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On the decomposition of generalized incomplete gamma functions with applications to Fourier transforms

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Abstract

In this paper we introduce decomposition functions $C_\Gamma(\alpha, x; \omega)$, $S_\Gamma(\alpha, x; \omega)$, $C_\gamma(\alpha, x; \omega)$ and $S_\gamma(\alpha, x; \omega)$ of the generalized gamma functions. These functions are found useful in the analytic study of the temperature distribution of a semi-infinite solid with periodic boundary conditions and to the theory of Fourier transforms. Several new identities involving the Fourier transforms are investigated and some of the classical ones are recovered as special cases. For numerical and scientific computations, tabular and graphical representations of the functions $C_\Gamma(\alpha, x; \omega)$ and $S_\Gamma(\alpha, x; \omega)$ are also given.

Keywords: Generalized incomplete gamma functions; Decompositions; Cosine and sine Fourier transforms

1. Introduction

The Fresnel integrals [2,6,9,11,19]

$$C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x t^{-1/2} \cos t \, dt \tag{1}$$

and

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x t^{-1/2} \sin t \, dt \tag{2}$$

occur in various branches of physics and engineering such as in diffraction theory, theory of vibrations, quantum optics and antenna theory, among others; (see [2,8,11,27,29,31,44]). The Fresnel integrals (1), (2) are related to the error functions via [19]

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$$C_2(x) = \frac{1}{2\sqrt{2}}[e^{-i\pi/4}\,\mathrm{Erf}(e^{i\pi/4}\sqrt{x}) + e^{i\pi/4}\,\mathrm{Erf}(e^{-i\pi/4}\sqrt{x})] \tag{3}$$

and

$$S_2(x) = \frac{i}{2\sqrt{2}}[e^{-i\pi/4}\,\mathrm{Erf}(e^{i\pi/4}\sqrt{x}) - e^{i\pi/4}\,\mathrm{Erf}(e^{-i\pi/4}\sqrt{x})], \tag{4}$$

where the error functions were originally introduced in 1799 by Kramp [30]. It should be noted that we define $\mathrm{Erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2}\,dt$ and $\mathrm{Erfc}(x) = (2/\sqrt{\pi})\int_x^\infty e^{-t^2}\,dt$. In 1939, Böhmer [7,19] generalized the Fresnel integrals and introduced a new pair of functions

$$C(x,\alpha) = \int_x^\infty t^{\alpha-1}\cos t\,dt \tag{5}$$

and

$$S(x,\alpha) = \int_x^\infty t^{\alpha-1}\sin t\,dt. \tag{6}$$

It can be seen that

$$C_2(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}C(x,\tfrac{1}{2}) \tag{7}$$

and

$$S_2(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}}S(x,\tfrac{1}{2}). \tag{8}$$

The Böhmer functions (5), (6) have also been discussed in [6]. It should be noted that the Böhmer functions are expressed in terms of incomplete gamma functions to give

$$C(x,\alpha) = \tfrac{1}{2}[e^{-i\pi\alpha/2}\,\Gamma(\alpha, ix) + e^{i\pi\alpha/2}\,\Gamma(\alpha, -ix)] \tag{9}$$

and

$$S(x,\alpha) = \frac{1}{2i}[e^{i\pi\alpha/2}\,\Gamma(\alpha, -ix) - e^{-i\pi\alpha/2}\,\Gamma(\alpha, ix)]. \tag{10}$$

The incomplete gamma functions defined by

$$\Gamma(\alpha,x) = \int_x^\infty t^{\alpha-1}e^{-t}\,dt \tag{11}$$

and

$$\gamma(\alpha,x) = \int_0^x t^{\alpha-1}e^{-t}\,dt \tag{12}$$

were first investigated for real x by Legendre [32,33]. The significance of the decomposition formula

$$\gamma(\alpha,x) + \Gamma(\alpha,x) = \Gamma(\alpha) \tag{13}$$

was recognized by Prym [19]. It can be seen that

$$C(x,\alpha) - i\,S(x,\alpha) = e^{-i\pi\alpha/2}\,\Gamma(\alpha, ix) \tag{14}$$

and

$$C(x,\alpha)+\mathrm{i}\,S(x,\alpha)=\mathrm{e}^{\mathrm{i}\pi\alpha/2}\,\Gamma(\alpha,-\mathrm{i}x). \tag{15}$$

It should be emphasized that the solutions to a considerable number of problems in applied mathematics and statistics, astrophysics and nuclear physics, diffraction theory, theory of vibrations, quantum optics and theory of heat conduction can be expressed in terms of incomplete gamma functions (see [2,8,13,15,17,18,21,25,29,38,42,43,50,51]). The older theory of the incomplete gamma functions is presented by Nielsen [37] and a relatively recent account can be found in Böhmer [7].

Chaudhry and Zubair [16] introduced generalized incomplete gamma functions

$$\gamma(\alpha,x;b)=\int_0^x t^{\alpha-1}\mathrm{e}^{-t-bt^{-1}}\,\mathrm{d}t \tag{16}$$

and

$$\Gamma(\alpha,x;b)=\int_x^\infty t^{\alpha-1}\mathrm{e}^{-t-bt^{-1}}\,\mathrm{d}t, \tag{17}$$

found useful in statistics, reliability theory, theory of heat conduction and thermoelasticity; (see [9,13,14,16–18,31,35,42,50–52]). Chaudhry and Zubair have shown that the functions

$$\Gamma\left(0,\frac{1}{4F_0},\frac{\tau}{4F_0}\right),\qquad \Gamma\left(\pm\frac{1}{2},\frac{1}{4F_0},\frac{\tau}{4F_0}\right),\qquad \Gamma\left(\pm\frac{3}{2},\frac{1}{4F_0},\frac{\tau}{4F_0}\right)$$

are solutions to heat conduction problems occurring in various physical applications with time-dependent boundary conditions. They have discussed several properties of these functions including special cases, asymptotic behavior, recurrence relations, Laplace transforms and decomposition formula. It should be noted that if we take $b=0$ in (16), (17), we get the classical incomplete gamma functions

$$\gamma(\alpha,x;0)=\gamma(\alpha,x) \tag{18}$$

and

$$\Gamma(\alpha,x;0)=\gamma(\alpha,x). \tag{19}$$

The generalized incomplete gamma functions (16), (17) satisfy the decomposition formula

$$\gamma(\alpha,x;b)+\Gamma(\alpha,x;b)=2b^{\alpha/2}K_\alpha(2\sqrt{b}), \tag{20}$$

where K_α is the Macdonald function [19].

In this paper, we have introduced two pairs of functions:

$$C_\Gamma(\alpha,x;\omega)=\int_x^\infty t^{\alpha-1}\mathrm{e}^{-t}\cos\left(\frac{\omega}{t}\right)\mathrm{d}t \tag{21}$$

and

$$S_\Gamma(\alpha,x;\omega)=\int_x^\infty t^{\alpha-1}\mathrm{e}^{-t}\sin\left(\frac{\omega}{t}\right)\mathrm{d}t; \tag{22}$$

respectively

$$C_\gamma(\alpha, x; \omega) = \int_0^x t^{\alpha-1}\mathrm{e}^{-t} \cos\left(\frac{\omega}{t}\right) \mathrm{d}t \tag{23}$$

and

$$S_\gamma(\alpha, x; \omega) = \int_0^x t^{\alpha-1}\mathrm{e}^{-t} \sin\left(\frac{\omega}{t}\right) \mathrm{d}t, \tag{24}$$

found useful in various branches of applied mathematics, physics and theory of heat conduction (see [13,17,25,38]). Chaudhry and Zubair [17] have shown that the nondimensional temperature solution to the heat conduction problem of a semi-infinite solid with periodic boundary conditions is given by

$$\theta = \left[\cos(\tau) C_\Gamma\left(\frac{1}{2}, \frac{1}{4F_0}; \chi\right) + \sin(\tau) S_\Gamma\left(\frac{1}{2}, \frac{1}{4F_0}; \chi\right)\right]. \tag{25}$$

It should be emphasized that the closed-form solution to the heat conduction problem of a semi-infinite solid subject to periodic boundary condition is not available in the literature (see [13,25,42]). The representation of the heat conduction solution in terms of these functions was helpful to find the closed-form representation of the heat flux and to discuss the steady-state solutions with rigorous mathematical justification.

We have also found the Fourier transforms of these functions. Some of the known identities are recovered as special cases of our results and several new identities are established.

Notation. For the sine, cosine and exponential Fourier transforms we shall follow the notations of Erdélyi et al. [20] as follows:

$$F_\mathrm{s}(\omega) = F_\mathrm{s}\{f(t); t \to \omega\} = \int_0^\infty f(t) \sin(\omega t)\, \mathrm{d}t, \tag{26}$$

$$F_\mathrm{c}(\omega) = F_\mathrm{c}\{f(t); t \to \omega\} = \int_0^\infty f(t) \cos(\omega t)\, \mathrm{d}t \tag{27}$$

and

$$F(\omega) = F\{f(t); t \to \omega\} = \int_{-\infty}^\infty f(t)\mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t. \tag{28}$$

The proofs of the several identities follow just from the simple manipulations of the definitions (16), (17) and (21)–(24). However, we have stated these identities as theorems for completeness.

It should be noted that the parameters α, x and ω are considered to be real in the following analysis.

2. Main results and applications

Theorem 1 (Relationship with generalized gamma functions).

$$C_\Gamma(\alpha, x; \omega) = \tfrac{1}{2}[\Gamma(\alpha, x; -\mathrm{i}\omega) + \Gamma(\alpha, x; \mathrm{i}\omega)], \tag{29}$$

$$S_\Gamma(\alpha, x; \omega) = \frac{1}{2\mathrm{i}}[\Gamma(\alpha, x; -\mathrm{i}\omega) - \Gamma(\alpha, x; \mathrm{i}\omega)], \tag{30}$$

$$C_\gamma(\alpha, x; \omega) = \tfrac{1}{2}[\gamma(\alpha, x; -\mathrm{i}\omega) + \gamma(\alpha, x; \mathrm{i}\omega)], \tag{31}$$

$$S_\gamma(\alpha, x; \omega) = \frac{1}{2\mathrm{i}}[\gamma(\alpha, x; -\mathrm{i}\omega) - \gamma(\alpha, x; \mathrm{i}\omega)]. \tag{32}$$

Theorem 2 (Special cases).

$$C_\Gamma(\tfrac{1}{2}, x; 0) = \sqrt{\pi}\,\mathrm{Erfc}(\sqrt{x}), \tag{33}$$

$$C_\gamma(\tfrac{1}{2}, x; 0) = \sqrt{\pi}\,\mathrm{Erf}(\sqrt{x}), \tag{34}$$

$$C_\Gamma(0, x; 0) = E_1(x) = -\mathrm{Ei}(-x), \tag{35}$$

$$C_\Gamma(\alpha, \mathrm{i}x; 0) = \mathrm{e}^{\mathrm{i}\pi\alpha/2}[C(x, \alpha) - \mathrm{i}\,S(x, \alpha)], \tag{36}$$

$$C_\Gamma(\alpha, x; 0) = \Gamma(\alpha, x), \tag{37}$$

$$C_\Gamma(\tfrac{1}{2}, x; \omega) = \tfrac{1}{2}\sqrt{\pi}\,\mathrm{Re}(U + V), \tag{38a}$$

where

$$U = \exp(-2(\mathrm{i}\omega)^{1/2})\,\mathrm{Erfc}\left(x^{1/2} - \left(\frac{\mathrm{i}\omega}{x}\right)^{1/2}\right) \tag{38b}$$

and

$$V = \exp(2(\mathrm{i}\omega)^{1/2})\,\mathrm{Erfc}\left(x^{1/2} + \left(\frac{\mathrm{i}\omega}{x}\right)^{1/2}\right). \tag{38c}$$

Remark. It follows from (29)-(32) that

$$C_\Gamma(\alpha, x; \omega) - \mathrm{i}\,S_\Gamma(\alpha, x; \omega) = \Gamma(\alpha, x; \mathrm{i}\omega) \tag{39a}$$

and

$$C_\gamma(\alpha, x; \omega) - \mathrm{i}\,S_\gamma(\alpha, x; \omega) = \gamma(\alpha, x; \mathrm{i}\omega). \tag{39b}$$

Theorem 3 (Evaluation of complete integrals).

$$C_\Gamma(\alpha, 0; \omega) = 2\omega^{\alpha/2}\,\mathrm{Re}[\mathrm{e}^{\mathrm{i}\pi\alpha/4} K_\alpha(2\sqrt{\mathrm{i}\omega})], \quad \omega > 0, \tag{40}$$

and

$$-S_\Gamma(\alpha, 0; \omega) = 2\omega^{\alpha/2}\,\mathrm{Im}[\mathrm{e}^{\mathrm{i}\pi\alpha/4} K_\alpha(2\sqrt{\mathrm{i}\omega})], \quad \omega > 0. \tag{41}$$

Proof. Substituting $x = 0$ in the decomposition formula (20), we get

$$\Gamma(\alpha, 0; b) = 2b^{\alpha/2} K_\alpha(2\sqrt{b}). \tag{42}$$

Substituting $x = 0$ in (39a) and using (42) for $b = \mathrm{i}\omega$, $\omega > 0$, we get

$$C_\Gamma(\alpha, 0; \omega) - \mathrm{i}\,S_\Gamma(\alpha, 0; \omega) = 2\omega^{\alpha/2}[\mathrm{e}^{\mathrm{i}\pi\alpha/4} K_\alpha(2\sqrt{\mathrm{i}\omega})]. \tag{43}$$

Separating the real and imaginary parts in (43), we get the proof of (40), (41). □

Corollary (Connection with trigonometric functions).

$$C_\Gamma(\tfrac{1}{2}+n,0;\omega)=\pi^{1/2}\mathrm{e}^{-\sqrt{2\omega}}\sum_{m=0}^{n}\frac{1}{m!\,2^{2m}}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\omega^{(n-m)/2}\cos[\sqrt{2\omega}-\tfrac{1}{4}\pi(n-m)] \tag{44}$$

and

$$S_\Gamma(\tfrac{1}{2}+n,0;\omega)=\pi^{1/2}\mathrm{e}^{-\sqrt{2\omega}}\sum_{m=0}^{n}\frac{1}{m!\,2^{2m}}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\omega^{(n-m)/2}\sin[\sqrt{2\omega}-\tfrac{1}{4}\pi(n-m)]. \tag{45}$$

Proof. Substituting $\alpha=\frac{1}{2}+n$ in (40), (41) and using the representation [19, p. 10]

$$K_{1/2+n}(z)=\left(\frac{\pi}{2z}\right)^{1/2}\mathrm{e}^{-z}\sum_{m=0}^{n}\frac{(2z)^{-m}}{m!}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}, \tag{46}$$

we get the proof of (44), (45) when we substitute the values of $K_{1/2+n}(z)$ for $z=\mathrm{i}\omega$ in (40), (41) from (46). □

Corollary.

$$C_\Gamma(\tfrac{1}{2}-n,0;\omega)=\pi^{1/2}\mathrm{e}^{-\sqrt{2\omega}}\sum_{m=0}^{n-1}\left[\frac{1}{m!\,2^{2m}}\frac{\Gamma(n+m)}{\Gamma(n-m)}\omega^{-(n+m)/2}\cos[\sqrt{2\omega}+\tfrac{1}{4}\pi(n+m)]\right] \tag{47}$$

and

$$S_\Gamma(\tfrac{1}{2}-n,0;\omega)=\pi^{1/2}\mathrm{e}^{-\sqrt{2\omega}}\sum_{m=0}^{n-1}\left[\frac{1}{m!\,2^{2m}}\frac{\Gamma(n+m)}{\Gamma(n-m)}\omega^{-(n+m)/2}\sin[\sqrt{2\omega}+\tfrac{1}{4}\pi(n+m)]\right]. \tag{48}$$

Proof. This follows from (40), (41) and from (44)–(46) when we use the fact $K_{1/2-n}(z)=K_{n-1/2}(z)$. □

Theorem 4 (Recurrence relations).

$$C_\Gamma(\alpha+1,x;\omega)=\alpha C_\Gamma(\alpha,x;\omega)+\omega S_\Gamma(\alpha-1,x;\omega)+x^\alpha\mathrm{e}^{-x}\cos\left(\frac{\omega}{x}\right) \tag{49}$$

and

$$S_\Gamma(\alpha+1,x;\omega)=\alpha S_\Gamma(\alpha,x;\omega)-\omega C_\Gamma(\alpha-1,x;\omega)+x^\alpha\mathrm{e}^{-x}\sin\left(\frac{\omega}{x}\right). \tag{50}$$

Proof. We note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t^\alpha\mathrm{e}^{-t}\cos\left(\frac{\omega}{t}\right)\right]=\alpha\left(t^{\alpha-1}\mathrm{e}^{-t}\cos\left(\frac{\omega}{t}\right)\right)+\omega t^{\alpha-2}\mathrm{e}^{-t}\sin\left(\frac{\omega}{t}\right)-t^\alpha\mathrm{e}^{-t}\cos\left(\frac{\omega}{t}\right). \tag{51}$$

Integrating both sides of (51) from $t = x$ to $t = \infty$ and using (21), (22), we get the proof of (49). The proof of (50) follows similarly. In particular, if we substitute $\omega = 0$ in (49), we get the recurrence relation (see [24])

$$\Gamma(\alpha+1,x) = \alpha\Gamma(\alpha,x) + x^{\alpha}e^{-x} \tag{52}$$

for the incomplete gamma function. □

Remark. According to (21), (23) and (22), (24), we have

$$C_{\Gamma}(\alpha,x;\omega) + C_{\gamma}(\alpha,x;\omega) = \mathrm{Re}[\Gamma(\alpha,0,-i\omega)] \tag{53}$$

and

$$S_{\Gamma}(\alpha,x;\omega) + S_{\gamma}(\alpha,x;\omega) = \mathrm{Im}[\Gamma(\alpha,0,-i\omega)]. \tag{54}$$

Therefore, in view of (53), (54), it is sufficient to discuss the properties of the functions $C_{\Gamma}(\alpha,x;\omega)$ and $S_{\Gamma}(\alpha,x;\omega)$.

Theorem 5 (Exponential Fourier transforms). *Let F be the Fourier transform operator as defined by* (28). *Then,*

$$F\{t^{\alpha-1}e^{-b/t}H(x-t)H(t);t\to\omega\} = \omega^{-\alpha}e^{-i\pi\alpha/2}\gamma(\alpha,i\omega x,i\omega b),\quad b>0, \tag{55}$$

where $H(t)$ is the Heaviside unit step function.

Proof. By definition of the Fourier transform,

$$\begin{aligned}F\{t^{\alpha-1}e^{-b/t}H(x-t)H(t);t\to\omega\} &= \int_{-\infty}^{\infty} t^{\alpha-1}e^{-b/t}H(x-t)H(t)e^{-i\omega t}\,dt\\ &= \int_0^x t^{\alpha-1}e^{-(i\omega)t-bt^{-1}}\,dt\\ &= (i\omega)^{-\alpha}\gamma(\alpha,i\omega x;i\omega b)\\ &= \omega^{-\alpha}e^{-i\pi\alpha/2}\gamma(\alpha,i\omega x;i\omega b).\quad\square\end{aligned}$$

Corollary (See [9, p. 154 (51)]).

$$\begin{aligned}F\{t^{\alpha-1}H(x-t)H(t);t\to\omega\} &= \omega^{-\alpha}e^{-i\pi\alpha/2}\gamma(\alpha,i\omega x)\\ &= \frac{1}{\alpha}x^{\alpha}\,{}_1F_1(\alpha,\alpha+1;-i\omega x),\quad \alpha>0.\end{aligned} \tag{56}$$

Proof. This follows from (55) when we take $b = 0$. It should be noted that when $b = 0$, we must have $\alpha > 0$ for convergence. In particular, when we take $\alpha = \frac{1}{2}$ in (56), we get (see [9, p. 173 (169)])

$$F\{t^{-1/2}H(x-t)H(t);t\to\omega\} = \omega^{-1/2}e^{-i\pi/4}\sqrt{\pi}\,\mathrm{Erf}\,((i\omega x)^{1/2}),$$

which can be simplified in terms of C and S functions to give

$$F\{t^{-1/2}H(x-t)H(t);t\to\omega\} = \sqrt{2\pi}\omega^{-1/2}[C_2(\omega x) - iS_2(\omega x)].\quad\square \tag{57}$$

Note. The Fourier transform F in [9] is defined differently by $F\{f(t) : t \to \omega\} = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t$.

Corollary.

$$F\{\omega^{-\alpha}\gamma(\alpha, \mathrm{i}\omega x; \mathrm{i}\omega b); \omega \to t\} = -2\pi \mathrm{e}^{-\mathrm{i}\pi\alpha/2} t^{\alpha-1} \mathrm{e}^{b/t} H(x+t)H(-t), \quad b > 0. \tag{58}$$

Proof. This follows from (55) when we use the Fourier inversion formula [48, p. 315]

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t, \tag{59}$$

where $F(\omega)$ is the Fourier transform of $f(t)$ as defined by (28). □

Theorem 6 (Exponential Fourier transform).

$$F\{t^{\alpha-1}\mathrm{e}^{-b/t}H(t-x); t \to \omega\} = \omega^{-\alpha}\mathrm{e}^{-\mathrm{i}\pi\alpha/2}\Gamma(\alpha, \mathrm{i}\omega x; \mathrm{i}\omega b), \quad x > 0,\ b \geqslant 0. \tag{60}$$

Proof. Similar to the proof of Theorem 5. □

Corollary.

$$F\{t^{\alpha-1}H(t-x); t \to \omega\} = \omega^{-\alpha}\mathrm{e}^{-\mathrm{i}\pi\alpha/2}\Gamma(\alpha, \mathrm{i}\omega x), \quad x > 0. \tag{61}$$

Proof. This follows from (60) when we substitute $b = 0$. It should be noted that (61), which is a special case of (60), does not seem to be known in the literature. □

Corollary.

$$F\{t^{-1/2}H(t-x); t \to \omega\} = \frac{\sqrt{\pi}}{\sqrt{\omega}}\mathrm{e}^{-\mathrm{i}\pi/4}\,\mathrm{Erfc}\,\left((\mathrm{i}\omega x)^{1/2}\right). \tag{62}$$

Proof. This follows from (61) when we substitute $\alpha = \frac{1}{2}$ and use (33). □

Corollary.

$$F\{t^{-1}H(t-x); t \to \omega\} = -\,\mathrm{Ei}(-\mathrm{i}\omega x) = E_1(\mathrm{i}\omega x), \quad x > 0. \tag{63}$$

Proof. This follows from (61) when we take $\alpha = b = 0$ and use (35). □

Corollary.

$$F\{\omega^{-\alpha}\Gamma(\alpha, \mathrm{i}\omega x; \mathrm{i}\omega b); \omega \to t\} = -2\pi \mathrm{e}^{-\mathrm{i}\pi\alpha/2} t^{\alpha-1}\mathrm{e}^{b/t}H(-t-x), \quad x > 0. \tag{64}$$

Proof. This follows from (60) when we apply the operators F on both sides and use the inversion formula (59). It should be noted that several special cases of (64) can be listed. In particular, substituting $\alpha = b = 0$ in (64), we get

$$F\{E_1(\mathrm{i}\omega x); \omega \to t\} = -2\pi t^{-1}H(-t-x), \tag{65}$$

which does not seem to be known in the literature. □

Theorem 7 (Fourier sine transforms). *Let F_s be the Fourier sine transform operator as defined by* (26). *Then,*

$$F_s\left\{t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right)H(t);t\to\omega\right\}=S_\Gamma(\alpha,x;\omega) \tag{66}$$

and

$$F_s\left\{t^{-\alpha-1}e^{-1/t}H\left(t-\frac{1}{x}\right)H(t);t\to\omega\right\}=S_\gamma(\alpha,x;\omega),\quad x>0,\ \alpha>-1. \tag{67}$$

Proof. By definition (26) of the Fourier sine transform,

$$F_s\left\{t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right)H(t);t\to\omega\right\}=\int_0^\infty t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right)\sin(\omega t)\,dt$$
$$=\int_0^{1/x} t^{-\alpha-1}e^{-1/t}\sin(\omega t)\,dt. \tag{68}$$

Substituting $t=1/\tau$, $dt=-d\tau/\tau^2$ in (68), we get

$$F_s\left\{t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right);t\to\omega\right\}=\int_x^\infty \tau^{\alpha-1}e^{-\tau}\sin\left(\frac{\omega}{\tau}\right)d\tau=S_\Gamma(\alpha,x;\omega).$$

The proof of (67) follows from (66) and from the fact that

$$H\left(\frac{1}{x}-t\right)+H\left(t-\frac{1}{x}\right)=1.\qquad\square \tag{69}$$

Corollary.

$$F_s\{S_\Gamma(\alpha,x;\omega);\omega\to t\}=\tfrac{1}{2}\pi t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right)H(-t) \tag{70}$$

and

$$F_s\{S_\gamma(\alpha,x;\omega);\omega\to t\}=\tfrac{1}{2}\pi t^{-\alpha-1}e^{-1/t}H\left(t-\frac{1}{x}\right)H(-t),\quad \alpha>-1. \tag{71}$$

Proof. The proofs of (70), (71) follow when we apply the operator F_s on both sides of (66), (67) and use the inversion formula [48, p. 32]

$$\tfrac{1}{2}\pi f(t)=\int_0^\infty F_s(\omega)\sin(\omega x)\,d\omega \tag{72}$$

of the Fourier sine transform. Here, $F_s(\omega)$ is the Fourier sine transform as defined by (26). $\square$

Remark. Several special cases of the identities (66) and (70) can be listed. Some of these special cases are known in the literature (see [20, p. 75 (31)]). It should be noted that

$$H\left(\frac{1}{x}-t\right)=1,\quad \text{for all } t,\ \text{ if } x\to 0^+. \tag{73}$$

However, the integral

$$\int_0^\infty t^{-\alpha-1}e^{-1/t}\sin(\omega t)\,dt \tag{74}$$

is convergent only for $\operatorname{Re}\alpha > -1$. Therefore, letting $x \to 0^+$ in (66) and using (73), (74), we get the following corollary.

Corollary (See [20, p. 75 (31)]). *For* $\operatorname{Re}\alpha > -1$,

$$F_s\{t^{-\alpha-1}e^{-1/t}; t \to \omega\} = S_\Gamma(\alpha, 0; \omega). \tag{75}$$

The value of $S_\Gamma(\alpha,0;\omega)$ *is given in* (41). *In particular, for* $\alpha = \frac{1}{2} + n$ *in* (75) *we get*

$$F_s\{t^{-n-1/2}e^{-1/t}; t \to \omega\}$$
$$= \pi^{1/2}e^{-\sqrt{2\omega}}\sum_{m=0}^{n}\left[\frac{1}{m!\,2^{2m}}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\omega^{(n-m)/2}\sin[\sqrt{2\omega}-\tfrac{1}{4}\pi(n-m)]\right],$$
$$n = 0, 1, 2, 3, \ldots . \tag{76}$$

It should be noted that in view of (41), $S_\Gamma(\alpha, 0;\omega)$ is defined for all α in (75) whereas the left-hand side is defined only for $\operatorname{Re}\alpha > -1$. Therefore, $S_\Gamma(\alpha,0;\omega)$ can be considered as an analytic continuation of the integral (74).

Theorem 8 (Fourier cosine transforms). *Let* F_c *be the Fourier cosine transform operator as defined by* (27). *Then,*

$$F_c\left\{t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right)H(t); t\to\omega\right\} = C_\Gamma(\alpha, x;\omega) \tag{77}$$

and

$$F_c\left\{t^{-\alpha-1}e^{-1/t}H\left(t-\frac{1}{x}\right)H(t); t\to\omega\right\} = C_\gamma(\alpha, x;\omega), \quad \operatorname{Re}\alpha > -1. \tag{78}$$

Proof. This is similar to (66), (67). □

Corollary (See [20, p. 16 (21)]).

$$F_c\{t^{-\alpha-1}e^{-1/t}H(t); t\to\omega\} = C_\Gamma(\alpha,0;\omega), \quad \operatorname{Re}\alpha > -1. \tag{79}$$

Proof. This follows from (77) when we let $x \to 0^+$ and use the fact that

$$H(\infty - t) = 1. \tag{80}$$

The value of $C_\Gamma(\alpha,0;\omega)$ is given by (40). For $\alpha = \frac{1}{2}+n$ in (79) we get (see (47))

$$F_c\{t^{-n-1/2}e^{-1/t}H(t); t\to\omega\}$$
$$= \pi^{1/2}e^{-\sqrt{2\omega}}\sum_{m=0}^{n}\left[\frac{1}{m!\,2^{2m}}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\omega^{(n-m)/2}\cos[\sqrt{2\omega}-\tfrac{1}{4}\pi(n-m)]\right]. \tag{81}$$

Some of the special cases of (81) are listed in the literature (see [9,20]). □

Corollary.

$$F_c\{C_\Gamma(\alpha,x;\omega);\omega\to t\} = \tfrac{1}{2}\pi t^{-\alpha-1}e^{-1/t}H\left(\frac{1}{x}-t\right)H(t) \tag{82}$$

and

$$F_c\{C_\gamma(\alpha,x;\omega);\omega\to t\} = \tfrac{1}{2}\pi t^{-\alpha-1}e^{-1/t}H\left(t-\frac{1}{x}\right)H(t). \tag{83}$$

Proof. The proofs of (82), (83) follow when we apply the operator F_c on both sides of (77), (78) and use the inversion formula [48, p. 321]

$$\tfrac{1}{2}\pi f(t) = \int_0^\infty F_c(\omega)\cos(\omega t)\,d\omega \tag{84}$$

of the Fourier cosine transform, where $F_c(\omega)$ is the Fourier cosine transform as defined by (27). □

Several special cases of (82), (83) can be listed. For example, we have the new identities

$$F_c\{\exp(-\sqrt{2\omega})\cos(\sqrt{2\omega});\omega\to t\} = \tfrac{1}{2}\pi^{1/2}t^{-3/2}e^{-1/t} \tag{85}$$

and

$$F_c\{\exp(-\sqrt{2\omega})\cos(\sqrt{2\omega}+\tfrac{1}{4}\pi);\omega\to t\} = \tfrac{1}{2}\pi^{1/2}t^{-1/2}e^{-1/t}, \tag{86}$$

which are obtained from (82) for $x=0$ and $\alpha=\pm\frac{1}{2}$.

Theorem 9.

$$S_\Gamma^2(\alpha,x;\omega)+C_\Gamma^2(\alpha,x;\omega) \leqslant 2\{\Gamma(\alpha)\}^2,\quad \alpha>0,\ x\geqslant 0,\ \omega\geqslant 0. \tag{87}$$

Proof. According to the definition,

$$S_\Gamma(\alpha,x;\omega) = \int_x^\infty t^{\alpha-1}e^{-t}\sin\left(\frac{\omega}{t}\right)dt,$$

which implies

$$|S_\Gamma(\alpha,x;\omega)| \leqslant \int_x^\infty t^{\alpha-1}e^{-t}\,dt \tag{88}$$

$$\leqslant \Gamma(\alpha,x),\quad \alpha>0,\ x>0,$$

$$\leqslant \Gamma(\alpha). \tag{89}$$

Similarly,

$$|C_\Gamma(\alpha,x;\omega)| \leqslant \Gamma(\alpha),\quad \alpha>0,\ x>0,\ \omega>0. \tag{90}$$

Adding (89) and (90) after taking the squares, we get the proof of (87). □

Theorem 10 (Fourier cosine and sine transforms).

$$F_c\left\{t^{-\alpha-1}\exp(-st-t^{-1})H\left(\frac{1}{t}-x\right)H(t);t\to\omega\right\}=\tfrac{1}{2}[\Gamma(\alpha,x;s-\mathrm{i}\omega)+\Gamma(\alpha,x;s+\mathrm{i}\omega)] \tag{91}$$

and

$$F_s\left\{t^{-\alpha-1}\exp(-st-t^{-1})H\left(\frac{1}{t}-x\right)H(t);t\to\omega\right\}=\frac{1}{2\mathrm{i}}[\Gamma(\alpha,x;s-\mathrm{i}\omega)-\Gamma(\alpha,x;s+\mathrm{i}\omega)],$$
$$s>0. \tag{92}$$

Proof. The proof of (91), (92) follows from (17) when b is replaced by $s+\mathrm{i}\omega$ and the real and imaginary parts are separated. □

Corollary.

$$F_c\{t^{-\alpha-1}\exp(-st-t^{-1});t\to\omega\}=\tfrac{1}{2}[\Gamma(\alpha,0;s+\mathrm{i}\omega)+\Gamma(\alpha,0;s-\mathrm{i}\omega)] \tag{93}$$

and

$$F_s\{t^{-\alpha-1}\exp(-st-t^{-1});t\to\omega\}=\frac{-1}{2\mathrm{i}}[\Gamma(\alpha,0;s+\mathrm{i}\omega)-\Gamma(\alpha,0;s-\mathrm{i}\omega)]. \tag{94}$$

Proof. This follows from (91), (92), when we substitute $x=0$ and use the fact that

$$H\left(\frac{1}{t}\right)=1,\quad t>0.\quad \square \tag{95}$$

Remark. It should be noted that the identities (91)-(94) do not seem to be listed in the literature (see [9,20,39,40]). It follows from (93), (94) that for $\alpha=n+\frac{1}{2}$, $n=0,\pm1,\pm2,\pm3,\ldots$, the cosine and sine transforms (93), (94) can be simplified in terms of trigonometric functions by using the identities (46)-(48).

Theorem 11.

$$\int_{-\infty}^{\infty}[C^2(\omega,\alpha)+S^2(\omega,\alpha)]\frac{\mathrm{d}\omega}{|\omega|^{2\alpha}}=\frac{2\pi}{1-2\alpha},\quad \alpha<\tfrac{1}{2}. \tag{96}$$

Proof. Substituting $x=1$ in (61) and using the Parseval equation [48, p. 312], we get

$$\int_{-\infty}^{\infty}|\Gamma(\alpha,\mathrm{i}\omega)|^2\frac{\mathrm{d}\omega}{|\omega|^{2\alpha}}=2\pi\int_{-\infty}^{\infty}t^{2\alpha-2}H^2(t-1)\,\mathrm{d}t=2\pi\int_1^{\infty}t^{2\alpha-2}\,\mathrm{d}t$$
$$=\frac{2\pi}{1-2\alpha},\quad \alpha<\tfrac{1}{2}. \tag{97}$$

However, it follows from (14) that

$$|\Gamma(\alpha,\mathrm{i}\omega)|^2=C^2(\omega,\alpha)+S^2(\omega,\alpha). \tag{98}$$

From (97), (98), we get the proof of (96). □

Theorem 12. *Let* Ci(x), si(x) *be the cosine and sine integral functions defined by* [19, p. 145]

$$\mathrm{Ci}(x) = \int_{\infty}^{x} \frac{\cos t}{t}\,\mathrm{d}t \quad \textit{and} \quad \mathrm{si}(x) = \int_{\infty}^{x} \frac{\sin t}{t}\,\mathrm{d}t.$$

Then,

$$\int_{-\infty}^{\infty} [\mathrm{Ci}^2(x) + \mathrm{si}^2(x)]\,\mathrm{d}x = 2\pi. \tag{99}$$

Proof. Substituting $\alpha = 0$ in (97), we get

$$\int_{-\infty}^{\infty} |\Gamma(0, \mathrm{i}\omega)|^2 \mathrm{d}\omega = 2\pi. \tag{100}$$

However, [19, p. 145]

$$\Gamma(0, \mathrm{i}\omega) = -\mathrm{Ei}(-\mathrm{i}\omega) = -[\mathrm{Ci}(\omega) - \mathrm{i}\ \mathrm{si}(\omega)]. \tag{101}$$

From (100) and (101) we get the proof of (99). □

Theorem 13. *Let* $C^*(x,\alpha)$ *and* $S^*(x,\alpha)$ *be the complementary Böhmer's integrals defined by*

$$C^*(x,\alpha) = \int_0^x t^{\alpha-1} \cos t\,\mathrm{d}t, \quad \alpha \geqslant \tfrac{1}{2}, \tag{102}$$

and

$$S^*(x,\alpha) = \int_0^x t^{\alpha-1} \sin t\,\mathrm{d}t, \quad \alpha \geqslant \tfrac{1}{2}. \tag{103}$$

Then,

$$\int_{-\infty}^{\infty} [C^{*^2}(x,\alpha) + S^{*^2}(x,\alpha)] \frac{\mathrm{d}x}{|x|^{2\alpha}} = \frac{2\pi}{2\alpha - 1}, \quad \alpha > \tfrac{1}{2}. \tag{104}$$

Proof. Substituting $x = 1$ in (56) and using the Parseval equation [48, p. 312], we get

$$\int_{-\infty}^{\infty} |\gamma(\alpha, \mathrm{i}\omega)|^2 \frac{\mathrm{d}\omega}{|\omega|^{2\alpha}} = 2\pi \int_{-\infty}^{\infty} |t|^{2\alpha-2} H^2(1-t)\,\mathrm{d}t = 2\pi \int_0^1 t^{2\alpha-2}\,\mathrm{d}t = \frac{2\pi}{2\alpha-1}, \quad \alpha > \tfrac{1}{2}. \tag{105}$$

However,

$$\gamma(\alpha, \mathrm{i}\omega) = \mathrm{e}^{\mathrm{i}\pi\alpha/2} [C^*(\omega,\alpha) - \mathrm{i}\,S^*(\omega,\alpha)]. \tag{106}$$

From (105) and (106) we get the proof of (104). □

Remark. It should be noted that the complementary Böhmer integrals (102), (103) have not been discussed in the literature. It follows from (12) and (102), (103) that

$$C^*(\omega,\alpha) = \tfrac{1}{2}[\mathrm{e}^{\mathrm{i}\pi\alpha/2}\gamma(\alpha, -\mathrm{i}\omega) + \mathrm{e}^{-\mathrm{i}\pi\alpha/2}\gamma(\alpha, \mathrm{i}\omega)] \tag{107}$$

and

$$S^*(\omega,\alpha) = \frac{1}{2\mathrm{i}}[\mathrm{e}^{\mathrm{i}\pi\alpha/2}\gamma(\alpha, -\mathrm{i}\omega) - \mathrm{e}^{-\mathrm{i}\pi\alpha/2}\gamma(\alpha, \mathrm{i}\omega)]. \tag{108}$$

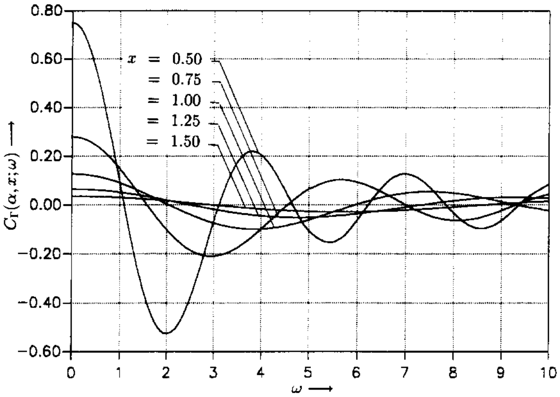

Fig. 1. Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = -1.50)$

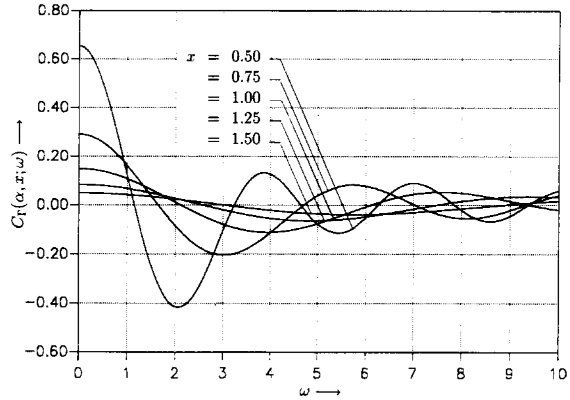

Fig. 2. Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = -1.00)$

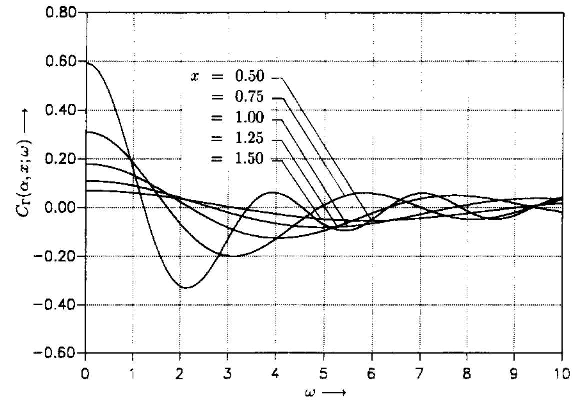

Fig. 3. Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = -0.50)$

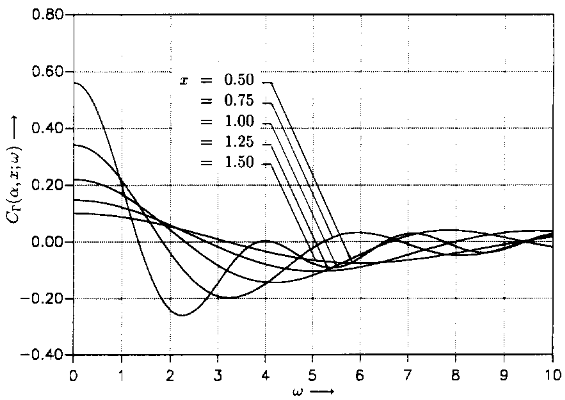

Fig. 4. Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = 0.00)$

Moreover, for $\alpha = \frac{1}{2}$ in (102), (103), we get the Fresnel integrals (1), (2)

$$C_2(x) = \frac{1}{\sqrt{2\pi}} C^*(x, \tfrac{1}{2}) \tag{109}$$

and

$$S_2(x) = \frac{1}{\sqrt{2\pi}} S^*(x, \tfrac{1}{2}). \tag{110}$$

Moreover, the identities (96), (99) and (104) are remarkably sharp and do not seem to be listed in the literature.

3. Graphical and tabular representations

For numerical and scientific computations, the decomposition functions $C_\Gamma(\alpha, x; \omega)$ and $S_\Gamma(\alpha, x; \omega)$ can easily be tabulated by using IMSL FORTRAN subroutines for mathematical ap-

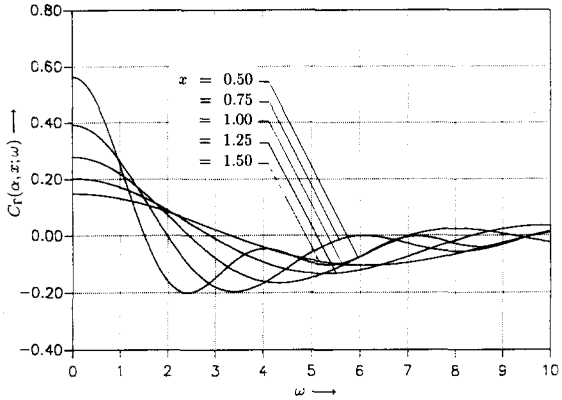

Fig. 5. Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = 0.50)$

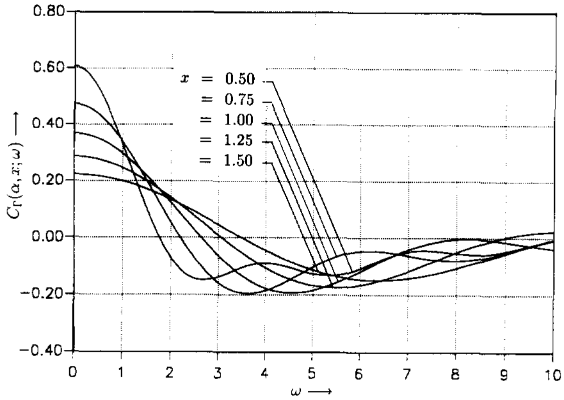

Fig. 6. Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = 1.00)$

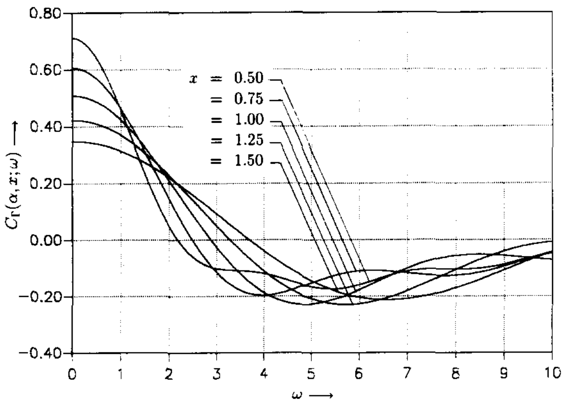

Fig. 7. Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = 1.50)$

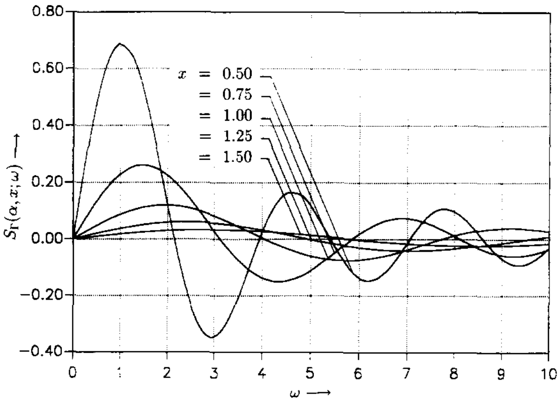

Fig. 8. Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = -1.50)$

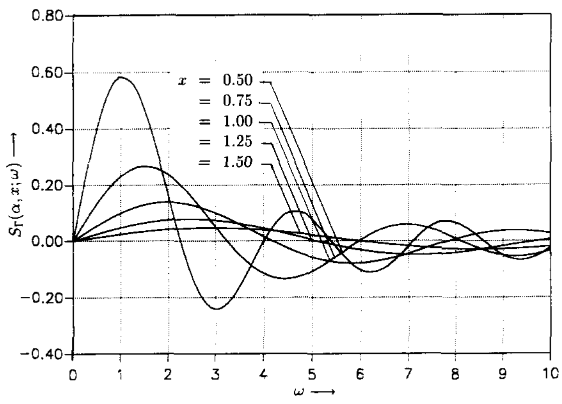

Fig. 9. Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = -1.00)$

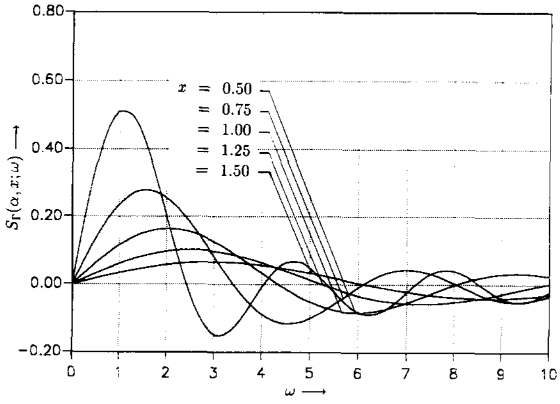

Fig. 10. Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = -0.50)$

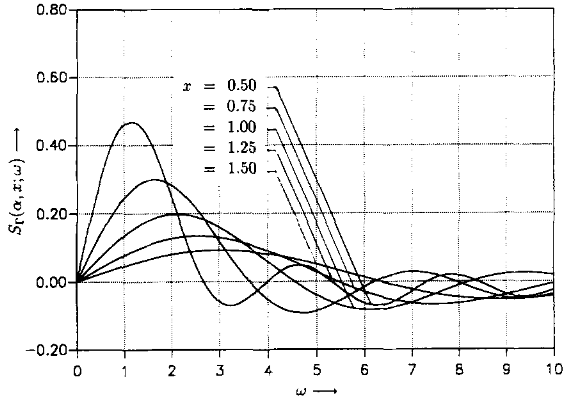

Fig. 11. Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = 0.00)$

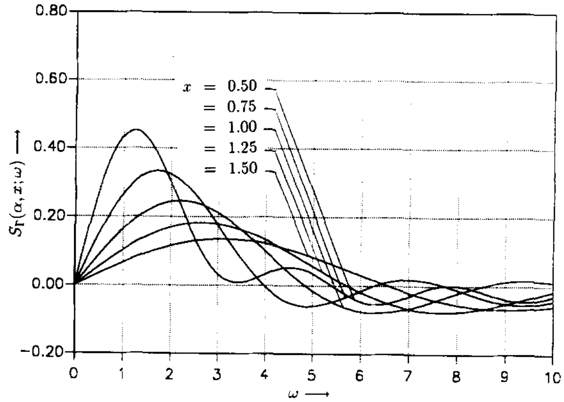

Fig. 12. Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = 0.50)$

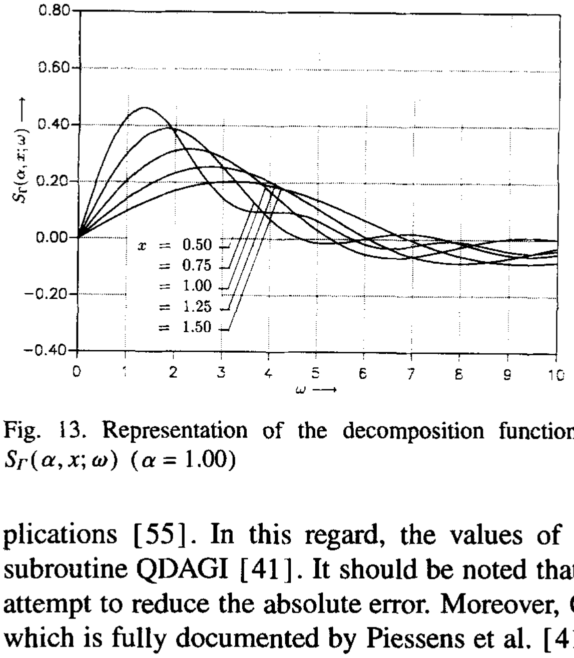

Fig. 13. Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = 1.00)$

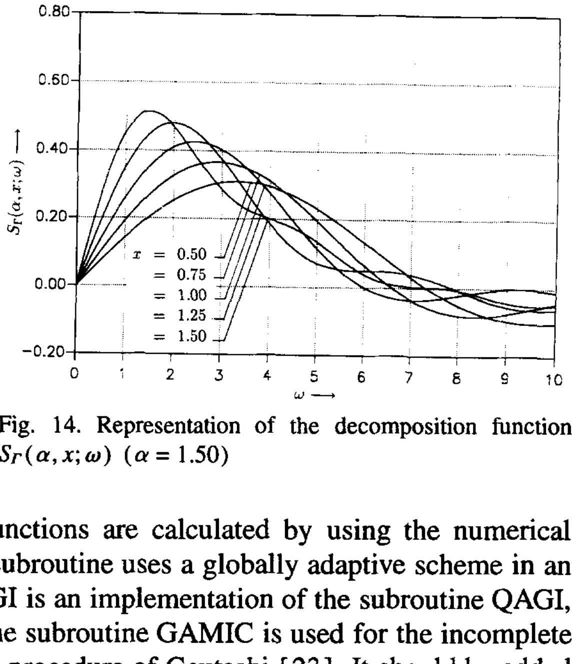

Fig. 14. Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = 1.50)$

plications [55]. In this regard, the values of the functions are calculated by using the numerical subroutine QDAGI [41]. It should be noted that the subroutine uses a globally adaptive scheme in an attempt to reduce the absolute error. Moreover, QDAGI is an implementation of the subroutine QAGI, which is fully documented by Piessens et al. [41]. The subroutine GAMIC is used for the incomplete gamma function, which is based on the computational procedure of Gautschi [23]. It should be added that $C_\Gamma(\alpha, x; 0)$, calculated by using the numerical integration scheme QDAGI, provides exactly the same results as those of the incomplete gamma function calculated by the subroutine GAMIC. The representation of $C_\Gamma(\alpha, x; \omega)$ for various values of α and x is given in Figs. 1–7 and Tables 1–7. Notice that the first row of each of the tables represents the value of the incomplete gamma function for $\alpha = -1.50, -1.00, -0.50, 0.00, 0.50, 1.00$ and 1.50, respectively, so that an easy comparison with the existing tables (or approximations) can be made. Similarly, the representation of $S_\Gamma(\alpha, x; \omega)$ for various values of α and x is given in Figs. 8–14 and Tables 8–14. It should be noted that the recurrence relations given by (49), (50) [cf. Theorem 4] can be exploited to calculate the numerical values of $C_\Gamma(\alpha, x; \omega)$ and $S_\Gamma(\alpha, x; \omega)$ for $\alpha = n$, $n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, with the help of the representations given in Tables 1–14.

Table 1
Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = -1.50)$

ω	$C_\Gamma(\alpha, 0.50; \omega)$	$C_\Gamma(\alpha, 0.75; \omega)$	$C_\Gamma(\alpha, 0.00; \omega)$	$C_\Gamma(\alpha, 1.25; \omega)$	$C_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.7499	0.2791	0.1265	0.0640	0.0348
0.0500	0.7478	0.2787	0.1264	0.0640	0.0348
0.1000	0.7416	0.2776	0.1261	0.0639	0.0348
0.1500	0.7314	0.2758	0.1256	0.0638	0.0347
0.2000	0.7171	0.2733	0.1249	0.0635	0.0346
0.2500	0.6989	0.2701	0.1241	0.0632	0.0345
0.3000	0.6770	0.2661	0.1230	0.0629	0.0344
0.3500	0.6514	0.2615	0.1218	0.0625	0.0342
0.4000	0.6223	0.2562	0.1203	0.0620	0.0340
0.4500	0.5899	0.2503	0.1187	0.0614	0.0338
0.5000	0.5546	0.2437	0.1169	0.0608	0.0336
0.5500	0.5163	0.2365	0.1150	0.0601	0.0333
0.6000	0.4756	0.2287	0.1128	0.0594	0.0330
0.6500	0.4325	0.2204	0.1105	0.0586	0.0327
0.7000	0.3875	0.2114	0.1080	0.0578	0.0324
0.7500	0.3408	0.2020	0.1054	0.0569	0.0320
0.8000	0.2926	0.1921	0.1026	0.0559	0.0316
0.8500	0.2434	0.1817	0.0997	0.0549	0.0312
0.9000	0.1935	0.1709	0.0966	0.0538	0.0308
0.9500	0.1431	0.1597	0.0933	0.0527	0.0304
1.0000	0.0926	0.1482	0.0900	0.0515	0.0299
1.2500	-0.1504	0.0863	0.0713	0.0449	0.0272
1.5000	-0.3509	0.0209	0.0503	0.0372	0.0241
1.7500	-0.4809	-0.0431	0.0277	0.0286	0.0205
2.0000	-0.5257	-0.1012	0.0046	0.0195	0.0166
2.2500	-0.4865	-0.1494	-0.0180	0.0099	0.0125
2.5000	-0.3787	-0.1846	-0.0392	0.0003	0.0081
2.7500	-0.2284	-0.2049	-0.0582	-0.0091	0.0037
3.0000	-0.0670	-0.2095	-0.0742	-0.0181	-0.0008
3.2500	0.0751	-0.1991	-0.0866	-0.0264	-0.0051
3.5000	0.1744	-0.1753	-0.0950	-0.0338	-0.0093
3.7500	0.2179	-0.1409	-0.0992	-0.0402	-0.0133
4.0000	0.2048	-0.0993	-0.0991	-0.0453	-0.0169
4.2500	0.1459	-0.0544	-0.0950	-0.0490	-0.0202
4.5000	0.0599	-0.0101	-0.0872	-0.0513	-0.0230
4.7500	-0.0309	0.0300	-0.0761	-0.0522	-0.0253
5.0000	-0.1054	0.0629	-0.0625	-0.0517	-0.0271
5.5000	-0.1525	0.0992	-0.0306	-0.0465	-0.0290
6.0000	-0.0623	0.0931	0.0022	-0.0368	-0.0287
6.5000	0.0695	0.0540	0.0298	-0.0239	-0.0262
7.0000	0.1274	0.0013	0.0478	-0.0096	-0.0218
7.5000	0.0729	-0.0432	0.0539	0.0044	-0.0160
8.0000	-0.0335	-0.0636	0.0482	0.0164	-0.0094
8.5000	-0.0954	-0.0550	0.0335	0.0252	-0.0025
9.0000	-0.0652	-0.0243	0.0136	0.0299	0.0040
9.5000	0.0208	0.0140	-0.0065	0.0303	0.0098
10.0000	0.0824	0.0439	-0.0225	0.0268	0.0142

Table 2
Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = -1.00)$

ω	$C_\Gamma(\alpha,0.50;\omega)$	$C_\Gamma(\alpha,0.75;\omega)$	$C_\Gamma(\alpha,0.00;\omega)$	$C_\Gamma(\alpha,1.25;\omega)$	$C_\Gamma(\alpha,1.50;\omega)$
0.0000	0.6533	0.2895	0.1485	0.0828	0.0487
0.0500	0.6516	0.2891	0.1484	0.0828	0.0487
0.1000	0.6467	0.2881	0.1481	0.0826	0.0487
0.1500	0.6385	0.2863	0.1475	0.0824	0.0486
0.2000	0.6271	0.2839	0.1468	0.0821	0.0485
0.2500	0.6125	0.2807	0.1458	0.0818	0.0483
0.3000	0.5950	0.2769	0.1446	0.0813	0.0481
0.3500	0.5745	0.2724	0.1433	0.0808	0.0479
0.4000	0.5512	0.2673	0.1417	0.0802	0.0476
0.4500	0.5252	0.2615	0.1399	0.0796	0.0474
0.5000	0.4968	0.2552	0.1379	0.0788	0.0470
0.5500	0.4662	0.2482	0.1357	0.0780	0.0467
0.6000	0.4334	0.2406	0.1333	0.0771	0.0463
0.6500	0.3987	0.2325	0.1308	0.0761	0.0459
0.7000	0.3624	0.2238	0.1280	0.0751	0.0454
0.7500	0.3247	0.2146	0.1251	0.0739	0.0450
0.8000	0.2858	0.2050	0.1220	0.0728	0.0444
0.8500	0.2459	0.1949	0.1187	0.0715	0.0439
0.9000	0.2053	0.1844	0.1152	0.0702	0.0433
0.9500	0.1643	0.1735	0.1116	0.0688	0.0427
1.0000	0.1231	0.1622	0.1079	0.0673	0.0421
1.2500	-0.0770	0.1016	0.0871	0.0591	0.0385
1.5000	-0.2466	0.0372	0.0635	0.0496	0.0343
1.7500	-0.3631	-0.0263	0.0382	0.0390	0.0295
2.0000	-0.4143	-0.0847	0.0122	0.0276	0.0242
2.2500	-0.3997	-0.1342	-0.0134	0.0157	0.0186
2.5000	-0.3297	-0.1717	-0.0377	0.0037	0.0127
2.7500	-0.2234	-0.1952	-0.0596	-0.0081	0.0067
3.0000	-0.1042	-0.2040	-0.0783	-0.0195	0.0007
3.2500	0.0049	-0.1983	-0.0933	-0.0301	-0.0053
3.5000	0.0854	-0.1796	-0.1039	-0.0396	-0.0111
3.7500	0.1265	-0.1504	-0.1100	-0.0478	-0.0165
4.0000	0.1264	-0.1138	-0.1115	-0.0546	-0.0216
4.2500	0.0917	-0.0732	-0.1085	-0.0598	-0.0261
4.5000	0.0354	-0.0321	-0.1013	-0.0632	-0.0301
4.7500	-0.0266	0.0058	-0.0905	-0.0649	-0.0335
5.0000	-0.0789	0.0380	-0.0768	-0.0648	-0.0361
5.5000	-0.1141	0.0768	-0.0436	-0.0597	-0.0392
6.0000	-0.0514	0.0776	-0.0085	-0.0489	-0.0393
6.5000	0.0440	0.0476	0.0222	-0.0341	-0.0364
7.0000	0.0891	0.0035	0.0433	-0.0171	-0.0311
7.5000	0.0539	-0.0353	0.0522	-0.0003	-0.0239
8.0000	-0.0207	-0.0542	0.0489	0.0146	-0.0154
8.5000	-0.0658	-0.0481	0.0356	0.0259	-0.0064
9.0000	-0.0460	-0.0221	0.0165	0.0326	0.0024
9.5000	0.0146	0.0112	-0.0036	0.0343	0.0101
10.0000	0.0592	0.0379	-0.0202	0.0312	0.0163

Table 3
Representation of the decomposition function $C_{\Gamma}(\alpha, x; \omega)$ $(\alpha = -0.50)$

ω	$C_{\Gamma}(\alpha, 0.50; \omega)$	$C_{\Gamma}(\alpha, 0.75; \omega)$	$C_{\Gamma}(\alpha, 0.00; \omega)$	$C_{\Gamma}(\alpha, 1.25; \omega)$	$C_{\Gamma}(\alpha, 1.50; \omega)$
0.0000	0.5907	0.3086	0.1781	0.1089	0.0692
0.0500	0.5894	0.3083	0.1780	0.1089	0.0692
0.1000	0.5853	0.3072	0.1777	0.1087	0.0691
0.1500	0.5787	0.3055	0.1771	0.1085	0.0690
0.2000	0.5694	0.3031	0.1762	0.1081	0.0688
0.2500	0.5576	0.3000	0.1751	0.1077	0.0686
0.3000	0.5433	0.2963	0.1738	0.1071	0.0684
0.3500	0.5267	0.2919	0.1723	0.1065	0.0681
0.4000	0.5077	0.2869	0.1705	0.1058	0.0677
0.4500	0.4866	0.2812	0.1685	0.1049	0.0673
0.5000	0.4635	0.2749	0.1662	0.1040	0.0669
0.5500	0.4384	0.2680	0.1638	0.1030	0.0664
0.6000	0.4116	0.2606	0.1611	0.1018	0.0659
0.6500	0.3833	0.2526	0.1582	0.1006	0.0653
0.7000	0.3535	0.2441	0.1551	0.0993	0.0647
0.7500	0.3226	0.2350	0.1518	0.0979	0.0641
0.8000	0.2906	0.2255	0.1483	0.0964	0.0634
0.8500	0.2577	0.2155	0.1446	0.0949	0.0627
0.9000	0.2242	0.2052	0.1408	0.0932	0.0619
0.9500	0.1902	0.1944	0.1367	0.0915	0.0611
1.0000	0.1560	0.1832	0.1325	0.0897	0.0602
1.2500	-0.0120	0.1231	0.1090	0.0794	0.0554
1.5000	-0.1585	0.0588	0.0824	0.0675	0.0496
1.7500	-0.2652	-0.0053	0.0537	0.0542	0.0431
2.0000	-0.3217	-0.0650	0.0240	0.0399	0.0360
2.2500	-0.3263	-0.1167	-0.0054	0.0250	0.0283
2.5000	-0.2860	-0.1573	-0.0335	0.0098	0.0202
2.7500	-0.2145	-0.1848	-0.0592	-0.0052	0.0120
3.0000	-0.1292	-0.1983	-0.0816	-0.0197	0.0037
3.2500	-0.0474	-0.1980	-0.0999	-0.0333	-0.0046
3.5000	0.0163	-0.1850	-0.1135	-0.0457	-0.0125
3.7500	0.0530	-0.1614	-0.1222	-0.0566	-0.0201
4.0000	0.0601	-0.1299	-0.1258	-0.0657	-0.0272
4.2500	0.0417	-0.0939	-0.1245	-0.0729	-0.0337
4.5000	0.0064	-0.0564	-0.1186	-0.0780	-0.0394
4.7500	-0.0345	-0.0208	-0.1085	-0.0809	-0.0442
5.0000	-0.0700	0.0102	-0.0950	-0.0817	-0.0482
5.5000	-0.0940	0.0508	-0.0609	-0.0772	-0.0531
6.0000	-0.0487	0.0579	-0.0235	-0.0656	-0.0541
6.5000	0.0218	0.0366	0.0103	-0.0486	-0.0512
7.0000	0.0579	0.0011	0.0349	-0.0288	-0.0448
7.5000	0.0366	-0.0317	0.0472	-0.0084	-0.0358
8.0000	-0.0146	-0.0483	0.0468	0.0099	-0.0249
8.5000	-0.0467	-0.0435	0.0356	0.0244	-0.0132
9.0000	-0.0332	-0.0212	0.0178	0.0337	-0.0016
9.5000	0.0098	0.0082	-0.0018	0.0373	0.0089
10.0000	0.0425	0.0324	-0.0184	0.0353	0.0175

Table 4
Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = 0.00)$

ω	$C_\Gamma(\alpha, 0.50; \omega)$	$C_\Gamma(\alpha, 0.75; \omega)$	$C_\Gamma(\alpha, 0.00; \omega)$	$C_\Gamma(\alpha, 1.25; \omega)$	$C_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.5598	0.3403	0.2194	0.1464	0.1000
0.0500	0.5587	0.3400	0.2192	0.1464	0.1000
0.1000	0.5553	0.3390	0.2188	0.1462	0.0999
0.1500	0.5498	0.3373	0.2182	0.1458	0.0997
0.2000	0.5422	0.3349	0.2172	0.1454	0.0995
0.2500	0.5324	0.3318	0.2160	0.1448	0.0992
0.3000	0.5206	0.3281	0.2145	0.1442	0.0989
0.3500	0.5068	0.3237	0.2127	0.1433	0.0985
0.4000	0.4911	0.3186	0.2107	0.1424	0.0980
0.4500	0.4735	0.3130	0.2084	0.1414	0.0975
0.5000	0.4543	0.3067	0.2059	0.1402	0.0969
0.5500	0.4335	0.2999	0.2031	0.1389	0.0962
0.6000	0.4112	0.2924	0.2000	0.1375	0.0955
0.6500	0.3875	0.2844	0.1967	0.1359	0.0947
0.7000	0.3626	0.2759	0.1932	0.1343	0.0939
0.7500	0.3367	0.2669	0.1894	0.1325	0.0930
0.8000	0.3098	0.2573	0.1855	0.1307	0.0921
0.8500	0.2822	0.2473	0.1813	0.1287	0.0911
0.9000	0.2539	0.2369	0.1768	0.1266	0.0900
0.9500	0.2252	0.2261	0.1722	0.1244	0.0889
1.0000	0.1961	0.2149	0.1674	0.1221	0.0877
1.2500	0.0518	0.1542	0.1406	0.1092	0.0811
1.5000	-0.0780	0.0889	0.1101	0.0942	0.0732
1.7500	-0.1782	0.0231	0.0770	0.0773	0.0643
2.0000	-0.2397	-0.0392	0.0426	0.0592	0.0544
2.2500	-0.2602	-0.0943	0.0083	0.0401	0.0439
2.5000	-0.2442	-0.1392	-0.0247	0.0207	0.0328
2.7500	-0.2011	-0.1719	-0.0553	0.0014	0.0213
3.0000	-0.1438	-0.1912	-0.0824	-0.0174	0.0098
3.2500	-0.0854	-0.1971	-0.1052	-0.0352	-0.0017
3.5000	-0.0374	-0.1906	-0.1230	-0.0515	-0.0129
3.7500	-0.0070	-0.1733	-0.1354	-0.0660	-0.0236
4.0000	0.0030	-0.1478	-0.1422	-0.0785	-0.0336
4.2500	-0.0055	-0.1169	-0.1434	-0.0886	-0.0428
4.5000	-0.0266	-0.0836	-0.1395	-0.0962	-0.0511
4.7500	-0.0527	-0.0510	-0.1308	-0.1012	-0.0583
5.0000	-0.0757	-0.0216	-0.1181	-0.1035	-0.0642
5.5000	-0.0901	0.0198	-0.0841	-0.1006	-0.0723
6.0000	-0.0555	0.0323	-0.0449	-0.0886	-0.0749
6.5000	-0.0018	0.0191	-0.0080	-0.0697	-0.0724
7.0000	0.0284	-0.0081	0.0205	-0.0466	-0.0651
7.5000	0.0175	-0.0345	0.0368	-0.0224	-0.0541
8.0000	-0.0160	-0.0479	0.0399	0.0003	-0.0403
8.5000	-0.0374	-0.0432	0.0316	0.0188	-0.0251
9.0000	-0.0272	-0.0232	0.0159	0.0316	-0.0098
9.5000	0.0043	0.0033	-0.0024	0.0378	0.0044
10.0000	0.0289	0.0257	-0.0185	0.0376	0.0164

Table 5
Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = 0.50)$

ω	$C_\Gamma(\alpha, 0.50; \omega)$	$C_\Gamma(\alpha, 0.75; \omega)$	$C_\Gamma(\alpha, 0.00; \omega)$	$C_\Gamma(\alpha, 1.25; \omega)$	$C_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.5624	0.3911	0.2788	0.2018	0.1476
0.0500	0.5615	0.3908	0.2786	0.2017	0.1475
0.1000	0.5587	0.3897	0.2782	0.2015	0.1474
0.1500	0.5540	0.3880	0.2774	0.2011	0.1472
0.2000	0.5475	0.3856	0.2763	0.2005	0.1469
0.2500	0.5392	0.3825	0.2749	0.1998	0.1465
0.3000	0.5292	0.3787	0.2731	0.1989	0.1460
0.3500	0.5175	0.3742	0.2711	0.1979	0.1455
0.4000	0.5042	0.3691	0.2688	0.1967	0.1448
0.4500	0.4893	0.3634	0.2661	0.1953	0.1441
0.5000	0.4729	0.3570	0.2632	0.1938	0.1433
0.5500	0.4551	0.3500	0.2600	0.1922	0.1424
0.6000	0.4361	0.3424	0.2565	0.1904	0.1414
0.6500	0.4158	0.3343	0.2527	0.1884	0.1403
0.7000	0.3945	0.3256	0.2486	0.1864	0.1391
0.7500	0.3722	0.3164	0.2442	0.1841	0.1379
0.8000	0.3491	0.3066	0.2396	0.1817	0.1366
0.8500	0.3252	0.2964	0.2348	0.1792	0.1352
0.9000	0.3007	0.2858	0.2297	0.1766	0.1337
0.9500	0.2758	0.2747	0.2243	0.1738	0.1322
1.0000	0.2505	0.2632	0.2187	0.1708	0.1306
1.2500	0.1229	0.2008	0.1876	0.1543	0.1214
1.5000	0.0045	0.1330	0.1520	0.1350	0.1105
1.7500	-0.0924	0.0639	0.1133	0.1134	0.0981
2.0000	-0.1596	-0.0025	0.0729	0.0900	0.0844
2.2500	-0.1944	-0.0626	0.0321	0.0654	0.0697
2.5000	-0.1992	-0.1135	-0.0075	0.0402	0.0542
2.7500	-0.1806	-0.1529	-0.0446	0.0149	0.0381
3.0000	-0.1477	-0.1796	-0.0782	-0.0097	0.0219
3.2500	-0.1105	-0.1933	-0.1071	-0.0333	0.0057
3.5000	-0.0778	-0.1946	-0.1307	-0.0552	-0.0102
3.7500	-0.0557	-0.1850	-0.1483	-0.0750	-0.0255
4.0000	-0.0469	-0.1667	-0.1598	-0.0923	-0.0399
4.2500	-0.0507	-0.1421	-0.1650	-0.1068	-0.0533
4.5000	-0.0634	-0.1142	-0.1644	-0.1182	-0.0655
4.7500	-0.0799	-0.0856	-0.1583	-0.1264	-0.0762
5.0000	-0.0945	-0.0588	-0.1475	-0.1314	-0.0853
5.5000	-0.1018	-0.0184	-0.1151	-0.1318	-0.0985
6.0000	-0.0741	-0.0019	-0.0751	-0.1207	-0.1044
6.5000	-0.0316	-0.0079	-0.0356	-0.1004	-0.1031
7.0000	-0.0049	-0.0272	-0.0033	-0.0741	-0.0954
7.5000	-0.0075	-0.0470	0.0174	-0.0455	-0.0823
8.0000	-0.0273	-0.0564	0.0247	-0.0179	-0.0651
8.5000	-0.0397	-0.0506	0.0203	0.0056	-0.0457
9.0000	-0.0302	-0.0315	0.0075	0.0230	-0.0255
9.5000	-0.0056	-0.0065	-0.0086	0.0330	-0.0064
10.0000	0.0143	0.0152	-0.0232	0.0354	0.0103

Table 6
Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = 1.00)$

ω	$C_\Gamma(\alpha, 0.50; \omega)$	$C_\Gamma(\alpha, 0.75; \omega)$	$C_\Gamma(\alpha, 0.00; \omega)$	$C_\Gamma(\alpha, 1.25; \omega)$	$C_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.6065	0.4724	0.3679	0.2865	0.2231
0.0500	0.6057	0.4720	0.3677	0.2864	0.2231
0.1000	0.6033	0.4709	0.3671	0.2861	0.2229
0.1500	0.5992	0.4691	0.3662	0.2856	0.2226
0.2000	0.5936	0.4666	0.3649	0.2849	0.2222
0.2500	0.5863	0.4634	0.3633	0.2839	0.2216
0.3000	0.5776	0.4594	0.3612	0.2828	0.2209
0.3500	0.5673	0.4548	0.3588	0.2815	0.2202
0.4000	0.5557	0.4495	0.3561	0.2799	0.2192
0.4500	0.5426	0.4435	0.3530	0.2782	0.2182
0.5000	0.5282	0.4369	0.3495	0.2762	0.2171
0.5500	0.5126	0.4296	0.3457	0.2741	0.2158
0.6000	0.4959	0.4218	0.3416	0.2718	0.2144
0.6500	0.4780	0.4133	0.3371	0.2693	0.2129
0.7000	0.4591	0.4042	0.3323	0.2665	0.2113
0.7500	0.4394	0.3946	0.3272	0.2636	0.2096
0.8000	0.4188	0.3844	0.3218	0.2606	0.2078
0.8500	0.3975	0.3737	0.3161	0.2573	0.2058
0.9000	0.3756	0.3626	0.3100	0.2538	0.2038
0.9500	0.3532	0.3510	0.3037	0.2502	0.2016
1.0000	0.3304	0.3389	0.2971	0.2464	0.1993
1.2500	0.2137	0.2731	0.2603	0.2250	0.1864
1.5000	0.1017	0.2010	0.2181	0.1999	0.1711
1.7500	0.0048	0.1266	0.1719	0.1717	0.1537
2.0000	-0.0698	0.0538	0.1233	0.1411	0.1344
2.2500	-0.1188	-0.0138	0.0739	0.1088	0.1136
2.5000	-0.1431	-0.0730	0.0254	0.0754	0.0916
2.7500	-0.1470	-0.1216	-0.0208	0.0419	0.0688
3.0000	-0.1370	-0.1581	-0.0633	0.0088	0.0457
3.2500	-0.1204	-0.1819	-0.1009	-0.0230	0.0224
3.5000	-0.1040	-0.1933	-0.1327	-0.0530	-0.0004
3.7500	-0.0927	-0.1936	-0.1581	-0.0805	-0.0226
4.0000	-0.0891	-0.1846	-0.1765	-0.1050	-0.0438
4.2500	-0.0933	-0.1684	-0.1881	-0.1261	-0.0636
4.5000	-0.1030	-0.1477	-0.1928	-0.1435	-0.0818
4.7500	-0.1149	-0.1249	-0.1913	-0.1570	-0.0981
5.0000	-0.1253	-0.1025	-0.1841	-0.1664	-0.1124
5.5000	-0.1295	-0.0661	-0.1564	-0.1732	-0.1339
6.0000	-0.1074	-0.0482	-0.1178	-0.1652	-0.1458
6.5000	-0.0731	-0.0489	-0.0769	-0.1450	-0.1478
7.0000	-0.0488	-0.0613	-0.0415	-0.1164	-0.1408
7.5000	-0.0449	-0.0747	-0.0166	-0.0835	-0.1261
8.0000	-0.0537	-0.0796	-0.0045	-0.0504	-0.1054
8.5000	-0.0580	-0.0716	-0.0044	-0.0210	-0.0809
9.0000	-0.0472	-0.0519	-0.0130	0.0021	-0.0548
9.5000	-0.0259	-0.0269	-0.0258	0.0171	-0.0293
10.0000	-0.0078	-0.0046	-0.0377	0.0236	-0.0063

Table 7
Representation of the decomposition function $C_\Gamma(\alpha, x; \omega)$ $(\alpha = 1.50)$

ω	$C_\Gamma(\alpha, 0.50; \omega)$	$C_\Gamma(\alpha, 0.75; \omega)$	$C_\Gamma(\alpha, 0.00; \omega)$	$C_\Gamma(\alpha, 1.25; \omega)$	$C_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.7101	0.6046	0.5073	0.4212	0.3471
0.0500	0.7094	0.6043	0.5071	0.4211	0.3470
0.1000	0.7071	0.6031	0.5064	0.4207	0.3467
0.1500	0.7035	0.6012	0.5053	0.4200	0.3463
0.2000	0.6983	0.5985	0.5037	0.4190	0.3457
0.2500	0.6918	0.5950	0.5017	0.4178	0.3449
0.3000	0.6839	0.5909	0.4993	0.4163	0.3440
0.3500	0.6746	0.5859	0.4964	0.4146	0.3428
0.4000	0.6640	0.5802	0.4931	0.4125	0.3416
0.4500	0.6521	0.5739	0.4894	0.4103	0.3401
0.5000	0.6390	0.5668	0.4853	0.4077	0.3385
0.5500	0.6247	0.5590	0.4807	0.4049	0.3367
0.6000	0.6094	0.5506	0.4757	0.4018	0.3347
0.6500	0.5930	0.5415	0.4704	0.3985	0.3326
0.7000	0.5757	0.5317	0.4646	0.3949	0.3303
0.7500	0.5575	0.5214	0.4584	0.3911	0.3278
0.8000	0.5384	0.5105	0.4519	0.3870	0.3252
0.8500	0.5187	0.4990	0.4450	0.3827	0.3225
0.9000	0.4983	0.4870	0.4377	0.3782	0.3195
0.9500	0.4773	0.4744	0.4301	0.3734	0.3165
1.0000	0.4559	0.4614	0.4221	0.3684	0.3132
1.2500	0.3446	0.3900	0.3776	0.3401	0.2948
1.5000	0.2339	0.3109	0.3262	0.3068	0.2730
1.7500	0.1329	0.2282	0.2698	0.2693	0.2481
2.0000	0.0483	0.1457	0.2099	0.2285	0.2205
2.2500	-0.0165	0.0673	0.1486	0.1851	0.1906
2.5000	-0.0612	-0.0039	0.0876	0.1402	0.1590
2.7500	-0.0882	-0.0653	0.0286	0.0947	0.1261
3.0000	-0.1018	-0.1153	-0.0266	0.0495	0.0924
3.2500	-0.1073	-0.1529	-0.0768	0.0055	0.0585
3.5000	-0.1098	-0.1782	-0.1207	-0.0364	0.0250
3.7500	-0.1132	-0.1921	-0.1576	-0.0754	-0.0079
4.0000	-0.1197	-0.1958	-0.1869	-0.1110	-0.0394
4.2500	-0.1298	-0.1915	-0.2083	-0.1424	-0.0693
4.5000	-0.1424	-0.1813	-0.2220	-0.1693	-0.0970
4.7500	-0.1555	-0.1676	-0.2282	-0.1913	-0.1223
5.0000	-0.1664	-0.1525	-0.2278	-0.2083	-0.1449
5.5000	-0.1739	-0.1258	-0.2100	-0.2270	-0.1808
6.0000	-0.1595	-0.1110	-0.1772	-0.2266	-0.2034
6.5000	-0.1335	-0.1101	-0.1383	-0.2099	-0.2127
7.0000	-0.1122	-0.1180	-0.1018	-0.1811	-0.2092
7.5000	-0.1040	-0.1263	-0.0738	-0.1451	-0.1946
8.0000	-0.1044	-0.1271	-0.0574	-0.1069	-0.1710
8.5000	-0.1016	-0.1162	-0.0524	-0.0712	-0.1413
9.0000	-0.0880	-0.0948	-0.0562	-0.0415	-0.1082
9.5000	-0.0669	-0.0682	-0.0644	-0.0201	-0.0748
10.0000	-0.0479	-0.0436	-0.0722	-0.0079	-0.0435

Table 8
Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = -1.50)$

ω	$S_\Gamma(\alpha, 0.50; \omega)$	$S_\Gamma(\alpha, 0.75; \omega)$	$S_\Gamma(\alpha, 0.00; \omega)$	$S_\Gamma(\alpha, 1.25; \omega)$	$S_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0500	0.0536	0.0138	0.0048	0.0020	0.0009
0.1000	0.1068	0.0276	0.0096	0.0040	0.0018
0.1500	0.1594	0.0413	0.0144	0.0060	0.0028
0.2000	0.2109	0.0548	0.0192	0.0080	0.0037
0.2500	0.2612	0.0682	0.0240	0.0100	0.0046
0.3000	0.3098	0.0814	0.0287	0.0119	0.0055
0.3500	0.3565	0.0944	0.0333	0.0139	0.0064
0.4000	0.4010	0.1071	0.0379	0.0158	0.0073
0.4500	0.4430	0.1195	0.0425	0.0177	0.0082
0.5000	0.4823	0.1316	0.0469	0.0196	0.0091
0.5500	0.5187	0.1433	0.0513	0.0215	0.0100
0.6000	0.5518	0.1545	0.0556	0.0234	0.0109
0.6500	0.5816	0.1654	0.0599	0.0252	0.0117
0.7000	0.6079	0.1757	0.0640	0.0270	0.0126
0.7500	0.6305	0.1856	0.0680	0.0288	0.0134
0.8000	0.6493	0.1949	0.0719	0.0305	0.0143
0.8500	0.6642	0.2037	0.0756	0.0322	0.0151
0.9000	0.6753	0.2119	0.0793	0.0339	0.0159
0.9500	0.6823	0.2195	0.0828	0.0355	0.0167
1.0000	0.6854	0.2265	0.0862	0.0371	0.0175
1.2500	0.6433	0.2517	0.1009	0.0445	0.0212
1.5000	0.5163	0.2596	0.1115	0.0506	0.0246
1.7500	0.3304	0.2503	0.1176	0.0554	0.0274
2.0000	0.1201	0.2250	0.1190	0.0587	0.0298
2.2500	-0.0786	0.1862	0.1158	0.0605	0.0315
2.5000	-0.2342	0.1373	0.1082	0.0607	0.0327
2.7500	-0.3257	0.0824	0.0966	0.0593	0.0332
3.0000	-0.3453	0.0259	0.0817	0.0565	0.0332
3.2500	-0.2990	-0.0278	0.0642	0.0522	0.0325
3.5000	-0.2047	-0.0749	0.0449	0.0467	0.0313
3.7500	-0.0871	-0.1121	0.0247	0.0401	0.0294
4.0000	0.0270	-0.1372	0.0044	0.0327	0.0271
4.2500	0.1145	-0.1492	-0.0149	0.0246	0.0243
4.5000	0.1603	-0.1480	-0.0325	0.0161	0.0212
4.7500	0.1596	-0.1349	-0.0477	0.0075	0.0176
5.0000	0.1180	-0.1119	-0.0599	-0.0010	0.0139
5.5000	-0.0278	-0.0481	-0.0739	-0.0167	0.0059
6.0000	-0.1370	0.0177	-0.0734	-0.0293	-0.0021
6.5000	-0.1229	0.0625	-0.0600	-0.0377	-0.0094
7.0000	-0.0133	0.0738	-0.0376	-0.0410	-0.0156
7.5000	0.0883	0.0526	-0.0115	-0.0394	-0.0202
8.0000	0.0984	0.0120	0.0127	-0.0334	-0.0229
8.5000	0.0204	-0.0298	0.0305	-0.0241	-0.0236
9.0000	-0.0681	-0.0558	0.0391	-0.0129	-0.0224
9.5000	-0.0893	-0.0574	0.0376	-0.0013	-0.0194
10.0000	-0.0315	-0.0361	0.0276	0.0090	-0.0150

Table 9
Representation of the decomposition function $S_{\Gamma}(\alpha, x; \omega)$ $(\alpha = -1.00)$

ω	$S_{\Gamma}(\alpha, 0.50; \omega)$	$S_{\Gamma}(\alpha, 0.75; \omega)$	$S_{\Gamma}(\alpha, 0.00; \omega)$	$S_{\Gamma}(\alpha, 1.25; \omega)$	$S_{\Gamma}(\alpha, 1.50; \omega)$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0500	0.0443	0.0138	0.0055	0.0025	0.0013
0.1000	0.0883	0.0275	0.0110	0.0050	0.0025
0.1500	0.1318	0.0411	0.0164	0.0075	0.0038
0.2000	0.1745	0.0546	0.0218	0.0100	0.0050
0.2500	0.2162	0.0680	0.0272	0.0125	0.0063
0.3000	0.2566	0.0812	0.0326	0.0150	0.0075
0.3500	0.2955	0.0942	0.0379	0.0174	0.0088
0.4000	0.3327	0.1069	0.0431	0.0199	0.0100
0.4500	0.3680	0.1193	0.0483	0.0223	0.0112
0.5000	0.4011	0.1314	0.0534	0.0247	0.0125
0.5500	0.4318	0.1431	0.0584	0.0271	0.0137
0.6000	0.4601	0.1544	0.0633	0.0294	0.0149
0.6500	0.4858	0.1653	0.0681	0.0317	0.0160
0.7000	0.5087	0.1758	0.0728	0.0340	0.0172
0.7500	0.5286	0.1858	0.0774	0.0362	0.0184
0.8000	0.5457	0.1953	0.0819	0.0384	0.0195
0.8500	0.5596	0.2043	0.0862	0.0406	0.0207
0.9000	0.5705	0.2127	0.0904	0.0427	0.0218
0.9500	0.5782	0.2205	0.0945	0.0448	0.0229
1.0000	0.5828	0.2278	0.0984	0.0468	0.0240
1.2500	0.5597	0.2547	0.1155	0.0562	0.0291
1.5000	0.4681	0.2651	0.1280	0.0641	0.0337
1.7500	0.3273	0.2588	0.1357	0.0703	0.0377
2.0000	0.1633	0.2369	0.1382	0.0747	0.0409
2.2500	0.0040	0.2017	0.1355	0.0773	0.0435
2.5000	-0.1261	0.1563	0.1280	0.0779	0.0452
2.7500	-0.2098	0.1044	0.1161	0.0765	0.0461
3.0000	-0.2398	0.0502	0.1003	0.0734	0.0462
3.2500	-0.2193	-0.0022	0.0815	0.0685	0.0454
3.5000	-0.1604	-0.0491	0.0605	0.0620	0.0439
3.7500	-0.0808	-0.0874	0.0383	0.0542	0.0416
4.0000	-0.0002	-0.1148	0.0160	0.0452	0.0387
4.2500	0.0640	-0.1303	-0.0056	0.0355	0.0351
4.5000	0.0999	-0.1336	-0.0256	0.0252	0.0309
4.7500	0.1031	-0.1256	-0.0432	0.0147	0.0263
5.0000	0.0767	-0.1081	-0.0577	0.0042	0.0213
5.5000	-0.0240	-0.0547	-0.0758	-0.0155	0.0108
6.0000	-0.1032	0.0034	-0.0785	-0.0317	0.0000
6.5000	-0.0959	0.0452	-0.0671	-0.0430	-0.0100
7.0000	-0.0188	0.0584	-0.0455	-0.0484	-0.0187
7.5000	0.0554	0.0429	-0.0192	-0.0478	-0.0253
8.0000	0.0659	0.0091	0.0061	-0.0419	-0.0294
8.5000	0.0128	-0.0268	0.0255	-0.0318	-0.0309
9.0000	-0.0497	-0.0501	0.0358	-0.0191	-0.0299
9.5000	-0.0658	-0.0523	0.0359	-0.0058	-0.0265
10.0000	-0.0255	-0.0345	0.0271	0.0065	-0.0213

Table 10
Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = -0.50)$

ω	$S_\Gamma(\alpha, 0.50; \omega)$	$S_\Gamma(\alpha, 0.75; \omega)$	$S_\Gamma(\alpha, 0.00; \omega)$	$S_\Gamma(\alpha, 1.25; \omega)$	$S_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0500	0.0375	0.0139	0.0063	0.0032	0.0017
0.1000	0.0747	0.0279	0.0126	0.0064	0.0035
0.1500	0.1116	0.0417	0.0189	0.0096	0.0052
0.2000	0.1478	0.0554	0.0252	0.0128	0.0070
0.2500	0.1832	0.0690	0.0314	0.0159	0.0087
0.3000	0.2176	0.0824	0.0376	0.0191	0.0104
0.3500	0.2508	0.0956	0.0437	0.0222	0.0121
0.4000	0.2827	0.1086	0.0498	0.0253	0.0138
0.4500	0.3130	0.1212	0.0558	0.0284	0.0155
0.5000	0.3416	0.1336	0.0616	0.0315	0.0172
0.5500	0.3684	0.1456	0.0674	0.0345	0.0189
0.6000	0.3932	0.1572	0.0731	0.0375	0.0205
0.6500	0.4159	0.1685	0.0787	0.0404	0.0222
0.7000	0.4364	0.1793	0.0842	0.0434	0.0238
0.7500	0.4547	0.1896	0.0895	0.0462	0.0254
0.8000	0.4705	0.1994	0.0947	0.0490	0.0270
0.8500	0.4839	0.2088	0.0998	0.0518	0.0286
0.9000	0.4948	0.2176	0.1047	0.0545	0.0301
0.9500	0.5032	0.2259	0.1094	0.0572	0.0317
1.0000	0.5091	0.2336	0.1140	0.0598	0.0332
1.2500	0.5014	0.2630	0.1342	0.0719	0.0403
1.5000	0.4375	0.2764	0.1495	0.0821	0.0467
1.7500	0.3319	0.2736	0.1592	0.0904	0.0523
2.0000	0.2042	0.2554	0.1633	0.0964	0.0570
2.2500	0.0760	0.2238	0.1616	0.1001	0.0606
2.5000	-0.0333	0.1818	0.1544	0.1014	0.0632
2.7500	-0.1098	0.1328	0.1422	0.1003	0.0647
3.0000	-0.1466	0.0806	0.1256	0.0969	0.0651
3.2500	-0.1449	0.0293	0.1054	0.0913	0.0643
3.5000	-0.1126	-0.0178	0.0826	0.0837	0.0625
3.7500	-0.0624	-0.0575	0.0582	0.0745	0.0597
4.0000	-0.0085	-0.0876	0.0334	0.0637	0.0559
4.2500	0.0362	-0.1069	0.0090	0.0519	0.0512
4.5000	0.0622	-0.1149	-0.0138	0.0394	0.0458
4.7500	0.0657	-0.1123	-0.0343	0.0264	0.0398
5.0000	0.0482	-0.1005	-0.0517	0.0134	0.0332
5.5000	-0.0227	-0.0582	-0.0751	-0.0114	0.0191
6.0000	-0.0807	-0.0084	-0.0821	-0.0324	0.0046
6.5000	-0.0780	0.0294	-0.0738	-0.0477	-0.0092
7.0000	-0.0241	0.0433	-0.0539	-0.0561	-0.0213
7.5000	0.0301	0.0321	-0.0280	-0.0573	-0.0308
8.0000	0.0400	0.0042	-0.0020	-0.0520	-0.0372
8.5000	0.0043	-0.0266	0.0187	-0.0415	-0.0402
9.0000	-0.0396	-0.0471	0.0306	-0.0275	-0.0398
9.5000	-0.0514	-0.0497	0.0323	-0.0123	-0.0363
10.0000	-0.0230	-0.0346	0.0248	0.0021	-0.0302

Table 11
Representation of the decomposition function $S_{\Gamma}(\alpha, x; \omega)$ $(\alpha = 0.00)$

ω	$S_{\Gamma}(\alpha,0.50;\omega)$	$S_{\Gamma}(\alpha,0.75;\omega)$	$S_{\Gamma}(\alpha,0.00;\omega)$	$S_{\Gamma}(\alpha,1.25;\omega)$	$S_{\Gamma}(\alpha,1.50;\omega)$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0500	0.0326	0.0145	0.0074	0.0041	0.0024
0.1000	0.0651	0.0289	0.0148	0.0083	0.0049
0.1500	0.0973	0.0433	0.0222	0.0124	0.0073
0.2000	0.1289	0.0575	0.0296	0.0165	0.0097
0.2500	0.1599	0.0716	0.0369	0.0206	0.0121
0.3000	0.1901	0.0856	0.0442	0.0247	0.0146
0.3500	0.2194	0.0993	0.0514	0.0287	0.0170
0.4000	0.2475	0.1128	0.0585	0.0328	0.0193
0.4500	0.2744	0.1260	0.0655	0.0368	0.0217
0.5000	0.3000	0.1390	0.0725	0.0407	0.0241
0.5500	0.3241	0.1515	0.0793	0.0446	0.0264
0.6000	0.3466	0.1638	0.0860	0.0485	0.0288
0.6500	0.3674	0.1756	0.0926	0.0524	0.0311
0.7000	0.3864	0.1870	0.0991	0.0561	0.0333
0.7500	0.4036	0.1980	0.1054	0.0599	0.0356
0.8000	0.4189	0.2085	0.1116	0.0635	0.0378
0.8500	0.4322	0.2185	0.1176	0.0671	0.0400
0.9000	0.4434	0.2279	0.1235	0.0707	0.0422
0.9500	0.4527	0.2369	0.1291	0.0742	0.0444
1.0000	0.4599	0.2453	0.1346	0.0776	0.0465
1.2500	0.4653	0.2784	0.1591	0.0934	0.0566
1.5000	0.4239	0.2958	0.1779	0.1070	0.0657
1.7500	0.3464	0.2971	0.1907	0.1181	0.0737
2.0000	0.2478	0.2830	0.1970	0.1264	0.0804
2.2500	0.1448	0.2554	0.1968	0.1318	0.0858
2.5000	0.0527	0.2169	0.1904	0.1343	0.0897
2.7500	-0.0170	0.1708	0.1782	0.1337	0.0921
3.0000	-0.0580	0.1206	0.1609	0.1303	0.0931
3.2500	-0.0700	0.0700	0.1393	0.1240	0.0925
3.5000	-0.0580	0.0225	0.1146	0.1153	0.0904
3.7500	-0.0306	-0.0190	0.0878	0.1044	0.0870
4.0000	0.0018	-0.0521	0.0600	0.0915	0.0822
4.2500	0.0297	-0.0755	0.0324	0.0772	0.0762
4.5000	0.0459	-0.0886	0.0061	0.0618	0.0692
4.7500	0.0469	-0.0918	-0.0180	0.0457	0.0612
5.0000	0.0334	-0.0862	-0.0389	0.0295	0.0525
5.5000	-0.0193	-0.0560	-0.0693	-0.0019	0.0335
6.0000	-0.0638	-0.0159	-0.0822	-0.0293	0.0138
6.5000	-0.0651	0.0164	-0.0785	-0.0501	-0.0053
7.0000	-0.0285	0.0294	-0.0616	-0.0630	-0.0222
7.5000	0.0102	0.0208	-0.0372	-0.0673	-0.0361
8.0000	0.0186	-0.0025	-0.0115	-0.0636	-0.0459
8.5000	-0.0054	-0.0291	0.0100	-0.0533	-0.0513
9.0000	-0.0360	-0.0473	0.0231	-0.0384	-0.0523
9.5000	-0.0444	-0.0500	0.0263	-0.0215	-0.0491
10.0000	-0.0242	-0.0372	0.0201	-0.0050	-0.0425

Table 12
Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = 0.50)$

ω	$S_\Gamma(\alpha, 0.50; \omega)$	$S_\Gamma(\alpha, 0.75; \omega)$	$S_\Gamma(\alpha, 0.00; \omega)$	$S_\Gamma(\alpha, 1.25; \omega)$	$S_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0C00
0.0500	0.0295	0.0154	0.0089	0.0054	0.0035
0.1000	0.0589	0.0308	0.0178	0.0109	0.0069
0.1500	0.0880	0.0461	0.0267	0.0163	0.0104
0.2000	0.1167	0.0614	0.0355	0.0217	0.0138
0.2500	0.1449	0.0764	0.0443	0.0271	0.0173
0.3000	0.1724	0.0914	0.0530	0.0325	0.0207
0.3500	0.1992	0.1061	0.0617	0.0378	0.0241
0.4000	0.2251	0.1205	0.0702	0.0432	0.0275
0.4500	0.2499	0.1347	0.0787	0.0484	0.0309
0.5000	0.2737	0.1486	0.0871	0.0536	0.0342
0.5500	0.2962	0.1622	0.0953	0.0588	0.0376
0.6000	0.3175	0.1754	0.1035	0.0639	0.0409
0.6500	0.3374	0.1883	0.1114	0.0690	0.0441
0.7000	0.3558	0.2007	0.1193	0.0740	0.0474
0.7500	0.3727	0.2127	0.1269	0.0789	0.0506
0.8000	0.3880	0.2242	0.1345	0.0838	0.0538
0.8500	0.4018	0.2352	0.1418	0.0886	0.0570
0.9000	0.4138	0.2457	0.1489	0.0933	0.0601
0.9500	0.4242	0.2557	0.1559	0.0979	0.0631
1.0000	0.4328	0.2652	0.1626	0.1024	0.0662
1.2500	0.4506	0.3036	0.1928	0.1236	0.0806
1.5000	0.4286	0.3264	0.2168	0.1420	0.0938
1.7500	0.3747	0.3330	0.2339	0.1572	0.1054
2.0000	0.3002	0.3241	0.2436	0.1690	0.1153
2.2500	0.2182	0.3012	0.2459	0.1772	0.1233
2.5000	0.1409	0.2667	0.2410	0.1815	0.1294
2.7500	0.0778	0.2236	0.2293	0.1821	0.1334
3.0000	0.0348	0.1754	0.2117	0.1789	0.1354
3.2500	0.0129	0.1256	0.1889	0.1723	0.1353
3.5000	0.0095	0.0775	0.1621	0.1624	0.1331
3.7500	0.0188	0.0340	0.1326	0.1496	0.1290
4.0000	0.0335	-0.0025	0.1014	0.1342	0.1231
4.2500	0.0467	-0.0306	0.0700	0.1169	0.1155
4.5000	0.0530	-0.0493	0.0396	0.0980	0.1063
4.7500	0.0495	-0.0589	0.0111	0.0781	0.0959
5.0000	0.0362	-0.0601	-0.0144	0.0577	0.0843
5.5000	-0.0079	-0.0436	-0.0537	0.0176	0.0588
6.0000	-0.0459	-0.0150	-0.0748	-0.0183	0.0319
6.5000	-0.0523	0.0096	-0.0778	-0.0471	0.0054
7.0000	-0.0300	0.0193	-0.0660	-0.0665	-0.0188
7.5000	-0.0043	0.0112	-0.0449	-0.0758	-0.0390
8.0000	0.0014	-0.0097	-0.0209	-0.0753	-0.0542
8.5000	-0.0155	-0.0335	0.0000	-0.0665	-0.0638
9.0000	-0.0374	-0.0502	0.0136	-0.0517	-0.0675
9.5000	-0.0437	-0.0535	0.0175	-0.0337	-0.0656
10.0000	-0.0294	-0.0429	0.0123	-0.0153	-0.0589

Table 13
Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = 1.00)$

ω	$S_\Gamma(\alpha, 0.50; \omega)$	$S_\Gamma(\alpha, 0.75; \omega)$	$S_\Gamma(\alpha, 0.00; \omega)$	$S_\Gamma(\alpha, 1.25; \omega)$	$S_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0500	0.0280	0.0170	0.0110	0.0073	0.0050
0.1000	0.0558	0.0340	0.0219	0.0146	0.0100
0.1500	0.0835	0.0509	0.0328	0.0219	0.0150
0.2000	0.1108	0.0677	0.0437	0.0292	0.0200
0.2500	0.1377	0.0844	0.0546	0.0365	0.0249
0.3000	0.1640	0.1009	0.0653	0.0437	0.0299
0.3500	0.1897	0.1172	0.0760	0.0509	0.0348
0.4000	0.2146	0.1332	0.0866	0.0580	0.0397
0.4500	0.2388	0.1490	0.0971	0.0651	0.0446
0.5000	0.2620	0.1645	0.1074	0.0722	0.0495
0.5500	0.2842	0.1797	0.1176	0.0791	0.0543
0.6000	0.3053	0.1945	0.1277	0.0861	0.0591
0.6500	0.3252	0.2089	0.1376	0.0929	0.0639
0.7000	0.3440	0.2229	0.1474	0.0996	0.0686
0.7500	0.3615	0.2365	0.1570	0.1063	0.0733
0.8000	0.3777	0.2496	0.1663	0.1129	0.0779
0.8500	0.3925	0.2622	0.1755	0.1194	0.0825
0.9000	0.4059	0.2743	0.1845	0.1258	0.0870
0.9500	0.4178	0.2859	0.1932	0.1320	0.0915
1.0000	0.4284	0.2969	0.2017	0.1382	0.0959
1.2500	0.4592	0.3432	0.2403	0.1672	0.1170
1.5000	0.4555	0.3737	0.2717	0.1926	0.1363
1.7500	0.4227	0.3876	0.2951	0.2141	0.1535
2.0000	0.3696	0.3855	0.3100	0.2312	0.1684
2.2500	0.3063	0.3686	0.3164	0.2436	0.1807
2.5000	0.2426	0.3392	0.3143	0.2512	0.1903
2.7500	0.1865	0.3001	0.3043	0.2540	0.1970
3.0000	0.1432	0.2544	0.2870	0.2520	0.2009
3.2500	0.1147	0.2056	0.2634	0.2454	0.2019
3.5000	0.0996	0.1569	0.2348	0.2345	0.2001
3.7500	0.0945	0.1112	0.2024	0.2198	0.1956
4.0000	0.0944	0.0709	0.1675	0.2017	0.1884
4.2500	0.0945	0.0378	0.1317	0.1807	0.1788
4.5000	0.0906	0.0127	0.0963	0.1576	0.1671
4.7500	0.0807	-0.0041	0.0624	0.1328	0.1534
5.0000	0.0645	-0.0131	0.0312	0.1072	0.1380
5.5000	0.0209	-0.0124	-0.0197	0.0558	0.1037
6.0000	-0.0171	0.0018	-0.0520	0.0081	0.0667
6.5000	-0.0313	0.0156	-0.0650	-0.0317	0.0296
7.0000	-0.0230	0.0186	-0.0614	-0.0609	-0.0049
7.5000	-0.0100	0.0076	-0.0466	-0.0782	-0.0349
8.0000	-0.0094	-0.0137	-0.0269	-0.0836	-0.0585
8.5000	-0.0238	-0.0372	-0.0086	-0.0786	-0.0749
9.0000	-0.0413	-0.0543	0.0035	-0.0657	-0.0836
9.5000	-0.0474	-0.0593	0.0069	-0.0481	-0.0849
10.0000	-0.0382	-0.0517	0.0014	-0.0290	-0.0796

Table 14
Representation of the decomposition function $S_\Gamma(\alpha, x; \omega)$ $(\alpha = 1.50)$

ω	$S_\Gamma(\alpha, 0.50; \omega)$	$S_\Gamma(\alpha, 0.75; \omega)$	$S_\Gamma(\alpha, 0.00; \omega)$	$S_\Gamma(\alpha, 1.25; \omega)$	$S_\Gamma(\alpha, 1.50; \omega)$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0500	0.0281	0.0196	0.0139	0.0101	0.0074
0.1000	0.0561	0.0391	0.0279	0.0202	0.0148
0.1500	0.0839	0.0585	0.0417	0.0302	0.0221
0.2000	0.1115	0.0779	0.0556	0.0403	0.0295
0.2500	0.1387	0.0971	0.0694	0.0503	0.0368
0.3000	0.1654	0.1161	0.0831	0.0602	0.0441
0.3500	0.1916	0.1349	0.0967	0.0702	0.0514
0.4000	0.2171	0.1535	0.1102	0.0800	0.0587
0.4500	0.2419	0.1718	0.1236	0.0898	0.0659
0.5000	0.2660	0.1898	0.1368	0.0996	0.0731
0.5500	0.2892	0.2075	0.1499	0.1092	0.0802
0.6000	0.3115	0.2248	0.1628	0.1188	0.0873
0.6500	0.3328	0.2417	0.1755	0.1283	0.0943
0.7000	0.3531	0.2582	0.1880	0.1376	0.1013
0.7500	0.3722	0.2743	0.2004	0.1469	0.1083
0.8000	0.3903	0.2899	0.2125	0.1560	0.1151
0.8500	0.4071	0.3049	0.2243	0.1651	0.1219
0.9000	0.4228	0.3195	0.2359	0.1740	0.1286
0.9500	0.4372	0.3335	0.2473	0.1827	0.1353
1.0000	0.4504	0.3470	0.2584	0.1913	0.1419
1.2500	0.4970	0.4051	0.3093	0.2320	0.1734
1.5000	0.5126	0.4469	0.3518	0.2683	0.2024
1.7500	0.5010	0.4715	0.3850	0.2994	0.2285
2.0000	0.4689	0.4791	0.4083	0.3248	0.2513
2.2500	0.4239	0.4708	0.4214	0.3443	0.2706
2.5000	0.3742	0.4486	0.4244	0.3575	0.2861
2.7500	0.3263	0.4150	0.4179	0.3644	0.2977
3.0000	0.2851	0.3732	0.4024	0.3650	0.3052
3.2500	0.2528	0.3263	0.3792	0.3596	0.3086
3.5000	0.2294	0.2775	0.3493	0.3485	0.3080
3.7500	0.2130	0.2299	0.3143	0.3322	0.3036
4.0000	0.2005	0.1857	0.2757	0.3112	0.2954
4.2500	0.1885	0.1470	0.2349	0.2862	0.2837
4.5000	0.1744	0.1150	0.1936	0.2580	0.2688
4.7500	0.1565	0.0900	0.1532	0.2274	0.2511
5.0000	0.1346	0.0720	0.1149	0.1951	0.2308
5.5000	0.0840	0.0536	0.0487	0.1288	0.1846
6.0000	0.0388	0.0496	0.0010	0.0652	0.1336
6.5000	0.0125	0.0479	-0.0265	0.0096	0.0814
7.0000	0.0045	0.0394	-0.0358	-0.0342	0.0316
7.5000	0.0026	0.0207	-0.0317	-0.0641	-0.0131
8.0000	-0.0059	-0.0058	-0.0206	-0.0798	-0.0500
8.5000	-0.0234	-0.0332	-0.0089	-0.0827	-0.0778
9.0000	-0.0418	-0.0542	-0.0018	-0.0753	-0.0956
9.5000	-0.0511	-0.0637	-0.0020	-0.0610	-0.1035
10.0000	-0.0483	-0.0613	-0.0101	-0.0436	-0.1024

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