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On the nullity and the matching number of unicyclic graphs

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ABSTRACT

Let G be a graph with n vertices and $\nu(G)$ be the matching number of G. Let $\eta(G)$ denote the nullity of G (the multiplicity of the eigenvalue zero of G). It is well known that if G is a tree, then $\eta(G) = n - 2\nu(G)$. Tan and Liu [X. Tan, B. Liu, On the nullity of unicyclic graphs, Linear Alg. Appl. 408 (2005) 212–220] proved that the nullity set of all unicyclic graphs with n vertices is $\{0, 1, \ldots, n-4\}$ and characterized the unicyclic graphs with $\eta(G) = n-4$. In this paper, we characterize the unicyclic graphs with $\eta(G) = n-5$, and we prove that if G is a unicyclic graph, then $\eta(G)$ equals $n-2\nu(G)-1$, $n-2\nu(G)$, or $n-2\nu(G)+2$. We also give a characterization of these three types of graphs. Furthermore, we determine the unicyclic graphs G with $\eta(G) = 0$, which answers affirmatively an open problem by Tan and Liu.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. A set M of edges in G is a matching if every vertex of G is incident with at most one edge in M. It is a perfect matching if every vertex of G is incident with exactly one edge in M. We denote by $m(G, i)$ the number of matchings of G with i edges and by $\nu(G)$ the matching number of G (i.e., the number of edges of a maximum matching in G). A subgraph Λ of G is an elementary subgraph if each component of Λ is a single edge or a cycle. We use $r(\Lambda)$ (resp. $s(\Lambda)$) to denote the number of even components

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(resp. the number of cycles) in an elementary subgraph Λ. A unicyclic graph is a connected graph with equal number of vertices and edges. Obviously, there exists exactly one cycle in a unicyclic graph. Denote by K_n, C_n, P_n, and S_n the complete graph, the cycle, the path, and the star $K_{1,n-1}$ with n vertices, respectively.

The adjacency matrix of a graph G with n vertices, denoted by $A(G) = (a_{ij})_{n\times n}$, is the $n \times n$ symmetric matrix such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent and 0 otherwise. The characteristic polynomial of G, denoted by $\phi(G, x)$, is defined as $\det(xI_n - A(G))$, where I_n is a unit matrix of order n. A graph G is said to be singular (resp. nonsingular) if $\phi(G, 0) = 0$ (resp. $\phi(G, 0) \neq 0$)). The roots of $\phi(G, x)$ are called the eigenvalues of G. The multiplicity of the eigenvalue zero of G is called the nullity of G, which is denoted by $\eta(G)$.

The following result, which will play a key role in the proofs of our main results, is well known and useful ([1,2,15–18]):

Proposition 1.1 [4]. *Suppose G is a graph with n vertices. Then the coefficients of the characteristic polynomial* $\phi(G, x) = \sum_{i=0}^{n} a_i x^{n-i}$ *of G is given by*

$$(-1)^i a_i = \sum (-1)^{r(\Lambda)} 2^{s(\Lambda)},$$

where the summation ranges over all elementary subgraphs Λ *of G with i vertices.*

Collatz and Sinogowitz [3] and Schwenk and Wilson [7] posed the problem of characterizing all singular or nonsingular graphs. This problem is very difficult. At present, only some particular cases are known [4–6,8–12,14]. On the other hand, this problem is very interesting in chemistry, because, as has been shown in Longuet-Higgins [6], the occurrence of a zero eigenvalue of a bipartite graph (corresponding to an alternant hydrocarbon) indicates the chemical instability of the molecule which such a graph represents. The question is of interest also for non-alternant hydrocarbons (non-bipartite graph), but a direct connection with the chemical stability in these cases is not so straightforward. The following result gives a concise formula for the nullity of a tree T in terms of the matching number of T:

Proposition 1.2 [4]. *Suppose T is a tree with n vertices and the matching number of T is* $\nu(T)$. *Then*

$$\eta(T) = n - 2\nu(T).$$

Recently, Tan and Liu [13] investigated the nullity of the unicyclic graphs and proved the following interesting results:

Proposition 1.3 [13]. *The nullity set of all unicyclic graphs with n vertices* ($n \geqslant 5$) *is* $\{0, 1, 2, \ldots, n-4\}$.

Proposition 1.4 [13]. *Let G be a unicyclic graph with n vertices* ($n \geqslant 5$). *Then* $\eta(G) = n - 4$ *if and only if* $G \cong U_1^*, U_2^*$ *or* U_3^*, *where* U_i^* *are illustrated in Fig. 1 for* $1 \leqslant i \leqslant 3$, *where* S_{n_i} *is the star* K_{1,n_i-1}.

In order to investigate the unicyclic graphs with $\eta(G) = 0$, Tan and Liu [13] introduced the definition of the elementary unicyclic graphs as follows. A unicyclic graph G is called an elementary unicyclic graph if G is a cycle of length l and $l \neq 0$ (*mod* 4), or G is obtained from a cycle C_l and t isolated vertices (where $0 < t \leqslant l$ and $l = t$ (*mod* 2)) by the rule: first select t vertices from C_l such that there are an even number (which may be 0) of vertices between any two consecutive such vertices. Then join an edge from each of the t vertices chosen in C_l to an isolated vertex. Tan and Liu [13] proved the following:

Proposition 1.5 [13]. *If U is an elementary unicyclic graph, or a graph obtained by joining a vertex of an elementary unicyclic graph with an arbitrary vertex of a tree with a perfect matching, then U is a nonsingular unicyclic graph.*

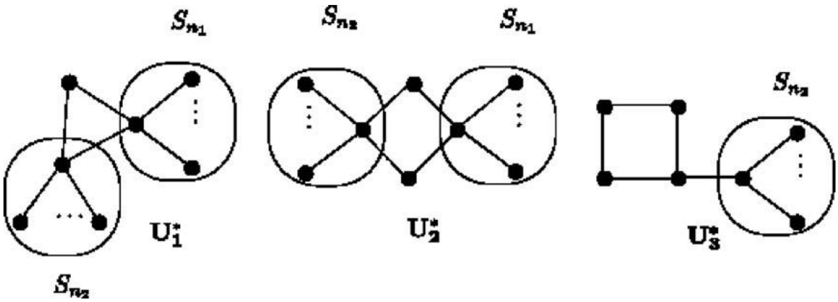

Fig. 1. Three graphs U_1^*, U_2^* and U_3^* in Proposition 1.4, where $n_1 \geqslant 1$ and $n_2 \geqslant 2$.

Let X_n be the set of the unicyclic graphs with n vertices and with $\eta(G) = 0$, and let Y_n be the set of unicyclic graphs with n vertices each of which is obtained from an arbitrary elementary unicyclic graph G' by joining a vertex of trees with a perfect matching to some or all vertices of G'. Clearly, we have $Y_n \subseteq X_n$. Tan and Liu [13] posed the following open problem:

Problem 1.1. *Does* $X_n = Y_n$*?*

In the next section, we prove that if G is a unicyclic graph, then $\eta(G)$ equals $n - 2\nu(G) - 1$, $n - 2\nu(G)$ or $n - 2\nu(G) + 2$. We also characterize these three types of graphs. In Section 3, we determine the unicyclic graphs G with $\eta(G) = n - 5$, and we also characterize the nonsingular unicyclic graphs, which answers affirmatively Problem 1.1.

2. The nullity of unicyclic graphs

For the sake of convenience, we will assume that G is a unicyclic graph with n vertices and the the cycle in G is denoted by C_l, where l is the length of the cycle. Let $G - C_l$ denote the induced subgraph of G by deleting, from G, vertices in C_l and their incident edges. We use the symbols $p = \nu(G)$ and $q = \nu(G - C_l)$ if not specified.

Now we can determine the nullity of non-bipartite unicyclic graphs as follows.

Lemma 2.1. *Suppose G is a unicyclic graph with n vertices and the length l of the cycle C_l in G is odd. Then $\eta(G) = n - 2\nu(G) - 1$ if $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$, and $\eta(G) = n - 2\nu(G)$ otherwise.*

Proof. Let $\nu(G) = p$ and $\nu(G - C_l) = q$. Suppose the characteristic polynomial of G is given by

$$\phi(G, x) = \sum_{i=0}^{n} a_i x^{n-i}. \tag{1}$$

By Proposition 1.1, we have

$$(-1)^i a_i = \sum (-1)^{r(\Lambda)} 2^{s(\Lambda)}, \tag{2}$$

where the summation ranges over all elementary subgraphs Λ of G with i vertices.

Note that $p \geqslant \frac{l-1}{2} + q$, that is, $2p \geqslant l - 1 + 2q$. Hence, if $i > 2p + 1 \geqslant l + 2q$, then G contains no elementary subgraphs with i vertices, which implies that if $i > 2p + 1$ we have $a_i = 0$. Since x^{n-2p-1} is a factor of $\phi(G, x)$, $\eta(G) \geqslant n - 2p - 1$.

If $p = \frac{l-1}{2} + q$, by (2) we have

$$(-1)^{2p+1} a_{2p+1} = \sum (-1)^{r(\Lambda)} 2^{s(\Lambda)},$$

where the summation ranges over all elementary subgraphs Λ of G with $2p+1$ vertices. Note that the set of the unions of the cycle C_l of G and every matching of $G-C_l$ with q edges equals exactly the set of elementary subgraphs of G with $2p+1$ vertices. Hence

$$a_{2p+1} = 2(-1)^{q+1}m(G-C_l,q) \neq 0,$$

where $m(G-C_l,q)$ is the number of matchings of $G-C_l$ with q edges. So we have proved that if $p=\frac{l-1}{2}+q$ then $\eta(G)=n-2\nu(G)-1$.

If $p \neq \frac{l-1}{2}+q$, then $2p+1 > l+2q$. Hence G contains no elementary subgraphs with $2p+1$ vertices, which shows that $\eta(G) \geqslant n-2p$. By a similar discussion, we have

$$a_{2p} = (-1)^p m(G,p) \neq 0,$$

which implies $\eta(G)=n-2p$ and the lemma follows. □

In Lemma 2.1 we have characterized the nullity of non-bipartite unicyclic graphs. Now we start to consider the case in which G is a bipartite unicyclic graph.

Lemma 2.2. *Suppose G is a unicyclic graph with n vertices and the length l of the cycle C_l in G is even. If $\nu(G) \neq \frac{l}{2}+\nu(G-C_l)$, or $\nu(G)=\frac{l}{2}+\nu(G-C_l)$ and $l=2 \ (mod\ 4)$, then $\eta(G)=n-2\nu(G)$.*

Proof. Let $\nu(G)=p$ and $\nu(G-C_l)=q$. Note that G is a bipartite graph. Hence the characteristic polynomial of G can be expressed by

$$\phi(G,x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} b_i x^{n-2i}. \tag{3}$$

By Proposition 1.1, we have

$$b_i = \sum (-1)^{r(\Lambda)} 2^{s(\Lambda)}, \tag{4}$$

where the summation ranges over all elementary subgraphs Λ of G with $2i$ vertices.

By a similar discussion as in the proof of Lemma 2.1, we can prove the following results:

(*i*) $\eta(G) \geqslant n-2p$;

(*ii*) $b_p = (-1)^p m(G,p) \neq 0$ if $p \neq \frac{l}{2}+q$;

(*iii*) $b_p = (-1)^p m(G,p) - 2(-1)^q m(G-C_l,q)$ if $p=\frac{l}{2}+q$.

Hence, by (*i*) and (*ii*), if $p \neq \frac{l}{2}+q$ then $\eta(G)=n-2p$. Note that, if $p=\frac{l}{2}+q$ and $l \neq 0 \ (mod\ 4)$, then $p \neq q \ (mod\ 2)$. So by (*iii*) we have $b_p = (-1)^p m(G,p) - 2(-1)^q m(G-C_l,q) \neq 0$, which also implies that $\eta(G)=n-2p$.

The lemma thus follows. □

Lemma 2.3. *Suppose G is a unicyclic graph with n vertices and cycle C_l of length $l=0 \ (mod\ 4)$, and $\nu(G)=\frac{l}{2}+\nu(G-C_l)$. Let E_1 be the set of edges of G between C_l and $G-C_l$ and E_2 the set of matchings of G with $\nu(G)$ edges. Then, $\eta(G)=n-2\nu(G)+2$ if $E_1 \cap M=\emptyset$ for all $M \in E_2$, and $\eta(G)=n-2\nu(G)$ otherwise.*

Proof. We use the notation in the proof of Lemma 2.2. Note that, by the proof of Lemma 2.2, we have shown that $\eta(G) \geqslant n-2p$.

First we prove that $\eta(G) \leqslant n-2p+2$. We only need to show that $b_{p-1} \neq 0$. Let

$\mathcal{M}_1$ = the set of matchings of G with $p-1$ edges;

$\mathcal{M}'_1$ = the set of matchings of $G-C_l$ with $q-1$ edges;

$\mathcal{M}_2 = \{\Lambda \mid \Lambda = C_l \cup M, M \in \mathcal{M}'_1\}$.

Obviously, the set of elementary subgraphs of G with $2p-2$ vertices equals exactly $\mathcal{M}_1 \cup \mathcal{M}_2$. Hence, by (4), we have

$$b_{p-1} = \sum_{\Lambda_1 \in \mathcal{M}_1} (-1)^{r(\Lambda_1)} + 2 \sum_{\Lambda_2 \in \mathcal{M}_2} (-1)^{r(\Lambda_2)} = (-1)^{p-1} m(G, p-1) - 2(-1)^{q-1} m(G - C_l, q-1).$$

In order to prove that $b_{p-1} \neq 0$, we only need to show that $m(G, p-1) > 2m(G - C_l, q-1)$, i.e., $|\mathcal{M}_1| > 2|\mathcal{M}_2|$.

Note that C_l contains exactly two perfect matchings, denoted by M_1 and M_2. It is obvious that

$$\mathcal{M}^* = \{M_1 \cup M | M \in \mathcal{M}_1'\} \cup \{M_2 \cup M | M \in \mathcal{M}_1'\} \subseteq \mathcal{M}_1.$$

Hence $|\mathcal{M}_1| \geqslant |\mathcal{M}^*| = 2|\mathcal{M}_1'| = 2|\mathcal{M}_2|$. Let M_3 be a matching of $G - C_l$ with q edges and M_4 a matching of C_l with $\frac{l}{2} - 1$ edges, then $M_3 \cup M_4$ is a matching of G with $\frac{l}{2} - 1 + q = p - 1$ edges. So $M_3 \cup M_4 \in \mathcal{M}_1$. Note that $M_3 \cup M_4 \notin \mathcal{M}^*$. Hence $|\mathcal{M}_1| \geqslant |\mathcal{M}^*| + 1 = 2|\mathcal{M}_2| + 1 > 2|\mathcal{M}_2|$.

Then the lemma follows from the following claim:

Claim we have $b_p = 0$ if $E_1 \cap M = \emptyset$ for an arbitrary $M \in E_2$ and $b_p \neq 0$ otherwise.

Let $\mathcal{M}_3$ be the set of matchings of $G - C_l$ with $\nu(G - C_l)$ edges. Let $\mathcal{M}_4$ be the set of matchings of G with $\nu(G)$ edges, each of which has at least one edge in E_1. Hence $\mathcal{M}_4 = \emptyset$ if $E_1 \cap M = \emptyset$ for an arbitrary $M \in E_2$ and $\mathcal{M}_4 \neq \emptyset$ otherwise. Note that we have

$$b_p = -2 \sum_{\Lambda_1 \in \mathcal{M}_3} (-1)^{r(\Lambda_1)} + \sum_{\Lambda_2 \in E_2} (-1)^{r(\Lambda_2)}.$$

For the case $\mathcal{M}_4 = \emptyset$, i.e., $E_1 \cap M = \emptyset$ for an arbitrary $M \in E_2$, since C_l have exactly two perfect matchings M_1 and M_2, we have $E_2 = \{M_1 \cup M | M \in \mathcal{M}_3\} \cup \{M_2 \cup M | M \in \mathcal{M}_3\}$. Hence

$$\sum_{\Lambda_2 \in E_2} (-1)^{r(\Lambda_2)} = (-1)^{|M_1|} \sum_{\Lambda \in \mathcal{M}_3} (-1)^{r(\Lambda)} + (-1)^{|M_2|} \sum_{\Lambda \in \mathcal{M}_3} (-1)^{r(\Lambda)} = 2(-1)^{\frac{l}{2}} \sum_{\Lambda \in \mathcal{M}_3} (-1)^{r(\Lambda)}.$$

Since $l = 0$ $(mod\ 4)$, we have

$$b_p = -2 \sum_{\Lambda_1 \in \mathcal{M}_3} (-1)^{r(\Lambda_1)} + 2 \sum_{\Lambda \in \mathcal{M}_3} (-1)^{r(\Lambda)} = 0.$$

For the case $\mathcal{M}_4 \neq \emptyset$, we have $E_2 = \{M_1 \cup M | M \in \mathcal{M}_3\} \cup \{M_2 \cup M | M \in \mathcal{M}_3\} \cup \mathcal{M}_4$. Hence

$$\begin{aligned}\sum_{\Lambda_2 \in E_2} (-1)^{r(\Lambda_2)} &= (-1)^{|M_1|} \sum_{\Lambda \in \mathcal{M}_3} (-1)^{r(\Lambda)} + (-1)^{|M_2|} \sum_{\Lambda \in \mathcal{M}_3} (-1)^{r(\Lambda)} + \sum_{\Lambda \in \mathcal{M}_4} (-1)^{r(\Lambda)} \\ &= 2 \sum_{\Lambda_2 \in \mathcal{M}_3} (-1)^{r(\Lambda_2)} + \sum_{\Lambda \in \mathcal{M}_4} (-1)^{r(\Lambda)},\end{aligned}$$

which implies that $b_p = \sum_{\Lambda \in \mathcal{M}_4} (-1)^{r(\Lambda)} \neq 0$. Thus the claim follows. □

By Lemmas 2.1–2.3, we have the following:

Theorem 2.1. *Suppose G is a unicyclic graph with n vertices and the cycle in G is C_l. Let E_1 be the set of edges of G between C_l and $G - C_l$ and E_2 the set of matchings of G with $\nu(G)$ edges. Then*

(1) $\eta(G) = n - 2\nu(G) - 1$ *if* $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$;

(2) $\eta(G) = n - 2\nu(G) + 2$ *if G satisfies properties:* $\nu(G) = \frac{l}{2} + \nu(G - C_l), l = 0$ $(mod\ 4)$ *and* $E_1 \cap M = \emptyset$ *for all* $M \in E_2$;

(3) $\eta(G) = n - 2\nu(G)$ *otherwise*.

Remark 2.1. Let G be a unicyclic graph with n vertices and C_l the cycle in G. If $\eta(G) = n - 2\nu(G) - 1$, then, by (1) in Theorem 2.1, l is odd and hence G is a non-bipartite graph.

Remark 2.2. Let G be a unicyclic graph with $n \geqslant 4$ vertices. Obviously, $\nu(G) \geqslant 2$. Hence, by Theorem 2.1, $\eta(G) \leqslant n - 4$. Let G be a unicyclic graph with n vertices and with $\eta(G) = n - 4$, and let the length of the cycle in G be l. If l is odd, by Lemma 2.1, $\eta(G) = n - 2\nu(G) - 1$, or $\eta(G) = n - 2\nu(G)$. Then we have $\nu(G) = 2$ and G must satisfy: $\nu(G) > \frac{l-1}{2} + \nu(G - C_l, q)$, i.e., $\nu(G) = 2, l = 3$ and $\nu(G - C_l) = 0$.

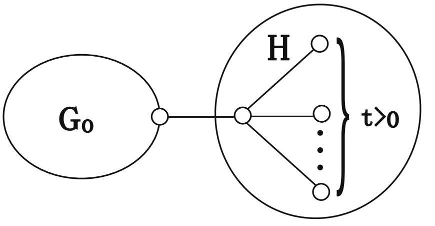

Fig. 2. A graph G and a pendant star H of G, where $G_0 = G - H$.

Hence G must have the form of U_1^* shown in Fig. 1. Similarly, if l is even, then, by Lemmas 2.2 and 2.3, we can show that G must have the form of U_2^* or U_3^* illustrated in Fig. 1. Hence, Proposition 1.4, which was proved by Tan and Liu [13], can be obtained from Theorem 2.1. In Section 3, we will characterize the unicyclic graph G with n vertices and with $\eta(G) = 0$ and $n - 5$, respectively.

A vertex-induced subgraph H of a graph G is called a pendant star of G if H is a star with at least two vertices and all pendant vertices of H are also pendant vertices in G. A graph G and a pendant star H of G are illustrated in Fig. 2, where G_0 is the graph $G - H$. The following result is immediate from the definition of the pendant star:

Lemma 2.4. *Suppose H is a pendant star of a graph G. Then $\nu(G) = \nu(G_0) + 1$, where $G_0 = G - H$.*

Suppose G is a unicyclic graph with n vertices. Let the length of the cycle in G be l. If G is a cycle C_l or a cycle C_l with pendant edges at some or all vertices of C_l, we call G a canonical unicyclic graph. If G is not canonical, G contains at least one pendant star H_1 such that $G_1^* = G - H_1$ is also a unicyclic graph. We call the procedure of obtaining $G - H_1$ from G a "deleting operator". With repeated applications of the "deleting operators", then a canonical unicyclic graph, denoted by G^*, is obtained from G.

Theorem 2.2. *Suppose G is a unicyclic graph with n vertices and G^* is the graph defined above. Then $\eta(G) = n - 2\nu(G) - 1$ if and only if $\eta(G^*) = |V(G^*)| - 2\nu(G^*) - 1$; $\eta(G) = n - 2\nu(G)$ if and only if $\eta(G^*) = |V(G^*)| - 2\nu(G^*)$; and $\eta(G) = n - 2\nu(G) + 2$ if and only if $\eta(G^*) = |V(G^*)| - 2\nu(G^*) + 2$.*

Proof. If G is a canonical unicyclic graph, then $G = G^*$ and the theorem holds. Hence we may assume that $G \neq G^*$. Then G has a pendant star H such that $G - H$ is a unicyclic graph. The theorem follows from the the following claims:

1. $\eta(G) = n - 2\nu(G) - 1$ if and only if $\eta(G - H) = n - |V(H)| - 2\nu(G - H) - 1$;
2. $\eta(G) = n - 2\nu(G)$ if and only if $\eta(G - H) = n - |V(H)| - 2\nu(G - H)$;
3. $\eta(G) = n - 2\nu(G) + 2$ if and only if $\eta(G - H) = n - |V(H)| - 2\nu(G - H) + 2$.

We prove that the second statement holds. Suppose that $\eta(G) = n - 2\nu(G)$. Note that, if G' is a graph obtained from G by deleting a pendant edge, then $\eta(G) = \eta(G')$, a result in [4]. Hence $\eta(G) = |V(H)| - 2 + \eta(G - H)$, which implies that $\eta(G - H) = n - |V(H)| + 2 - 2\nu(G)$. By Lemma 2.4, we have $\eta(G - H) = n - |V(H)| + 2 - 2(\nu(G - H) + 1) = n - |V(H)| - 2\nu(G - H)$. Similarly, we can show that if $\eta(G - H) = n - |V(H)| - 2\nu(G - H)$ then $\eta(G) = n - 2\nu(G)$. By a similar discussion, we can prove the first and the third statements. □

The following corollary, which can be obtained from Theorems 2.1 and 2.2, characterizes the unicyclic graphs G with $\eta(G) = |V(G)| - 2\nu(G) - 1$, $|V(G)| - 2\nu(G)$ and $|V(G)| - \nu(G) + 2$, respectively.

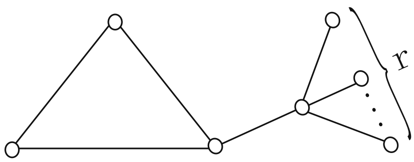

Fig. 3. The graph U_4^* in Theorem 3.3.

Corollary 2.1. *Suppose G is a unicyclic graph with n vertices and the length of the cycle in G is l. Let* G^* *be the graph defined above. Then* $\eta(G) = n - 2\nu(G) - 1$ *if* $G^* = C_l$ *and l is odd,* $\eta(G) = n - 2\nu(G) + 2$ *if* $G^* = C_l$ *and* $l = 0 \pmod 4$*, and* $\eta(G) = n - 2\nu(G)$ *otherwise.*

3. The unicyclic graphs with extremal nullity

In this section, we use some results in Section 2 to characterize the unicyclic graphs G with $\eta(G) = 0$ and $n - 5$, respectively.

Theorem 3.3. *Let G be a unicyclic graph with n vertices* $(n \geqslant 5)$ *and with* $\eta(G) = n - 5$*. Then G must have the form of* U_4^* *illustrated in Fig. 3 or* $G = C_5$*, where* $r > 0$*.*

Proof. Suppose the length of the cycle in G is l. Note that, by Theorem 2.1, $\eta(G) = n - 2\nu(G) - 1$, $n - 2\nu(G)$ or $n - 2\nu(G) + 2$. Hence if $\eta(G) = n - 5$ then G must satisfy:

(*i*) l is odd;
(*ii*) $\eta(G) = n - 2\nu(G) - 1$;
(*iii*) $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$.

By (*i*), (*ii*) and (*iii*), we have: $\nu(G) = 2$, and $l = 3$ or $l = 5$. If $l = 5$ then $\nu(G - C_l) = 0$. Note that $\nu(G) = 2$. Hence if $l = 5$ then $G = C_5$. If $l = 3$, it is trivial to show that G must have the form of U_4^* illustrated in Fig. 3. The theorem has thus been proved. □

Now we start to characterize the nonsingular unicyclic graphs. First we consider the case in which G is not bipartite.

Lemma 3.5. *Let G be a unicyclic graph with n vertices and the length l of the cycle* C_l *in G be odd. Then G is nonsingular if and only if G has a perfect matching or* $G - C_l$ *has a perfect matching.*

Proof. "⇐". If G has a perfect matching, then $n = 2\nu(G)$. If $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$, then, by Lemma 2.1, we have $\eta(G) = n - 2\nu(G) - 1 = -1$, a contradiction. Hence, by Lemma 2.1, $\eta(G) = n - 2\nu(G) = 0$. If $G - C_l$ contains a perfect matching, then $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$ and $n = 2\nu(G) + 1$. Hence, by Lemma 2.1, we have $\eta(G) = n - 2\nu(G) - 1 = 0$. Hence we have proved that sufficiency holds.

"⇒". Let G be nonsingular (i.e., $\eta(G) = 0$). By Lemma 2.1, either we have $n = 2\nu(G)$ and $\nu(G) > \frac{l-1}{2} + \nu(G - C_l)$ or we have $n = 2\nu(G) + 1$ and $\nu(G) = \frac{l-1}{2} + \nu(G - C_l)$, which implies that G has a perfect matching or $G - C_l$ has a perfect matching.

The lemma thus follows. □

For the bipartite unicyclic graphs G with $\eta(G) = 0$, we have the following:

Lemma 3.6. *Let G be a unicyclic graph with n vertices and the length l of the cycle* C_l *in G be even. Then G is nonsingular if and only if G contains a unique perfect matching or* $l \neq 0 \pmod 4$ *and G has two perfect matchings.*

Proof. Since G is a bipartite graph with n vertices, the characteristic polynomial of G can be expressed by

$$\phi(G,x)=\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor} b_i x^{n-2i}.$$

"$\Leftarrow$". If G contains a unique perfect matching, by Proposition 1.1 we have $b_{\frac{n}{2}}=(-1)^{\frac{n}{2}}$. If G contains two perfect matchings, $G-C_l$ contains a (unique) perfect matching. By Proposition 1.1, we have $b_{\frac{n}{2}}=2(-1)^{\frac{n}{2}}+2(-1)^{\frac{n-l}{2}+1}$. Note that $l\neq 0\ (mod\ 4)$. Hence we have $b_{\frac{n}{2}}=4(-1)^{\frac{n}{2}}$. So we have shown that if G contains a unique perfect matching or $l\neq 0\ (mod\ 4)$ and G has two perfect matchings then $\eta(G)\neq 0$. Sufficiency thus follows.

"$\Rightarrow$". We assume that G is nonsingular. Hence $\eta(G)=0$. By Lemmas 2.2 and 2.3, $\eta(G)=n-2p$ or $\eta(G)=n-2p+2$, where $p=\nu(G)$. Hence $n=2p$ or $n=2p-2$. Note that $n\geqslant 2p$, thus it is impossible that $n=2p-2$. So $n=2p$, which shows that G contains perfect matchings. Note that G contains at most two perfect matchings. Thus either $l=0\ (mod\ 4), E_1\cap M\neq\emptyset$ for arbitrary $M\in E_2, n=2p$ or $l=2\ (mod\ 4), n=2p$. Hence we only need to prove that if G contains two perfect matchings then $l\neq 0\ (mod\ 4)$. We prove this by contradiction. If $l=0\ (mod\ 4)$, by a similar way as in the proof of Lemma 2.2, we have

$$b_{\frac{n}{2}}=(-1)^{\frac{n}{2}}m\left(G,\frac{n}{2}\right)+2(-1)^{\frac{n-l}{2}+1}m\left(G-C_l,\frac{n-l}{2}\right).$$

Since G has two perfect matchings and $G-C_l$ contains a unique perfect matching (a matching with $\frac{n-l}{2}$ edges), we have

$$b_{\frac{n}{2}}=(-1)^{\frac{n}{2}}m\left(G,\frac{n}{2}\right)+2(-1)^{\frac{n-l}{2}+1}m\left(G-C_l,\frac{n-l}{2}\right)=0.$$

This contradicts $\eta(G)=0$.

So we have finished the proof of the lemma. □

The following result is immediate from Lemmas 3.5 and 3.6.

Theorem 3.4. *Suppose G is a unicyclic graph and the cycle in G is denoted by C_l. Then G is nonsingular if and only if G satisfies one of the following properties*:

(1) *l is odd and $G-C_l$ contains a perfect matching*;
(2) *G contains a unique perfect matching*;
(3) *$l\neq 0\ (mod\ 4)$ and G contains two perfect matchings*.

Corollary 3.2. *Let X_n and Y_n be as in Problem 1.1. Then $X_n=Y_n$.*

Proof. Note that, by Theorem 2.2, $Y_n\subseteq X_n$. We only need to prove $X_n\subseteq Y_n$. Suppose that G is a nonsingular unicyclic graph with n vertices. Let the cycle in G be denoted by C_l (i.e., $G\in X_n$). By Theorem 3.4, G must satisfy one of the following properties:

(1) l is odd and $G-C_l$ contains a perfect matching;
(2) G contains a unique perfect matching;
(3) $l\neq 0\ (mod\ 4)$ and G contains two perfect matching.

If G satisfies the property (1), then $G-C_l$ is a forest with a perfect matching. By the definition of $Y_n, G\in Y_n$. If G satisfies the properties (2) or (3), then, by Theorem 2.2, either G is an elementary unicyclic graph or each of its pendant stars, obtained when we obtain G^* from G by the "deleting operators", is a P_2. Then $G\in Y_n$. Hence $X_n\subseteq Y_n$ and the corollary follows. □

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References

[1] G.G. Cash, The permanental polynomial, J. Chem. Inf. Comput. Sci. 40 (2000) 1203–1206.
[2] R.S. Chen, A note on the relations between the permanental and characteristic polynomials of coronoid hydrocarbons, MATCH Commun. Math. Comput. Chem., 51 (2004) 137–148.
[3] L. Collat, U. Sinogowitz, Spektren endlicher Grafen, Abh. Math. Sem. Univ., Hamburg 21 (1957) 63–77.
[4] D.M. Cvetkocić, M. Doob, H. Sachs, Spectra of Graphs – Theory and Application, Academic Press, New York, 1980.
[5] S. Fiorini, I. Gutman, I. Sciriha, Trees with maximum nullity, Linear Algebra Appl. 397 (2005) 245–251.
[6] H.C. Longuet-Higgins, Resonance structures and MO in unsaturated hydrocarbons, J. Chem. Phys. 18 (1950) 265–274.
[7] A.J. Schwenk, R.J. Wilson, Selected Topics in Graph Theory, Academic Press, New York, 1978, pp. 307–336.
[8] I. Sciriha, On the construction of graphs of nullity one, Discrete Math. 181 (1998) 193–211.
[9] I. Sciriha, On the rank of grpahs, in: Y. Alavi, D.R. Lick, A. Schwenk (Eds.), Combinatorics, Graph Theory and Algorithms, vol. 2, Michigan, 1999, pp. 769–778.
[10] I. Sciriha, On singular line graphs of trees, Congr. Numer. 135 (1998) 73–91.
[11] I. Sciriha, I. Gutman, On the nullity of line graphs of trees, Discrete Math. 232 (2001) 35–45.
[12] I. Sciriha, I. Gutman, Nut graphs-maximally extending cores, Util. Math. 54 (1998) 257–272.
[13] X. Tan, B. Liu, On the nullity of unicyclic graphs, Linear Algebra Appl. 408 (2005) 212–220.
[14] Y.N. Yeh, I. Gutman, C.M. Fu, Graph transformations which preserve the multiplicity of an eigenvalue, Discrete Appl. Math. 67 (1996) 221–228.
[15] W. Yan, L. Ye, On the minimal energy of trees with a given diameter, Appl. Math. Lett. 18 (2005) 1046–1052.
[16] W. Yan, F. Zhang, On the permanental polynomials of some graphs, J. Math. Chem. 35 (2004) 175–188.
[17] W. Yan, F. Zhang, Enumeration of perfect matchings of graphs with reflective symmetry by Pfaffians, Adv. Appl. Math. 32 (2004) 655–668.
[18] W. Yan, F. Zhang, Enumeration of perfect matchings of a type of Cartesian products of graphs, Discrete Appl. Math. 154 (2006) 145–157.