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Enhanced multilevel linear sampling methods for inverse scattering problems

Jingzhi Li [a,*], Hongyu Liu [b], Qi Wang [c]

[a] *Faculty of Science, South University of Science and Technology of China, Shenzhen, 518055, PR China*
[b] *Department of Mathematics and Statistics, University of North Carolina, Charlotte, NC 28223, USA*
[c] *Department of Computing Sciences, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, PR China*

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ABSTRACT

We develop two enhanced techniques for the multilevel linear sampling method (MLSM) proposed in [32] for inverse scattering problems. Under some practical situations, the MLSM suffers certain undesirable "breakage cells" problem. We propose to avoid the curse of "breakage cells" by incorporating "expanding" and "searching" techniques. The new techniques are shown to significantly improve the robustness of the MLSM, and meanwhile they possess the same optimal computational complexity as the MLSM. Numerical experiments are presented to illustrate the promising features of the enhanced MLSMs.

1. Introduction

We consider the inverse problems of reconstructing the shape of an unknown scatterer by measuring the corresponding scattered acoustic or electromagnetic wave fields away from the target object. These inverse scattering problems are of central importance in many areas of science and technology, such as radar and sonar, geophysical exploration, non-destructing testing and medical imaging to name just a few (see [4,3,19,24,38]).

In this paper, we shall be mainly concerned with the effective and efficient numerical reconstruction algorithms for attacking the inverse scattering problems (ISPs). There are extensive studies in the literature in this aspect. One effective inversion approach is to treat the ISPs as non-linear ill-posed operator equations and to adapt constrained optimization techniques to estimate the shape [11,34]. Among others, we would like to mention the work [1,26,37] of using integral equation techniques, and a more recent one [21] of using fast solvers and sophisticated preconditioned iterative techniques. We also refer interested readers to the optimal control approach [6,8] for solving inverse scattering with extended targets (of characteristic size larger than the operating wavelength) for an alternative strategy in this category. The non-linear optimization and optimal control approaches can produce fine reconstructions for the ISPs in many practical situations. However, the iterative procedure involved requires the solution of forward scattering problems at each iteration, which makes the whole process rather computationally slow. Moreover, such approaches require the *a priori* knowledge of the physical properties of the underlying scatterer, which for many practical applications may not be available.

An alternative approach that has received wide attentions in the literature is more "indirect". The idea behind is to develop various "indicator functions" to characterize the profile of the scattering support, namely the boundary of the underlying obstacle or the inhomogeneous medium. From the qualitative behaviors of the indicator functions, one can reconstruct

* Corresponding author.
E-mail address: li.jz@sustc.edu.cn (J. Li).

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the shape of the underlying scatterer. In this approach, the reconstruction process is independent of physical properties of the underlying scatterers, which is of significant practical importance when little *a priori* information is available about the target object. There are a lot of methods developed in this category by various researchers. We would like to mention the linear sampling method [17], the enclosure method [22], the probe method [23,35,30], the MUSIC type method [9,10,15,27], the factorization method [28], the direct sampling method [25,33], and a polarization tensor-based approach for shape reconstruction in the low-frequency case [7]. In particular, the relation between the MUSIC and factorization methods has been clarified in [5]. We refer to [12,13,19,20,28,35,36,38] and the references therein for these and other developments in the literature.

In this work, we shall consider the enhanced techniques of the linear sampling method (LSM) for ISPs. The LSM was originated in [17] by Colton and Kirsch. It makes use of the blowup behavior of an indicator function that can be solved from a linear far-field integral equation. The method is computationally faster than the non-linear optimization approach since only linear inversions would be involved. Moreover, the LSM requires no *a priori* knowledge of the physical properties of the underlying target object, which possibly includes, at the same time, sound-soft, sound-hard or impedance-type impenetrable obstacles, or penetrable inhomogeneous mediums. The method has also been shown to work for both inverse acoustic and electromagnetic scattering problems. We refer to [13] for a comprehensive review. In [2,16,31], strategies on how to choose the critical cut-off values and how to avoid the interior eigenvalues for the LSM have been developed. In [32], a multilevel technique was developed for the LSM, which was shown to possess the optimal computational complexity. Indeed, for an $n \times n$ sampling mesh in R^2 or an $n \times n \times n$ sampling mesh in R^3, the multilevel linear sampling method (MLSM) requires to solve only $O(n^{N-1})$ far-field equations for an R^N problem ($N = 2, 3$). This is in sharp contrast to the original LSM which requires to solve n^N far-field equations.

However, the MLSM is shown to suffer some "breakage cells" problem in the present paper. That is, some cells on the sampling mesh which lie on the boundary of the scatterer would be trimmed down undesirably in the MLSM processing. This happens particularly at the boundary of a scatterer where the curvature is very large and at the later stage of the MLSM. The major goal of this work is to develop two enhanced techniques to avoid the "breakage cells". Specifically, we propose the "searching" and "expanding" strategies which are shown to effectively defeat the undesirable "breakage cells" problem, even in some extreme situations. On the other hand, the enhanced MLSMs (EMLSMs) are shown to possess the same optimal computational complexity as the original MLSM. Hence, the proposed methods significantly improve the robustness of the MLSM, and meanwhile they bring no extra computational complexity.

We shall develop our EMLSMs based on a model inverse scattering problem of reconstructing the shapes of acoustic obstacles. However, we would like to emphasize that since our proposed techniques are mainly some computational strategies, they can be straightforwardly extended to the LSM for reconstructing the supports of inhomogeneous mediums, and for inverse electromagnetic scattering problems as well.

The rest of the paper is organized as follows. In Section 2, we introduce the inverse acoustic obstacle scattering problem that we shall take as the model problem for our subsequent study. Section 3 is devoted to a brief review of the LSM and MLSM. In Section 4, we present the EMLSMs, together with the computational complexity analysis. Section 5 contains the numerical results, which illustrate the effectiveness of the proposed EMLSMs. Our work is concluded in Section 6.

2. Inverse acoustic obstacle scattering problem

In this section, we briefly introduce the time harmonic inverse acoustic obstacle scattering problem that we shall use as a model problem for our subsequent study on the EMLSMs.

Consider an impenetrable obstacle D, which is assumed to be the open complement of an unbounded domain of class C^2 in R^N ($N = 2, 3$); that is, we include into our study the scattering from obstacles with more than one (but finitely many) components. Given an incident wave field u^i, the presence of the obstacle will give rise to a scattered wave field u^s. Throughout, we take $u^i(x) = \exp\{\mathrm{i}kx \cdot d\}$ to be a time-harmonic plane wave, where $\mathrm{i} = \sqrt{-1}$, $d \in \mathrm{R}^{N-1}$ and $k \in \mathrm{R}_+$ are, respectively, the incident direction and wave number. We define $u(x) = u^i(x) + u^s(x)$ to be the total field, which satisfies the following Helmholtz system (cf. [18,19]),

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathrm{R}^N \backslash \bar{\mathrm{D}}, \\ \lim_{r\to\infty} r^{\frac{N-1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0, & r = |x|. \end{cases} \tag{1}$$

To complete the description of the direct scattering problem, we need to impose suitable boundary conditions on $\partial\mathrm{D}$, which depend on the physical properties of the underlying scatterer. For a sound-soft obstacle, the pressure of the total wave vanishes on the boundary of the obstacle, resulting in a *Dirichlet boundary condition*

$$u = 0 \quad \text{on } \partial\mathrm{D}; \tag{2}$$

whereas for a sound-hard obstacle, the normal velocity of the acoustic wave field vanishes on the boundary of the obstacle, leading to a *Neumann boundary condition*

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\mathrm{D}, \tag{3}$$