance to the solution from each iterate. The convergence of the sequences obtained by these methods in Banach spaces are derived from the convergence of majorizing sequences. In [14], a new approach is used for the convergence of these methods by recurrence relations to

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get a priori error bounds for them. This paper is concerned with convergence of third order methods for $F(x)=0$ in Banach spaces. For many applications, third order methods are used inspite of high computational cost in evaluating the involved second order derivatives. They can also be used in stiff systems [12], where a quick convergence is required. Moreover, they are important from the theoretical standpoint as they provide results on existence and uniqueness of solutions that improve the results obtained from Newton's method [1,7]. Some of the well-known third order methods are Chebyshev's method, Halley's method and super Halley's method. Candela and Marquina [2,3] proposed recurrence relations to study the convergence of Halley's method and Chebyshev's method. Also Gutuérrez and Hernández [8,9] and Ezquerro and Hernández [4] used recurrence relations to study the convergence of Super Halley and Chebyshev-type methods.

In this paper, we shall use recurrence relations to establish the convergence of a third order Newton-like method for solving a nonlinear operator equation $F(x)=0$, where $F:\Omega\subseteq\mathbb{X}\to\mathbb{Y}$ be a nonlinear operator on an open convex subset Ω of a Banach space $\mathbb{X}$ with values in a Banach space $\mathbb{Y}$. The recurrence relations based on two constants which depend on the operator F are derived. Then, based on this recurrence relations a priori error bounds are obtained for the said iterative method. Finally, some numerical examples are worked out for demonstrating our work.

The paper is organized as follows. Section 1 is the introduction. In Section 2, recurrence relations for a Newton-like third order method are derived. The convergence analysis based on recurrence relations of the method derived in Section 2 is given in Section 3. In Section 4, some numerical examples are worked out. Finally, conclusions form the Section 5.

2. Recurrence relations for a third order method

In this section, we shall discuss a third order Newton-like method for solving the nonlinear operator equation

$$F(x)=0, \tag{1}$$

where $F:\Omega\subseteq\mathbb{X}\to\mathbb{Y}$ be a nonlinear operator on a open convex subset Ω of a Banach space $\mathbb{X}$ with values in a Banach space $\mathbb{Y}$. Recently, Frontini and Sormani [6,5] developed a family of third order iterative methods for solving (1). This family involves only the vue of F and it's first derivative F'. One of the most well-known member of this family is

$$\left.\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ x_{n+1} &= x_n - \left(\frac{F'(x_n)+F'(y_n)}{2}\right)^{-1}F(x_n). \end{aligned}\right\} \tag{2}$$

Let F be a twice Fréchet differentiable operator in Ω and $BL(\mathbb{Y},\mathbb{X})$ be the set of bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$. It is assumed that $\Gamma_0=F'(x_0)^{-1}\in BL(\mathbb{Y},\mathbb{X})$ exists at some point $x_0\in\Omega$ and let the following conditions hold on F:

$$\left.\begin{aligned} &1.\ \|F'(x_0)^{-1}\|\leqslant\beta,\\ &2.\ \|F'(x_0)^{-1}F(x_0)\|\leqslant\eta,\\ &3.\ \|F''(x)\|\leqslant M,\ \ x\in\Omega,\\ &4.\ \|F''(x)-F''(y)\|\leqslant N\|x-y\|\quad\forall x,y\in\Omega. \end{aligned}\right\} \tag{3}$$

Now taking $a=M\beta\eta$ and $b=N\beta\eta^2$, $a_0=1$, $b_0=1$, $c_0=a/2$, $d_0=2/(2-a)$, we define for $n=0,1,\ldots$

$$\left.\begin{aligned} a_{n+1} &= \frac{a_n}{1-aa_nd_n},\\ b_{n+1} &= a_{n+1}d_n^2\left[\frac{a}{2}(2c_n^2-7c_n+6)+\frac{5b}{12}(1-c_n)^3d_n\right],\\ c_{n+1} &= \frac{aa_{n+1}b_{n+1}}{2},\\ d_{n+1} &= \frac{b_{n+1}}{1-c_{n+1}}. \end{aligned}\right\} \tag{4}$$

In these conditions, for $n \geqslant 0$, we prove the following inequalities:

$$\left.\begin{aligned}
&\text{(I) } \|\Gamma_n\| = \|F'(x_n)^{-1}\| \leqslant a_n\beta,\\
&\text{(II) } \|\Gamma_n F(x_n)\| = \|F'(x_n)^{-1}F(x_n)\| \leqslant b_n\eta,\\
&\text{(III) } \|L_F(x_n)\| = \left\| I - \Gamma_n \frac{F'(y_n)+F'(x_n)}{2} \right\| \leqslant c_n,\\
&\text{(IV) } \|x_{n+1}-x_n\| \leqslant d_n\eta,\\
&\text{(V) } \|x_{n+1}-y_n\| \leqslant (b_n+d_n)\eta.
\end{aligned}\right\} \tag{5}$$

The proof of the above inequalities will require the following Lemma 1.

Lemma 1. *Let $F : \Omega \subseteq \mathbb{X} \to \mathbb{Y}$ be a twice Fréchet differentiable nonlinear operator in an open convex domain Ω of a Banach space $\mathbb{X}$ with values in a Banach space $\mathbb{Y}$. Let the sequences $\{x_n\}$ and $\{y_n\}$ are generated by* (2). *Then $\forall n \in \mathbb{N}$, we have*

$$\begin{aligned}
F(x_{n+1}) = {}& \int_0^1 F''(y_n + t(x_{n+1}-y_n))(1-t)\,\mathrm{d}t\,(x_{n+1}-y_n)^2\\
&+ \frac{1}{2}\int_0^1 F''(x_n + t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)(x_{n+1}-y_n)\\
&- \frac{1}{2}\int_0^1 F''(x_n + t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)^2\\
&+ \int_0^1 F''(x_n + t(y_n-x_n))(1-t)\,\mathrm{d}t\,(y_n-x_n)^2.
\end{aligned} \tag{6}$$

Proof. Using (2), we get

$$\begin{aligned}
F'(y_n)(x_{n+1}-y_n) &= \frac{1}{2}(F'(y_n)-F'(x_n))(x_{n+1}-y_n) + \frac{1}{2}(F'(y_n)+F'(x_n))(x_{n+1}-y_n)\\
&= \frac{1}{2}\int_0^1 F''(x_n + t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)(x_{n+1}-y_n)\\
&\quad + \frac{F'(y_n)+F'(x_n)}{2}\left[F'(x_n)^{-1}F(x_n) - \left(\frac{F'(y_n)+F'(x_n)}{2}\right)^{-1} F(x_n)\right]\\
&= \frac{1}{2}\int_0^1 F''(x_n + t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)(x_{n+1}-y_n)\\
&\quad + \frac{F'(y_n)+F'(x_n)}{2} F'(x_n)^{-1}F(x_n) - F(x_n)\\
&= \frac{1}{2}\int_0^1 F''(x_n + t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)(x_{n+1}-y_n)\\
&\quad + \left[\frac{F'(y_n)+F'(x_n)}{2} - F'(x_n)\right] F'(x_n)^{-1}F(x_n)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 F''(x_n+t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)(x_{n+1}-y_n)\\
&\quad - \frac{F'(y_n)-F'(x_n)}{2}(y_n-x_n)\\
&= \frac{1}{2}\int_0^1 F''(x_n+t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)(x_{n+1}-y_n)\\
&\quad - \frac{1}{2}\int_0^1 F''(x_n+t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)^2.
\end{aligned}
$$

Again reusing (2), we get

$$
\begin{aligned}
F(y_n) &= F(y_n)-F(x_n)-F'(x_n)(y_n-x_n)\\
&= \int_0^1 F''(x_n+t(y_n-x_n))(1-t)\,\mathrm{d}t\,(y_n-x_n)^2.
\end{aligned}
$$

This yields

$$
\begin{aligned}
F(x_{n+1}) &= F(x_{n+1})-F(y_n)-F'(y_n)(x_{n+1}-y_n)+F'(y_n)(x_{n+1}-y_n)+F(y_n)\\
&= \int_0^1 F''(y_n+t(x_{n+1}-y_n))(1-t)\,\mathrm{d}t\,(x_{n+1}-y_n)^2\\
&\quad + \frac{1}{2}\int_0^1 F''(x_n+t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)(x_{n+1}-y_n)\\
&\quad - \frac{1}{2}\int_0^1 F''(x_n+t(y_n-x_n))\,\mathrm{d}t\,(y_n-x_n)^2\\
&\quad + \int_0^1 F''(x_n+t(y_n-x_n))(1-t)\,\mathrm{d}t\,(y_n-x_n)^2. \qquad \square
\end{aligned}
$$

Now the conditions (I)–(V) can be proved by induction. For $n=0$, (I) and (II) follow from the assumptions. Here the existence of $\Gamma_0 = F'(x_0)^{-1}$ implies the existence of y_0. This gives us

$$
\begin{aligned}
\left\| I - F'(x_0)^{-1}\,\frac{F'(y_0)+F'(x_0)}{2}\right\| &= \left\| F'(x_0)^{-1}\,\frac{F'(x_0)-F'(y_0)}{2}\right\|\\
&\leqslant \frac{1}{2} M \|F'(x_0)^{-1}\|\,\|x_0-y_0\|\\
&\leqslant \frac{M\beta\eta}{2} = \frac{a}{2} = c_0 < 1.
\end{aligned} \tag{7}
$$

Hence by Banach's theorem [11],

$$
\left(\frac{F'(y_0)+F'(x_0)}{2}\right)^{-1} F'(x_0)
$$

exists and

$$
\left\| \left(\frac{F'(y_0)+F'(x_0)}{2}\right)^{-1} F'(x_0)\right\| \leqslant \frac{1}{1-a/2} = \frac{2}{2-a}.
$$

Now we have

$$\begin{aligned}\|x_1 - x_0\| &= \left\| \left(\frac{F'(y_0) + F'(x_0)}{2} \right)^{-1} F(x_0) \right\| \\ &\leqslant \left\| \left(\frac{F'(y_0) + F'(x_0)}{2} \right)^{-1} F'(x_0) \right\| \|F'(x_0)^{-1} F(x_0)\| \\ &\leqslant \frac{2\eta}{2-a} = d_0 \eta \end{aligned} \tag{8}$$

and

$$\begin{aligned}\|x_1 - y_0\| &\leqslant \|x_1 - x_0\| + \|x_0 - y_0\| \\ &\leqslant \frac{2\eta}{2-a} + \eta = (b_0 + d_0)\eta. \end{aligned} \tag{9}$$

Thus, for $n = 0$, the existence of the conditions (III)–(V) follow from (7)–(9), respectively. Assume that the conditions (I)–(V) hold for $n = 1, 2, \ldots, k$ and consider $x_k \in \Omega$, $c_{k+1} < 1$ and $aa_k d_k < 1$. We now have

$$\begin{aligned}\|I - \Gamma_k F'(x_{k+1)}\| &= \|\Gamma_k (F'(x_k) - F'(x_{k+1}))\| \\ &\leqslant M \|\Gamma_k\| \|x_k - x_{k+1}\| \\ &\leqslant M a_k \beta d_k \eta \\ &\leqslant aa_k d_k < 1. \end{aligned}$$

Hence by Banach's theorem [11],

$$\Gamma_{k+1} = F'(x_{k+1})^{-1}$$

exists and

$$\begin{aligned}\|\Gamma_{k+1}\| &\leqslant \frac{\|\Gamma_k\|}{1 - \|\Gamma_k\| \|F'(x_k) - F'(x_{k+1})\|} \\ &\leqslant \frac{a_k \beta}{1 - aa_k d_k} = a_{k+1}\beta. \end{aligned} \tag{10}$$

Also

$$\begin{aligned}&\left\| \int_0^1 F''(x_k + t(y_k - x_k))(1-t)\,\mathrm{d}t\,(y_k - x_k)^2 - \frac{1}{2} \int_0^1 F''(x_k + t(y_k - x_k))\,\mathrm{d}t\,(y_k - x_k)^2 \right\| \\ &\quad \leqslant \left\| \int_0^1 F''(x_k + t(y_k - x_k))(1-t)\,\mathrm{d}t\,(y_k - x_k)^2 - \frac{1}{2} F''(x_k)(y_k - x_k)^2 \right\| \\ &\qquad + \left\| \frac{1}{2} \int_0^1 F''(x_k + t(y_k - x_k))\,\mathrm{d}t\,(y_k - x_k)^2 - \frac{1}{2} F''(x_k)(y_k - x_k)^2 \right\| \end{aligned}$$

$$\leqslant \left\| \int_0^1 [F''(x_k + t(y_k - x_k)) - F''(x_k)](1-t)\,\mathrm{d}t \right\| \|y_k - x_k\|^2$$
$$+ \left\| \frac{1}{2}\int_0^1 [F''(x_k + t(y_k - x_k)) - F''(x_k)]\,\mathrm{d}t \right\| \|y_k - x_k\|^2$$
$$\leqslant N \int_0^1 (t - t^2)\,\mathrm{d}t \|y_k - x_k\|^3 + \frac{N}{2}\int_0^1 t\,\mathrm{d}t \|y_k - x_k\|^3$$
$$= \frac{5N}{12}\|y_k - x_k\|^3.$$

Hence,

$$\begin{aligned}\|F(x_{k+1})\| &\leqslant \frac{M}{2}\|x_{k+1} - y_k\|^2 + \frac{M}{2}\|y_k - x_k\|\|x_{k+1} - y_k\| + \frac{5N}{12}\|y_k - x_k\|^3 \\ &\leqslant \frac{M}{2}(b_k + d_k)^2\eta^2 + \frac{M}{2}b_k(b_k + d_k)\eta^2 + \frac{5N}{12}b_k^3\eta^3. \end{aligned} \tag{11}$$

From this, we get

$$\begin{aligned}\|\Gamma_{k+1}F(x_{k+1})\| &\leqslant \|\Gamma_{k+1}\|\|F(x_{k+1})\| \\ &\leqslant a_{k+1}\beta\left(\frac{M}{2}(b_k + d_k)^2\eta^2 + \frac{M}{2}\,b_k(b_k + d_k)\eta^2 + \frac{5N}{12}\,b_k^3\eta^3\right) \\ &= a_{k+1}\left(\frac{M\beta\eta}{2}(b_k + d_k)^2\eta + \frac{M\beta\eta}{2}\,b_k(b_k + d_k)\eta + \frac{5N\beta\eta^2}{12}\,b_k^3\eta\right) \\ &= a_{k+1}\left(\frac{a}{2}(b_k + d_k)^2\eta + \frac{a}{2}\,b_k(b_k + d_k)\eta + \frac{5b}{12}\,b_k^3\eta\right) \\ &= a_{k+1}\left(\frac{a}{2}(2b_k^2 + d_k^2 + 3b_kd_k) + \frac{5b}{12}\,b_k^3\right)\eta \\ &= a_{k+1}\left(\frac{a}{2}(2(1-c_k)^2d_k^2 + d_k^2 + 3(1-c_k)d_k^2) + \frac{5b}{12}(1-c_k)^3d_k^3\right)\eta \\ &= a_{k+1}d_k^2\left(\frac{a}{2}(2c_k^2 - 7c_k + 6) + \frac{5b}{12}(1-c_k)^3d_k\right)\eta \\ &= b_{k+1}\eta. \end{aligned} \tag{12}$$

As $\Gamma_{k+1} = F'(x_{k+1})^{-1}$ exists, so y_{k+1} exists.

Hence,

$$\begin{aligned}\left\| I - F'(x_{k+1})^{-1}\,\frac{F'(y_{k+1}) + F'(x_{k+1})}{2} \right\| &= \left\| F'(x_{k+1})^{-1}\,\frac{F'(x_{k+1}) - F'(y_{k+1})}{2} \right\| \\ &\leqslant \frac{1}{2}\,M\|F'(x_{k+1})^{-1}\|\|x_{k+1} - y_{k+1}\| \\ &\leqslant \frac{1}{2}\,a_{k+1}\beta M b_{k+1}\eta = \frac{aa_{k+1}b_{k+1}}{2} = c_{k+1} < 1. \end{aligned} \tag{13}$$

Thus, by Banach's theorem [11],

$$\left(\frac{F'(y_{k+1}) + F'(x_{k+1})}{2}\right)^{-1} F'(x_{k+1})$$

exists and

$$\left\| \left(\frac{F'(y_{k+1}) + F'(x_{k+1})}{2} \right)^{-1} F'(x_{k+1}) \right\| \leqslant \frac{1}{1 - c_{k+1}}.$$

This implies

$$\begin{aligned} \|x_{k+2} - x_{k+1}\| &= \left\| \left(\frac{F'(y_{k+1}) + F'(x_{k+1})}{2} \right)^{-1} F(x_{k+1}) \right\| \\ &\leqslant \left\| \left(\frac{F'(y_{k+1}) + F'(x_{k+1})}{2} \right)^{-1} F'(x_{k+1}) \right\| \|F'(x_{k+1})^{-1} F(x_{k+1})\| \\ &\leqslant \frac{b_{k+1}\eta}{1 - c_{k+1}} \\ &= d_{k+1}\eta. \end{aligned} \tag{14}$$

Also,

$$\begin{aligned} \|x_{k+2} - y_{k+1}\| &\leqslant \|x_{k+2} - x_{k+1}\| + \|x_{k+1} - y_{k+1}\| \\ &\leqslant d_{k+1}\eta + b_{k+1}\eta = (d_{k+1} + b_{k+1})\eta. \end{aligned} \tag{15}$$

From Eqs. (10), (12)–(15), we conclude that the conditions (I)–(V) hold for $k+1$, respectively. Hence, by induction it holds for all n.

3. Convergence analysis

In this section, we shall establish the convergence of our third order Newton-like method discussed in Section 2. This can be done by establishing the convergence of $\{x_n\}$. In order to prove the convergence of $\{x_n\}$, from (IV) of (5) it is sufficient to prove that the sequence $\{d_n\}$ is a Cauchy sequence and the following assumptions hold:

$$c_n < 1, \quad n \in \mathbb{N},$$

$$x_n \in \Omega, \quad n \in \mathbb{N},$$

$$a a_n d_n < 1, \quad n \in \mathbb{N}.$$

For this purpose, we will use the following lemmas.

Lemma 2. *Let* $r_0 = 0.0952980448\ldots$ *be the smallest positive root of* $p(x) = 0$, *where* $p(x) = 1 - 12x + 16x^2 - 2x^3$. *Define the functions*

$$\begin{aligned} \Phi(x) &= \frac{3}{5}\,\frac{(1 - 12x + 16x^2 - 2x^3)}{(1 - x)^2}, \\ g(x, y) &= \frac{1}{(1 - 3x)^2}\left(2x^2 - 7x + 6 + \frac{5b}{3a^2 y}(1 - x)^2 x\right), \\ h(x, y) &= \frac{x^2}{(1 - 3x)^2}\left(2x^2 - 7x + 6 + \frac{5b}{3a^2 y}(1 - x)^2 x\right), \\ h_0(x) &= h(x, 1), \end{aligned} \tag{16}$$

then

(i) $\Phi(x)$ *is a decreasing function in* $[0, r_0]$,

(ii) *$g(x, y)$ and $h(x, y)$ are increasing functions of x in $[0, r_0]$, for $y > 1$,*
(iii) *$g(x, y)$ and $h(x, y)$ are decreasing functions of y,*
(iv) *$h_0(x)$ and $h_0'(x)$ are increasing functions in $[0, r_0]$.*

Proof. Since

$$\Phi'(x) = \frac{3}{5}\left(\frac{-12+32x-6x^2}{(1-x)^2} + \frac{2(1-12x+16x^2-2x^3)}{(1-x)^3}\right) \leqslant 0, \quad \forall x \in [0, r_0]$$

this implies $\Phi(x)$ is a decreasing function. The proof of others follow from similar reasonings as given for $\Phi(x)$. □

Lemma 3. *The following recurrence relations hold for the sequences $\{a_n\}$ and $\{c_n\}$.*

$$a_{n+1} = \prod_{k=0}^{n}\left(1 + \frac{2c_k}{1-3c_k}\right), \tag{17}$$

$$c_{n+1} = \frac{c_n^2(2c_n^2-7c_n+6)}{(1-3c_n)^2}\left(1 + \frac{5b}{3a^2a_n}\frac{(1-c_n)^2c_n}{(2c_n^2-7c_n+6)}\right). \tag{18}$$

Proof. As

$$c_n = \frac{aa_nb_n}{2} \Rightarrow b_n = \frac{2c_n}{aa_n}.$$

We get

$$d_n = \frac{b_n}{1-c_n} = \frac{2c_n}{aa_n(1-c_n)}. \tag{19}$$

As $a_0 = 1$, this gives

$$a_{n+1} = \frac{a_n}{1-aa_nd_n} = \frac{a_n}{1-\frac{2c_n}{1-c_n}} = \frac{a_n(1-c_n)}{1-3c_n} = a_n\left(1+\frac{2c_n}{1-3c_n}\right) = \prod_{k=0}^{n}\left(1+\frac{2c_k}{1-3c_k}\right). \tag{20}$$

Also

$$\begin{aligned}
c_{n+1} &= \frac{aa_{n+1}b_{n+1}}{2} \\
&= \frac{a}{2}a_{n+1}^2d_n^2\left(\frac{a}{2}(2c_n^2-7c_n+6) + \frac{5b}{12}(1-c_n)^3d_n\right) \\
&= \frac{a}{2}\left(\frac{a_n(1-c_n)}{1-3c_n}\right)^2\left(\frac{2c_n}{aa_n(1-c_n)}\right)^2\left(\frac{a}{2}(2c_n^2-7c_n+6) + \frac{5b}{12}(1-c_n)^3\left(\frac{2c_n}{aa_n(1-c_n)}\right)\right) \\
&= \frac{2}{a}\frac{c_n^2}{(1-3c_n)^2}\left(\frac{a}{2}(2c_n^2-7c_n+6) + \frac{5b}{6}(1-c_n)^2\frac{c_n}{aa_n}\right) \\
&= \frac{c_n^2(2c_n^2-7c_n+6)}{(1-3c_n)^2}\left(1+\frac{5b}{3a^2a_n}\frac{(1-c_n)^2c_n}{(2c_n^2-7c_n+6)}\right).
\end{aligned}$$

Hence the lemma is proved. □

Lemma 4. *Let* $0 < a < 2r_0$ *and* $0 \leqslant b \leqslant 4\Phi(\frac{a}{2})$. *Then* $\{a_n\}$ *and* $\{c_n\}$ *are increasing and decreasing sequences, respectively. We also have* $c_n < 1$, $a_n > 1$, *and* $aa_n d_n < 1$ $\forall n \in \mathbb{N}$.

Proof. The proof follows by induction.

From

$$c_{n+1} = \frac{c_n^2(2c_n^2 - 7c_n + 6)}{(1-3c_n)^2}\left(1 + \frac{5b}{3a^2a_n}\frac{(1-c_n)^2c_n}{(2c_n^2 - 7c_n + 6)}\right),$$

we get $c_{n+1} \leqslant c_n$ if

$$\frac{c_n(2c_n^2 - 7c_n + 6)}{(1-3c_n)^2}\left(1 + \frac{5b}{3a^2a_n}\frac{(1-c_n)^2c_n}{(2c_n^2 - 7c_n + 6)}\right) \leqslant 1$$

or, if

$$\left(\frac{5b}{3a^2a_n}\frac{(1-c_n)^2c_n}{(2c_n^2 - 7c_n + 6)}\right) \leqslant \frac{(1-3c_n)^2}{c_n(2c_n^2 - 7c_n + 6)} - 1$$

$$= \frac{1 - 12c_n + 16c_n^2 - 2c_n^3}{c_n(2c_n^2 - 7c_n + 6)}$$

or, if

$$\frac{b}{a^2a_n} \leqslant \frac{3}{5}\frac{(1 - 12c_n + 16c_n^2 - 2c_n^3)}{(1-c_n)^2c_n^2}$$

$$= \frac{\Phi(c_n)}{c_n^2}. \tag{21}$$

From our assumption

$$\frac{b}{a^2a_0} = \frac{b}{a^2} \leqslant \frac{4\Phi(\frac{a}{2})}{a^2} = \frac{4\Phi(c_0)}{4c_0^2} = \frac{\Phi(c_0)}{c_0^2}.$$

Hence $c_1 \leqslant c_0$. We also get $0 \leqslant c_1 \leqslant c_0 = a/2 < 0.5 \Rightarrow 1/(1-(a/2)) < 2$ and

$$aa_nd_n = aa_n\frac{b_n}{1-c_n} = \frac{2c_n}{1-c_n}.$$

Hence

$$aa_0d_0 = \frac{2c_0}{1-c_0} = \frac{a}{1-\frac{a}{2}} < 1.$$

Therefore, $a_1 = a_0/(1 - aa_0d_0) > a_0 = 1$. Also as $c_1 < 0.5$, so $aa_1d_1 < 1$, this implies $a_2 > a_1 > 1$.

Let the lemma holds for $n = 1, 2, \ldots k$.

Now $c_k \leqslant c_{k-1} \leqslant a/2 < 0.5$. This gives us $aa_kd_k < 1$ and hence

$$a_{k+1} = \frac{a_k}{1 - aa_kd_k} > a_k > \cdots > a_0 = 1.$$

Also

$$\frac{b}{a^2a_k} < \frac{b}{a^2} \leqslant \frac{4\Phi(\frac{a}{2})}{a^2} \leqslant \frac{4\Phi(c_k)}{4c_k^2} = \frac{\Phi(c_k)}{c_k^2}.$$

So $c_{k+1} \leqslant c_k \leqslant a/2 < 1$. Hence the lemma is proved for all $n \in \mathbb{N}$. $\square$

Lemma 5. *Under the assumptions of Lemma* 4, *let us define* $\gamma = c_2/c_1$, *then*

$$c_{n+1} \leqslant \gamma^{2^{n+1}} c_0 \quad \forall n \in \mathbb{N}. \tag{22}$$

Hence the sequence $\{c_n\}$ *converges to* 0. *Also* $\sum_{n=0}^{\infty} c_n < \infty$.

Proof. From Lemmas 2–4, we have

$$c_2 = h(c_1, a_1) < h(c_1, 1) \leqslant h(c_0, 1) = h(c_0, a_0) = c_1.$$

So we have $c_2 = \gamma c_1$, where $\gamma < 1$. Suppose $c_k \leqslant \gamma c_{k-1}$. Since $g(x, y)$ increases as a function of x and decreases as function of y, so we get

$$\begin{aligned} c_{k+1} &= \frac{c_k^2(2c_k^2 - 7c_k + 6)}{(1 - 3c_k)^2} \left(1 + \frac{5b}{3a^2 a_k} \frac{(1 - c_k)^2 c_k}{(2c_k^2 - 7c_k + 6)}\right) \\ &= c_k^2 g(c_k, a_k) \\ &\leqslant \gamma^2 c_{k-1}^2 g(c_k, a_k) \\ &\leqslant \gamma^2 c_{k-1}^2 g(c_{k-1}, a_k), \\ &\leqslant \gamma^2 c_{k-1}^2 g(c_{k-1}, a_{k-1}), \\ &= \gamma^2 c_k = \gamma^{2^{k+1}} c_0. \end{aligned}$$

Hence $c_{n+1} \leqslant \gamma^{2^{n+1}} c_0$ $\forall n \in \mathbb{N}$. This gives $c_n \to 0$, as $\gamma < 1$.

Again $h_0'(x) \geqslant 0$ as $h_0(x)$ increases in $[0, r_0]$. Also as $h_0'(x)$ is continuous in $[0, r_0]$ and $c_n \to 0$, there exists a positive integer n_0 and $\alpha \in [0, 1)$, such that

$$h_0'(c_n) \leqslant \alpha < 1 \quad \forall n \geqslant n_0.$$

By using the Mean value theorem, we get

$$\begin{aligned} c_{n_0+k+1} &= h(c_{n_0+k}, a_{n_0+k}) \leqslant h(c_{n_0+k}, a_0) = h_0(c_{n_0+k}) \\ &= h_0(c_{n_0+k}) - h_0(0) \leqslant h_0'(c_{n_0+k}) c_{n_0+k} \leqslant \alpha c_{n_0+k}. \end{aligned}$$

Using recurrence relation $c_{n_0+k} \leqslant \alpha^k c_{n_0}$. This gives,

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{n_0-1} c_n + \sum_{n=n_0}^{\infty} c_n \leqslant \sum_{n=0}^{n_0-1} c_n + c_{n_0} \sum_{n=n_0}^{\infty} \alpha^{n-n_0} < \infty.$$

Hence lemma is proved. □

Lemma 6. *The sequence* $\{a_n\}$ *is bounded above*, *that is there exist a constant* $K_1 > 0$ *such that* $a_n \leqslant K_1$, $\forall n \in \mathbb{N}$.

Proof. From Eq. (20), we have

$$a_{n+1} = \prod_{k=0}^{n} \left(1 + \frac{2c_k}{1 - 3c_k}\right).$$

For $0 \leqslant c_k < 0.25$ we get

$$\prod_{k=0}^{n} (1 + 2c_k) \leqslant a_{n+1} < \prod_{k=0}^{n} (1 + 8c_k). \tag{23}$$

Taking logarithm of (23), we get

$$\ln \prod_{k=0}^{n}(1+2c_k) \leqslant \ln(a_{n+1}) < \ln \prod_{k=0}^{n}(1+8c_k)$$

This reduces to

$$\sum_{k=0}^{n} \ln(1+2c_k) \leqslant \ln(a_{n+1}) < \sum_{k=0}^{n} \ln(1+8c_k) \leqslant 8 \sum_{k=0}^{n} c_k < \infty. \qquad \square$$

Lemma 7. *The sequence* $\{d_n\}$ *is a Cauchy sequence, as it satisfies the condition* $d_n < (8/3a)\gamma^{2^n} c_0$. *Also* $\sum_{n=0}^{\infty} d_n < \infty$.

Proof. From Eq. (19), we have

$$d_n = \frac{b_n}{1-c_n} = \frac{2c_n}{aa_n(1-c_n)}.$$

Again as $a_n > 1$, and $0 < c_n \leqslant r_0 < 0.25$, we get

$$d_n < \frac{4}{3}\frac{2c_n}{aa_n} = \frac{8c_n}{3aa_n} < \frac{8c_n}{3a}.$$

Therefore,

$$\sum_{n=0}^{\infty} d_n < \frac{8}{3a} \sum_{n=0}^{\infty} c_n < \infty.$$

So the lemma is proved. $\square$

The theorem given below will establish the convergence of the sequence $\{x_n\}$ and give a priori error bounds for it. Let us denote $r = \sum_{n=0}^{\infty} d_n$ and $B(x_0, r\eta) = \{x \in \mathbb{X} : \|x - x_0\| < r\eta\}$ and $\bar{B}(x_0, r\eta) = \{x \in \mathbb{X} : \|x - x_0\| \leqslant r\eta\}$.

Theorem 1. *Let* $F : \Omega \subseteq \mathbb{X} \to \mathbb{Y}$ *be a nonlinear twice Fréchet differentiable operator in an open convex subset* Ω *of a Banach space* $\mathbb{X}$ *with values in a Banach space* $\mathbb{Y}$ *and* $BL(\mathbb{Y}, \mathbb{X})$ *is the set of bounded linear operators from* $\mathbb{Y}$ *into* $\mathbb{X}$. *For* $\Gamma_0 = F'(x_0)^{-1} \in BL(\mathbb{Y}, \mathbb{X})$ *defined at some point* $x_0 \in \Omega$, *assume that the conditions* (3), $0 < a \leqslant 2r_0$ *and* $0 \leqslant b \leqslant 4\Phi(a/2)$ *hold, where* $\Phi(x)$ *is the function defined by Eq.* (16). *If* $\bar{B}(x_0, r\eta) \subseteq \Omega$, *then starting from* x_0, *the sequence* $\{x_n\}$ *defined by method* (2) *converges to a solution* x^* *of the equation* $F(x) = 0$ *with* x_n, y_n *and* x^* *belonging to* $\bar{B}(x_0, r\eta)$ *and* x^* *is the unique solution of* (1) *in* $B(x_0, 2/(M\beta) - r\eta) \cap \Omega$.

Furthermore, the error bounds on x^* *depend on the sequence* $\{d_n\}$ *given by*

$$\|x^* - x_{n+1}\| \leqslant \sum_{k=n+1}^{\infty} d_k \eta < r\eta. \tag{24}$$

Proof. Let $0 < a < 2r_0$. Now using Lemma 7, it is easy to show that $\{d_n\}$ is Cauchy sequence. This makes the sequence $\{x_n\}$ a Cauchy sequence.

Also when $a = 2r_0$, $b = \Phi(a/2) = \Phi(r_0) = 0$. This implies $c_n = c_0 = a/2$, for $n \geqslant 0$.

Now

$$a_{n+1} = a_n\left(1 + \frac{2c_n}{1-3c_n}\right) = a_n\left(1 + \frac{2c_0}{1-3c_0}\right) = \omega a_n = \omega^{n+1}, \quad \text{where } \omega = 1 + \frac{2c_0}{1-3c_0} > 1.$$

Again,

$$d_n = \frac{2c_n}{aa_n(1-c_n)} = \frac{2c_0}{aa_n(1-c_0)} = \frac{1}{\omega^n(1-c_0)} = \frac{d_0}{\omega^n},$$

hence $\{d_n\}$ is a cauchy sequence.

Also, $aa_nd_n = ad_0 = 2a/(2-a) < 1$. Hence all the conditions on the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ hold true. Thus, in both cases $\{d_n\}$ is a Cauchy sequence, and hence the sequence $\{x_n\}$ is convergent.

Hence, there exists a x^* such that, $\lim_{n\to\infty} x_n = x^*$.

Now from Eq. (11), we get

$$\begin{aligned}\|F(x_{n+1})\| &\leqslant \frac{M}{2}(b_n+d_n)^2\eta^2 + \frac{M}{2}\,b_n(b_n+d_n)\eta^2 + \frac{5N}{12}\,b_n^3\eta^3\\ &= \frac{1}{\beta}\,d_n^2\left(\frac{a}{2}(2c_n^2-7c_n+6) + \frac{5b}{12}(1-c_n)^3 d_n\right)\eta.\end{aligned}$$

Since the sequence $\{d_n\}$ converges to 0, $\{c_n\}$ is bounded and F is continuous, we get

$$\|F(x^*)\| = \lim_{n\to\infty} \leqslant \lim_{n\to\infty}\left[\frac{1}{\beta}\,d_n^2\left(\frac{a}{2}(2c_n^2-7c_n+6)+\frac{5b}{12}(1-c_n)^3 d_n\right)\eta\right] = 0.$$

Also

$$\begin{aligned}\|x_{n+1}-x_0\| &\leqslant \|x_{n+1}-x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\|\\ &\leqslant \sum_{k=0}^{n} d_k\eta \leqslant r\eta,\end{aligned}$$

From this, we conclude that x_n lies in $\bar{B}(x_0, r\eta)$ and similarly one can easily prove that y_n lies in $\bar{B}(x_0, r\eta)$. Now taking limit as $n\to\infty$ we get $x^*\in \bar{B}(x_0, r\eta)$.

Again, for every $m \geqslant n+1$,

$$\begin{aligned}\|x_m - x_{n+1}\| &\leqslant \|x_m - x_{m-1}\| + \|x_{m-1}-x_{m-2}\| + \cdots + \|x_{n+2}-x_{n+1}\|\\ &\leqslant \sum_{k=n+1}^{m-1} d_k\eta\\ &\leqslant \sum_{k=n+1}^{\infty} d_k\eta < r\eta,\end{aligned}$$

By taking $m\to\infty$ we get

$$\|x^* - x_{n+1}\| \leqslant \sum_{k=n+1}^{\infty} d_k\eta < r\eta,$$

Now for the uniqueness of x^*, let y^* is another zero of Eq. (1). Then

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^*-x^*))\,\mathrm{d}t\,(y^*-x^*).$$

To show $y^* = x^*$, we have to show that $\int_0^1 F'(x^*+t(y^*-x^*))\,\mathrm{d}t$ is invertible. Now for

$$\begin{aligned}\|\Gamma_0\|\left\|\int_0^1 [F'(x^*+t(y^*-x^*)) - F'(x_0)]\,\mathrm{d}t\right\| &\leqslant M\beta\int_0^1 \|x^*+t(y^*-x^*)-x_0\|\,\mathrm{d}t\\ &\leqslant M\beta\int_0^1 (1-t)\|x^*-x_0\| + t\|y^*-x_0\|\,\mathrm{d}t\\ &< \frac{M\beta}{2}\left(r\eta + \frac{2}{M\beta} - r\eta\right) = 1,\end{aligned}$$

it follows from Banach's theorem [11], the operator $\int_0^1 F'(x^* + t(y^* - x^*))\,\mathrm{d}t$ is invertible and hence $y^* = x^*$. Hence the theorem is proved. □

4. Numerical examples

In this section, numerical examples are given for demonstrating the convergence of our third order Newton-like method based on recurrence relations.

Example 1. Let $\mathbb{X} = C[0, 1]$ be the space of continuous functions on [0, 1] and consider the integral equation $F(x) = 0$, where

$$F(x)(s) = -1 + x(s) + \lambda x(s) \int_0^1 \frac{s}{s+t} x(t)\,\mathrm{d}t, \tag{25}$$

with $s \in [0, 1]$, $x \in C[0, 1]$ and $0 < \lambda \leqslant 2$. Integral equations of this kind called Chandrasekhar equations arise in elasticity or neutron transport problems [12]. The norm is taken as sup-norm.

Now it is easy to find the first derivative of F as

$$F'(x)u(s) = u(s) + \lambda u(s) \int_0^1 \frac{s}{s+t} x(t)\,\mathrm{d}t + \lambda x(s) \int_0^1 \frac{s}{s+t} u(t)\,\mathrm{d}t, \quad u \in \Omega$$

and the second derivative as

$$F''(x)(uv)(s) = \lambda u(s) \int_0^1 \frac{s}{s+t} v(t)\,\mathrm{d}t + \lambda v(s) \int_0^1 \frac{s}{s+t} u(t)\,\mathrm{d}t, \quad v \in \Omega.$$

The second derivative F'' satisfies the Lipschitz condition as,

$$\|[F''(x) - F''(y)](uv)\| = 0\|x - y\| \quad \forall x, y \in \Omega.$$

Now one can easily compute

$$\begin{aligned}
\|F(x_0)\| &= \left\| -1 + x_0(s) + \lambda x_0(s) \int_0^1 \frac{s}{s+t} x_0(t)\,\mathrm{d}t \right\| \\
&\leqslant \|x_0 - 1\| + |\lambda| \max_{0 \leqslant s \leqslant 1} \left| \int_0^1 \frac{s}{s+t}\,\mathrm{d}t \right| \|x_0\|^2 \\
&\leqslant \|x_0 - 1\| + |\lambda| \log 2 \|x_0\|^2
\end{aligned}$$

and

$$\|F''(x)\| \leqslant 2|\lambda| \max_{0 \leqslant s \leqslant 1} \left| \int_0^1 \frac{s}{s+t}\,\mathrm{d}t \right| = 2|\lambda| \log 2.$$

Also notice that

$$\begin{aligned}
\|I - F'(x_0)\| &= \left\| \lambda \int_0^1 \frac{s}{s+t} x_0(t)\,\mathrm{d}t + \lambda x_0(s) \int_0^1 \frac{s}{s+t}\,\mathrm{d}t \right\| \\
&\leqslant 2|\lambda| \max_{0 \leqslant s \leqslant 1} \left| \int_0^1 \frac{s}{s+t}\,\mathrm{d}t \right| \|x_0\| \\
&\leqslant 2|\lambda| \log 2 \|x_0\|
\end{aligned} \tag{26}$$

Table 1

n	a_n	b_n	c_n	d_n	Σd_n	$(\Sigma d_n) * \eta$
0	1	1	7.829572e − 003	1.007891	1.007891	8.929102e − 02
1	1.016036	4.804513e − 002	3.822051e − 004	4.806350e − 002	1.055955	9.354906e − 02
2	1.016813	1.102980e − 004	8.781060e − 007	1.102981e − 004	1.056065	9.355883e − 02
3	1.016815	5.811215e − 010	4.626441e − 012	5.811215e − 010	1.056065	9.355883e − 02
4	1.016815	1.613115e − 020	1.284237e − 022	1.613115e − 020	1.056065	9.355883e − 02
5	1.016815	1.242973e − 041	9.895592e − 044	1.242973e − 041	1.056065	9.355883e − 02
6	1.016815	7.379973e − 084	5.875364e − 086	7.379973e − 084	1.056065	9.355883e − 02
7	1.016815	2.601602e − 168	2.071194e − 170	2.601602e − 168	1.056065	9.355883e − 02
8	1.016815	0	0	0	1.056065	9.355883e − 02

and, if $2|\lambda| \log 2\|x_0\| < 1$, then by Banach's theorem [11], we obtain

$$\|\Gamma_0\| = \|F'(x_0)^{-1}\| \leqslant \frac{1}{1 - 2|\lambda| \log 2\|x_0\|}.$$

Hence

$$\|\Gamma_0 F(x_0)\| \leqslant \frac{\|x_0 - 1\| + |\lambda| \log 2\|x_0\|^2}{1 - 2|\lambda| \log 2\|x_0\|}$$

Now for $\lambda = 1/4$, and the initial point $x_0 = x_0(s) = 1$, we obtain
$\|\Gamma_0\| \leqslant \beta = 1.17718382$, $\|\Gamma_0 F(x_0)\| \leqslant \eta = 0.08859191$, $\|F''(x)\| \leqslant M = 0.150514997$, $N = 0$.

Hence $a = M\beta\eta = 0.015697052$ and $b = N\beta\eta^2 = 0$.

As $a \leqslant 2r_0 = 0.19059609$ and $0 = b \leqslant 4\Phi(a/2) = 2.210893861$, so the hypotheses of Theorem 1 is satisfied. Hence the recurrence relations for our method is given in Table 1.

From Table 1, we have $r = \Sigma d_n = 1.056065$. Hence a solution of Eq. (25) exists in $\bar{B}(1, 0.09355883) \subseteq \Omega$ and this solution is unique in $B(1, 11.22148) \cap \Omega$.

But in [15], $K = M[1+5N/3M^2\beta] = M = 0.150514997$. Hence $h = K\beta\eta = 0.015697052$. So $t^* = ((1-\sqrt{1-2h})/h)\eta = 0.089298359$ and $t^{**} = ((1+\sqrt{1-2h})/h)\eta = 11.19841477$. Hence by the convergence method given in [15], the solution of Eq. (25) exists in $\bar{B}(1, 0.089298359) \subseteq \Omega$, and the unique solution exists in the ball $B(1, 11.19841477) \cap \Omega$ both of which are inferior to our result.

Example 2. Consider the root of the equation

$$F(x) = x^3 - 2x - 5 = 0 \tag{27}$$

on [1, 3]. Now for the initial point $x_0 = 2$, it is very easy to get

$$\beta = \|F'(x_0)^{-1}\| = 0.1, \quad \eta = \|F'(x_0)^{-1} F(x_0)\| = 0.1, \quad M = 18, \quad N = 6.$$

Therefore, $a = M\beta\eta = 0.18 < 2r_0 = 0.19059609$ and $b = N\beta\eta^2 = 0.006 \leqslant 4\Phi(a/2) = 0.139525178$. Hence the hypotheses of the Theorem 1 holds true. Hence the recurrence relations for our method is given in Table 2.

Hence from Table 2 we have $r = \Sigma d_n = 2.702070$. So a solution of Eq. (25) exists in $\bar{B}(1, 0.2702070) \subseteq \Omega$ and this solution is unique in $B(1, 0.8409041) \cap \Omega$.

But by [15], $K = M[1+5N/3M^2\beta] = 23.55555556$. Hence $h = K\beta\eta = 0.235555556$. So $t^* = ((1-\sqrt{1-2h})/h)\eta = 0.115791166$ and $t^{**} = ((1+\sqrt{1-2h})/h)\eta = 0.733265437$. Hence by the convergence method given in [15], the solution of Eq. (27) exists in $\bar{B}(1, 0.115791166) \subseteq \Omega$, and the unique solution exists in the ball $B(1, 0.733265437) \cap \Omega$ both of which are inferior to our result.

Table 2

n	a_n	b_n	c_n	d_n	Σd_n	$(\Sigma d_n) * \eta$
0	1	1	9.000000e − 002	1.098901	1.098901	1.098901e − 01
1	1.246575	7.328440e − 001	8.221907e − 002	7.984955e − 001	1.897397	1.897397e − 01
2	1.518675	4.753985e − 001	6.497783e − 002	5.084355e − 001	2.405832	2.405832e − 01
3	1.763823	2.283736e − 001	3.625295e − 002	2.369642e − 001	2.642796	2.642796e − 01
4	1.907317	5.546990e − 002	9.521882e − 003	5.600315e − 002	2.698800	2.698800e − 01
5	1.944708	3.257958e − 003	5.702198e − 004	3.259817e − 003	2.702059	2.702059e − 01
6	1.946929	1.116472e − 005	1.956323e − 006	1.116474e − 005	2.702070	2.702070e − 01
7	1.946937	1.310515e − 010	2.296342e − 011	1.310515e − 010	2.702070	2.702070e − 01
8	1.946937	1.805635e − 020	3.163911e − 021	1.805635e − 020	2.702070	2.702070e − 01
9	1.946937	3.427721e − 040	6.006201e − 041	3.427721e − 040	2.702070	2.702070e − 01
10	1.946937	1.235255e − 079	2.164467e − 080	1.235255e − 079	2.702070	2.702070e − 01
11	1.946937	1.604201e − 158	2.810951e − 159	1.604201e − 158	2.702070	2.702070e − 01
12	1.946937	2.705599e − 316	4.740867e − 317	2.705599e − 316	2.702070	2.702070e − 01

5. Conclusions

In this paper, recurrence relations are developed for establishing the convergence of a third order Newton-like method for solving $F(x) = 0$ in Banach spaces. Based on this recurrence relations an existence-uniqueness theorem and a priori error bounds are established for this method. This approach is simple and efficient in comparison with the usual approach used for this purpose. Numerical examples are worked out to demonstrate our approach.

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