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On a Conjecture on the Sperner Property

XIAOYA ZHA

1. INTRODUCTION

Let P be a finite partially ordered set with rank function $r(x)$, and P_k be all the elements in P whose ranks are k. A set of elements of P is called an antichain if no two elements of the set are ordered. P has the Sperner property if $\max_k |P_k| = \max_A \{|A|, A$ is an antichain in $P\}$. Let $G = \{a_1, a_2, \ldots, a_m\} \subset P$, then $F = \{x, x \in P, \exists a_i, x \geqslant a_i\}$ is called an ordered filter generated by G. We denote it as $F = \langle a_1, a_2, \ldots, a_m \rangle$. Let P be a Boolean algebra, i.e. P is the family of all the subsets of $\{1, 2, \ldots, n\}$ whose partial order relation is inclusion, and its rank function is $r(x) = |x|$, where $|x|$ is the cardinality of x.

Ko-wei Lih [5] proved that if $|a_i| = t = 1$, $i = 1, 2, \ldots, m$, then F has the Sperner property; he also conjectured that it is true for all $t > 1$.

Yingxian Zhu has proved in [6] that; if n is odd, $t = 2$, and if n is even, $t = 2, 3$, $\bigcap_{i=1}^{m} a_i \neq \varnothing$, then F has the Sperner property. Furthermore, he has given counter-examples in [6] which show that Ko-wei Lih's conjecture is not true for $t > n/2$. In Section 2 of this paper we will show that such a conjecture does not hold for $t \geqslant 4$, in Section 3, the conjecture will be confirmed in some special cases for $t = 2, 3$.

2. COUNTER-EXAMPLES FOR $t \geqslant 4$

THEOREM 1. *Let $t \geqslant 4$ and $n \geqslant 2t - 1$. Then there is a filter F in B_n, generated by a collection of elements of fixed rank t, which has no Sperner property.*

PROOF. Let $Q(n) = \{x \cup \{n\}, x \supset \{1, 2, \ldots, n-1\}\}$, then

$$B_n = B_{n-1} \cup Q(n), \tag{2.1}$$

and $Q(n)$ is isomorphic with B_{n-1}.

Find p so that the following are satisfied:

$$\binom{n-t-1}{p+1-t} > \binom{n-t-1}{p-t}, \qquad \binom{n-1}{p-1} > \binom{n-1}{p}. \tag{2.2, 2.3}$$

The numbers of left- and right-hand parts of (2.2) are the numbers of all $(p+1)$- and p-subsets of B_{n-1} which contain a fixed t-subset, and those of (2.3) are the number of all p- and $(p+1)$-subsets of $Q(n)$. Denote the corresponding collections as A_1, A_2 and B_1, B_2, respectively. Then

$$|A_1 \cup B_1| > \max\{|A_1 \cup B_2|, |A_2 \cup B_1|\},$$

and $A_1 \cup B_1$ is an antichain.

(i) For $n = 2s + 1$, let $p = s + 1$; (2.3) holds, and (2.2) becomes

$$\binom{2s-t}{s+2-t} > \binom{2s-t}{s+1-t},$$

which is implied by $t \geqslant 4$ and $2(s + 2 - t) \leqslant 2s - t$.

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Suppose $t=4$ for a moment and let F be the ordered filter generated by any 4-subset of $\{1,2,\ldots,n-1\}$ and all the 4-subsets of $Q(n)$; then $|F|$ is unimodal, i.e.

$$|F_4|<|F_5|<\cdots<|F_{s+1}|>|F_{s+2}|>\cdots>|F_n|,$$

$F_{s+1}=A_2\cup B_1$, $F_{s+2}=A_1\cup B_2$, $|A_1\cup B_1|>|F_{s+1}|$, and therefore F has no Sperner property.

If $4<t<\lfloor n/2\rfloor+1=s+1$ then the filter generated by F_t of F is also a counter-example for the Sperner property.

(ii) For $n=2s$, let $p=s+1$; (2.2) and (2.3) are satisfied for $t\geqslant 5$ in the same way. Similarly, for $t=5$, let F be the ordered filter generated by any 5-subsets of $\{1,2,\ldots,n-1\}$ and all the 5-subsets of $Q(n)$; then F has no Sperner property.

The case $5<t<\lfloor n/2\rfloor+1=s+1$ is settled as in the previous example.

(iii) For $n=2s$, $t=4$, let the generating elements elements be $\{1,2,3,4\}$ and all the 4-subsets of $\{1,2,\ldots,2s\}$ which contain '$2s-1$' or '$2s$'; thus

$$|F_i|=\binom{2s-6}{i-4}+\binom{2s-2}{i-1}+\binom{2s-1}{i-1},$$

$$|F_4|<|F_5|<\cdots<|F_s|>|F_{s+1}|>\cdots>|F_n|.$$

Let $A=\{x,\ |x|=s+1,\ \ \{1,2,3,4\}\subset x\subset\{1,2,\ldots,2s-2\}\}$, $B=\{x,\ |x|=s,\ \ x\cap\{2s-1,2s\}\neq\varnothing,\ x\subset\{1,2,\ldots,2s\}\}$. $A\cup B$ is an antichain, $A\cap B=\varnothing$, and

$$|A|=\binom{2s-6}{s-3},\qquad |B|=\binom{2s-1}{s-1}+\binom{2s-2}{s-2},$$

$|A\cup B|>|F_s|$; F has no Sperner property.

3. Cases for $t=2,3$

The cases for $n=2,3$ and $n=2s+1$, $t=3$ will be discussed in this section. We cannot give counter-examples to deny Lih's conjecture: on the contrary, some special cases will be considered which is just support Lih's conjecture. In fact, we think that the conjecture is true in these three cases.

The well known Hall's matching theorem concerning bipartite graphs will be used in our proof.

Theorem (Hall [4]). *Let G be a bipartite graph with bipartition (X,Y). Then G contains a matching from X to Y if*

$$|A|\leqslant|R(A)|\quad \textit{for all } A\subset X,$$

where $R(A)$ is the set of vertices in Y joined with at least one vertex in A.

Let F be the ordered filter generated by $\{a_1,a_2,\ldots,a_m\}$, $a_i\in B_n$, $|a_i|=t$. Let $H\subset F$, then $H_i=H\cap F_i$. Write

$$H_i^-=\{x\in F,\ |x|=i-1,\ \exists y\in H_i,\ y\supset x\},$$

$$H_i^+=\{x\in F,\ |x|=i+1,\ \exists y\in H,\ x\supset y\}.$$

Any element in H_i contains at least $i-t$ elements in H_i^-. However, any element in H_i^- is contained in at most $n-i+1$ elements of F_i; thus

$$(i-t)\,|H_i|\leqslant(n-i+1)\,|H_i^-|. \tag{3.1}$$

When $n=2s$, $t=2$, $i\geqslant s+2$, we have $i-t\geqslant n-i+1$, $|H_i^-|>|H_i|$. By Hall's theorem, there is a matching from F_i to F_{i-1}; hence

$$|F_{s+1}| > |F_{s+2}| > \cdots > |F_n|,$$

and F_j can be matched into F_{j-1} for $j \geqslant s+2$.

Similarly,

$$(n-i)\,|H_i| \leqslant (i+1)\,|H_i^+|, \tag{3.2}$$

there is a matching from F_i to F_{i+1} for $2 \leqslant i < s-1$, $n = 2s$, and

$$|F_2| < |F_3| <, \ldots, < |F_s|.$$

For $t = 3$, it can be treated analogously, and we have:

THEOREM 2. *Let F be the ordered filter generated by $\{a_1, a_2, \ldots, a_m\}$, $a_i \in B_n$, $|a_i| = t$, and let A be any antichain in F. Then there is an antichain A', $|A'| \geqslant |A|$, such that*:

(i) $A' \subset F_s \cup F_{s+1}$ *for* $t = 2, 3$, $n = 2s$;

(ii) $A' \subset F_{s+1} \cup F_{s+2}$ *for* $t = 3$, $n = 2s+1$.

DEFINITION. Let $\{a_1, a_2, \ldots, a_m\}$ be a family of 2-element subsets of $\{1, 2, \ldots, n\}$. Introduce the notation $d_i = |\{a_j,\ i \in a_j\}|$; d_i is called the degree of i.

THEOREM 3. *Let $F = \langle a_1, a_2, \ldots, a_m \rangle$ be an ordered filter in B_n. Let n be even, $|a_i| = t = 2$. If* $\max d_i \geqslant n/2 + 1$, *then F has the Sperner property.*

PROOF. Let $n = 2s$. By Theorem 2, we need to consider only the maximum antichain in $F_s \cup D_{s+1}$. Without any loss of generality, let $d_{2s} \geqslant s+1$; otherwise a permutation will be used.

We know that $B_{2s} = B_{2s-1} \cup Q(2s)$, and $F_2 = \{a_1, a_2, \ldots, a_m\}$. Let $F_2 \cap B_{2s-1} = F_2'$, $F_2 \cap Q(2s) = F_2''$; if $F_2' = \varnothing$, then $F_2 = F_2''$. This case has already been solved in [6], and we may assume that $F_2 \neq \varnothing$.

Take $Q(2s)$ as B_{2s-1} by deleting $(2s)$ from the elements in $Q(2s)$, and F_2'' becomes a family of 1-element subsets. Let $H = \bigcup_{j \in F_2''} \{j\} - \{2s\}$; then $|H| \geqslant s+1$. For any $x \in B_{2s-1}$, $|x| \geqslant s-1$, we have $x \cap H \neq \varnothing$. Thus $Q(2s)_s \subset F$. Let n be $2s-1$ and t be 1 in (3.1); we can see in the same way as earlier that there is a matching from $(B_{2s-1})_s$ to $(B_{2s-1})_{s-1}$, so that the matching exists from $(Q(2s))_{s+1}$ to $(Q(2s))_s$.

Let n be $2s-1$ and t be 2 in (3.1); it is easy to see that there is a matching from $(B_{2s-1})_{s+1}$ to $(B_{2s-1})_s$, and the theorem is proven. □

THEOREM 4. *Let $F = \langle a_1, a_2, \ldots, a_m \rangle$ be the ordered filter in B_n, n even, and let $|a_i| = t = 2$. If $m > n^2/4$, then F has the Sperner property.*

PROOF. If F has no sperner property, by Theorem 3, we have $d_i \leqslant n/2$, $i = 1, 2, \ldots, n$.

$$m = \sum_{i=1}^{n} d_i/2 \leqslant n^2/4;$$

this contradicts the assumption. □

If x and y are two subsets of $\{1, 2, \ldots, n\}$, neither of which contains the other, write $x <_L Y$ if $\max\{i, i \in x - y\} > \max\{i, i \in y - x\}$. This defines a partial order in which the sets of the same size are totally ordered. We call $<_L$ the antilexicographic order. Let $D \subset B_n$, $\mathscr{L}D_k$ be the first $|D_k|$ elements of $(B_n)_k$ in the antilexicographic order. Kruskal [3] and, independently, Katona [1] have obtained the following:

THEOREM 5. *$(\mathscr{L}D_k)^+ \subset \mathscr{L}(D_k^+)$, $k = 0, 1, \ldots, n-1$, where D_k^+ is defined in B_n as in Section 3.*

Using Theorem 5 we can obtain:

THEOREM 6. *Let $F = \langle a_1, a_2, \ldots, a_m \rangle$ be an ordered filter in B_n, where $n = 2s$ is even, $|a_i) = t \leq 3$. If $|F_{n/2}| \leq \sum_{i=2}^{s} \binom{2i-1}{i}$, then F has the Sperner property.*

PROOF. Let $n = 2s$. We have

$$|Q_{s+1}(2s)| = \binom{2s-1}{s} = \binom{2s-1}{s-1} = |Q_s(2s)|, \tag{3.3}$$

where $Q_1(2s) = (Q(2s))_i$.

Following (3.2), by deleting $\{2s\}$, we know that there is a matching from $Q_s(2s)$ to $Q_{s+1}(2s)$. Consider the following pairs of equal binomial coefficients:

$$\left\{\binom{2s-3}{s-2}, \binom{2s-3}{s-1}\right\}, \left\{\binom{2s-5}{s-3}, \binom{2s-5}{s-2}\right\}, \ldots, \left\{\binom{3}{1}, \binom{3}{2}\right\}.$$

These pairs of binomial coefficients correspond to the number of s- and $(s+1)$-subsets containing $\{2s-1, 2s-2\}$, $\{2s-1, 2s-3, 2s-4\}$, $\{2s-1, 2s-3, 2s-5, 2s-6\}, \ldots, \{2s-1, 2s-3, \ldots, 7, 5, 4\}$ respectively, where the other elements of these s- and $(s+1)$-subsets are taken from $\{1, 2, \ldots, 2s-3\}$, $\{1, 2, \ldots, 2s-5\}, \ldots, \{1, 2, 3\}$. Denote these s- and $(s+1)$-subsets as $P_{s,2s-i}$ and $P_{s+1,2s-i}$, $i = 3, 5, \ldots, 2s-3$. They are arranged hierarchically in turn in $(B_n)_s$ and $(B_n)_{s+1}$, respectively, according to the antilexicographic order $<_L$. Obviously, a matching from $P_{s,2s-i}$ to $P_{s+1,2s-i}$ exists, $i = 3, 5, \ldots, 2s-3$. Write

$$G_l = Q_l(2s) \cup \left(\bigcup_{i=3,5,\ldots,2s-3} P_{l,2s-i}\right), \qquad l = s, s+1.$$

We known that $|G_s| = |G_{s+1}| = \sum_{i=2}^{s} \binom{2i-1}{i}$ and a matching from G_s to G_{s+1} exists. Therefore, for any $E \subset G_s$,

$$|E^+| \geq |E|. \tag{3.6}$$

Now we will show that, under the assumptions of the theorem, there is a matching from F_s to F_{s+1}, and the conclusion follows. Otherwise, there is $D \subset F_s$, such that

$$|D| > |D^+|, \tag{3.7}$$

Since $|D| \leq \sum_{i=2}^{s} \binom{2i-1}{i}$, $(\mathscr{L}D) \subset G_s$; thus $|\mathscr{L}D| \leq |(\mathscr{L}D)^+|$. By Theorem 5, $|(\mathscr{L}D)^+| \leq |\mathscr{L}(D^+)|$. Since $|D| = |\mathscr{L}(D)|$, $|D^+| = |\mathscr{L}(D^+)|$, this contradicts (3.7). □

COROLLARY. *Let F, t be the same as in Theorem 6. If $m \leq n$ for $t = 2$ or $m \leq n^2/2 - n/2 - 1$ for $t = 3$, respectively, and there is a permutation π on $\{1, 2, \ldots, n\}$ such that $\pi F_2 = \mathscr{L}F_2$ or $\pi F_3 = \mathscr{L}F_3$, respectively, then F has the Sperner property.*

Now the cases in which $\{a_i\}$ are pairwise non-disjoint will be discussed. The following is a theorem about intersecting systems, due to Katona:

THEOREM 7 (Katona [2]). *Let $D \in (B_n)_i$. If $|x \cap y| \geqslant k$ for any $x, y \in D$; then*

$$|D^-| \geqslant \left[\binom{2i-k}{i-1} \Big/ \binom{2i-k}{i}\right] |D|.$$

Using Theorem 7, we have:

THEOREM 8. *For $F = \langle a_1, a_2, \ldots, a_m \rangle$ the same as before $n = 2s$, $|a_i| = t = 2, 3$, if $a_i \cap a_j \neq \varnothing$ for any i, j, then f has the Sperner property.*

PROOF. By Theorem 2, we need only to consider in $F_s \cup F_{s+1}$. For any $D \subset F_s$ and any $x_1, x_2 \in D$,

$$|x_1 \cup x_2| = |x_1| + |x_2| - |x_1 \cap x_2| \leqslant 2s - 1, \qquad \text{thus } |\bar{x}_1 \cap \bar{x}_2| \geqslant 1.$$

$\bar{D} = \{\bar{x}, x \in D\}$ is also an interecting family. Since D^+ in F coincides with D^+ in B_n, then $(\bar{D})^-$ is also the same in $\bar{F}$ and $\bar{B}_n$. By Theorem 7 we have

$$|(\bar{D})^-| \geqslant \left[\binom{2s-1}{s-1} \Big/ \binom{2s-1}{s}\right] |\bar{D}| = |\bar{D}| = |D|.$$

$(\bar{D})^- = (\overline{D^+})$ implies $|D^+| = |\overline{D^+}| \geqslant |D|$. Consequently there is a matching from F_s to F_{s+1}. □

The following theorem can be obtained in the same way.

THEOREM 9. *$F = \langle a_1, a_2, \ldots, a_m \rangle$ the same as before, if $|a_i| = t \leqslant 4$, $|a_i \cap a_j| \geqslant 2$, then F has the Sperner property.*

4. CONCLUDING REMARK

Ko–wei Lih's conjecture has negative answers in most cases; i.e. for $t \geqslant 4$, counter-examples can be constructed. The following three cases are still open $n = 2s$, $t = 2$ or $n = 2s + 1$, $t = 3$. The author is inclined to agree with Lih's conjecture in these cases. In Section 3 of this paper, the conjecture has been confirmed in some special cases. For example, when n is even, $t = 2$, the conjecture is true for $\max d_i \geqslant n/2 + 1$. By the way, it should be pointed out that if the conjecture does not hold in the remaining three cases, it must be $|F_s| > |F_{s+1}|$ (for $n = 2s$, $t = 2, 3$) or $|F_{s+1}| > |F_{s+2}|$ (for $n = 2s + 1$, $t = 3$).

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XIAOYA ZHA
Department of Mathematics,
Ohio State University,
Columbus, Ohio 43210, U.S.A.