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Optimal average sample number of the SPRT chart for monitoring fraction nonconforming

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ABSTRACT

The Sequential Probability Ratio Test (SPRT) control chart is a powerful tool for monitoring manufacturing processes. It is highly suitable for the applications where testing is destructive or very expensive, such as the automobile airbags test. This article studies the effect of the Average Sample Number (*ASN*) (i.e., the average sample size) on the chart's performance. A design algorithm is proposed to develop the optimal SPRT chart for monitoring the fraction nonconforming p of Bernoulli processes. By optimizing the *ASN* and other charting parameters, the average detection speed of the SPRT chart is almost doubled. It is also found that the optimal SPRT chart significantly outperforms the optimal *np* and binomial CUSUM charts, in terms of Average Number of Defectives (*AND*), under different combinations of the design specifications. It is observed that the SPRT chart using a relatively smaller *ASN* and a shorter sampling interval (h) has a higher overall detection effectiveness.

1. Introduction

Since its introduction in Statistical Process Control (SPC) as an effective tool to monitor processes and ensure quality, the control chart had fast become a necessity and has been increasingly adopted in modern industries and beyond the manufacturing sectors (Chen and Cheng, 2009; Wang, 2012; Chen, 2013). The *np* chart is the most popular and simple control chart for attributes. It is used to examine the number d of nonconforming units found in a sample or to monitor the fraction nonconforming p. The process is considered to be in control if $d \leqslant UCL$, where UCL is the upper control limit of the *np* chart. However, if $d > UCL$, then an upward p shift is signaled.

More advanced charts for attributes including the binomial CUSUM chart and SPRT chart have also been developed to monitor p (Gan, 1993; Reynolds and Stoumbos, 1998; Wu and Luo, 2003; Wu et al., 2006, 2008). The widespread applications of the *np* chart and other attribute charts are due to several factors, such as the simplicity of handling attribute quality characteristics, the ease of communication between people at different levels, the capability of checking multiple quality requirements, and the prevalence of count data in many applications, especially in non-manufacturing sectors. Many quality characteristics cannot be measured on a numerical or a quantitative scale.

Unlike the *np* chart that only uses the information of d in the current sample, the binomial cumulative sum (CUSUM) chart incorporates all the information in the sequence of observed values of d (Lucas, 1985). While the CUSUM chart is more sensitive to small and moderate shifts in fraction nonconforming p, it is less effective than the *np* chart for detecting large p shifts. A statistic E_i is updated and plotted for the ith sample in a binomial CUSUM chart for detecting upward p shifts.

$$E_0 = 0$$
$$E_i = \max(0, E_{i-1} + d_i - k) \quad (1)$$

where k is the reference parameter and d_i is the number of nonconforming units found in the ith sample. When an increasing p shift occurs, E_i tends to increase. Eventually, a sample point will exceed the control limit U of the CUSUM chart, and thereby an out-of-control signal is produced.

The Bernoulli CUSUM chart is a special case of the binomial CUSUM chart. If the sample size n of a binomial CUSUM chart is set at one, the binomial CUSUM chart is termed as the Bernoulli CUSUM chart. Reynolds and Stoumbos (1999) found that the Bernoulli CUSUM chart substantially outperforms the p chart for detecting shifts in p. Reynolds and Stoumbos (2000) showed that there is a little difference between the Bernoulli CUSUM chart and the binomial CUSUM chart in terms of the expected time required to detect small and moderate shifts in p, but the Bernoulli CUSUM chart is better for detecting large shifts in p. Bourke (2001) also highlighted

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that the advantage of using $n = 1$ for the CUSUM chart is greater for larger shifts when the in-control Average Run Length is small.

Traditional control charts are operated by using a Fixed Sampling Rate (FSR) obtaining samples of fixed size n from the process using a fixed sampling interval h. Contrary to these traditional charts, n may be varied based on the data observed in the current sample. This is the concept of sequential analysis (Wald, 1947; Ghosh, 1970). Wald (1947) first defined the Sequential Probability Ratio Test (SPRT) and showed that it is optimal to produce a lower expected sampling size than any other tests with the same probability of error. The SPRT charts are usually much more effective than the FSR charts, but they are more difficult to implement.

Woodall and Reynolds (1983) used tests that can be represented exactly by discrete Markov chains to approximate the properties of the SPRT. Stoumbos and Reynolds (1997b) employed the Corrected Diffusion Theory (CDT) to evaluate the properties and statistical design of SPRT charts. Researchers also applied the SPRT chart to different areas and applications, such as sampling plan, arc welding, failure analysis and radiation (Bagchi, 1992; Stefan et al., 1996; Kwon et al., 2008; Luo et al., 2010). Stoumbos and Reynolds (1996) generalized the development and applications of the SPRT chart. The SPRT chart is especially appropriate for applications where testing is destructive and/or expensive, such as deployment-rate testing of automobile airbags and durability testing for batch-produced plastic eyeglass lenses (Stoumbos and Reynolds, 1997a). Usually, the time required to test an airbag is short, compared with the interval between the sampling inspections, which are often no more than once for every work shift. In such applications, the efficient use of sampling resources is very important. With the adoption of the SPRT chart, the sampling rate will be very low when the process is in control, and therefore the sampling cost can be reduced substantially.

An attribute SPRT control chart for detecting shifts in fraction nonconforming p was proposed by Reynolds and Stoumbos (1998). Inside a sample of an SPRT chart, individual observations are taken sequentially, with the possibility of a decision about the process after each observation. The statistical properties of the SPRT chart are evaluated based on the assumption that the time required to obtain an individual observation is short enough to be neglected relative to the sampling interval h between two samples. The SPRT chart has the administrative advantage of using a fixed sampling interval (FSI). Since the SPRT chart allows the sample size used at each sample to vary, it is similar to a variable sample size (VSS) chart. However, while the sample size of a VSS chart used at the current sample point depends on the data obtained in the last sample (Chen and Hsieh, 2007; Lee, 2013), the sampling size of an SPRT chart is determined based on the data observed at the current sample point.

Reynolds and Stoumbos (1998) found that the SPRT chart is substantially more effective than the FSR charts, such as the p chart, the binomial CUSUM chart, or the Bernoulli CUSUM chart. However, neither the p chart, CUSUM chart, nor SPRT chart had been optimized in their study. Consequently, none of these charts performs at its highest detection effectiveness. Moreover, no systematic procedure was provided to determine the charting parameters of the SPRT chart. If these parameters are optimized, the overall effectiveness of the SPRT chart increases.

The objective of this article is to find the optimal Average Sample Number (*ASN*) of the attribute SPRT chart that results in the best overall performance. Moreover, the influence of *ASN* on the overall performance of the SPRT chart is studied. The results of performance studies reveal that the SPRT chart using a relatively small *ASN* (together with a short h) has a very high effectiveness. However, the optimal value of *ASN* of each SPRT chart needs to be determined by the optimal design. The sample size n and sampling interval h of the np and binomial CUSUM charts will also be optimized in order to achieve their best overall performance. The optimal n and the corresponding h are determined by an exhaustive search that tests all the possible values of n (starting from $n = 1$). The optimal n should be explored in a range as broad as possible until no further improvement in the overall performance can be expected. Optimizing n and h for these traditional attribute control charts will also significantly improve their detection effectiveness so that they can stand as firm competitors to the SPRT chart.

The SPRT chart proposed by Reynolds and Stoumbos (1998) is called the basic SPRT chart in this article. In the design of a basic SPRT chart, *ASN* and the reference value γ are given in advance. Two new SPRT charts are proposed in this article. The first one is a semi-optimal SPRT chart in which only γ is optimized. The second one is called the optimal SPRT chart in which both γ and *ASN* are optimized. In both new SPRT charts, the optimal design aims at minimizing an objective function *AND* (Average Number of Defectives) (Haridy et al., 2013). As the *AND* is minimized, the overall performance of the SPRT chart will be improved.

The in-control and out-of-control performance of a control chart is usually measured by the Average Time to Signal (*ATS*). The in-control ATS_0 must be large enough so that a false alarm occurs infrequently. At the same time, the out-of-control *ATS* should be small enough to detect the process shifts quickly. There is always a trade-off between a larger in-control ATS_0 and a smaller out-of-control *ATS*. In this article, *ATS* is calculated by using the steady-state mode. This mode implies that the process starts and stays in an in-control condition for a long time and then a process shift occurs at some random time. This random time is assumed to have a uniform distribution between two samples (Reynolds et al., 1990).

In this article, it is assumed that the random number d (the number of nonconforming units in a sample) follows a binomial distribution. The focus of the research is to monitor the fraction nonconforming p of Bernoulli processes. The time required to take an observation inside each sample is negligible compared with the sampling interval h between samples.

Since the control charts for attributes are most often used to detect an increase in fraction nonconforming p or deterioration in product quality (Lucas, 1985; Reynolds and Stoumbos, 1999), the focus of this research is to detect increasing p shifts. If detecting decreasing p shifts is desired, a symmetric SPRT scheme can be built as well.

The remainder of this article proceeds as follows. In Section 2, the SPRT chart is introduced. The objective function *AND* is presented in Section 3, and the determination of the optimal *ASN* of the SPRT chart is also detailed in this section. In Section 4, a comparative study is conducted and the effect of *ASN* on the performance of the SPRT chart is analyzed. An illustrative example is given in Section 5. Finally, the discussions and conclusions are drawn.

2. SPRT chart

An SPRT chart has five charting parameters: Average Sample Number (*ASN*) (or average sample size), reference value (γ), sampling interval (h), lower limit (g) and upper limit (H). A sample is taken at the end of each fixed sampling interval h. Within a sample, when the ith observation x_i has been taken, it is used to update the test statistic C_i. For an upper one-sided SPRT chart (where $-\infty < g < H < \infty$),

$$\begin{aligned} C_0 &= 0 \\ C_i &= C_{i-1} + x_i - \gamma \end{aligned} \tag{2}$$

where x_i is a Bernoulli random variable which is defined as $x_i = 1$ if the ith item is nonconforming and $x_i = 0$ otherwise. The choice of γ